Exchange of momentum between moving matter induced by the zero-point fluctuations of the electromagnetic field

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We propose a quantum theory for the characterization of the momentum of moving media at zero temperature. It is demonstrated that the zero-point quantum fluctuations of the electromagnetic field may cause a net transfer of momentum between polarizable bodies moving at different velocities in Casimir-type geometries. However, the total net momentum induced by the quantum fluctuations, i.e., the total additional momentum imparted to all the matter in the system, vanishes. It is proven that the exchanged momentum can be calculated from the zero-point interaction energy of the system.

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I. INTRODUCTION

The Casimir phenomenon is one of the most impressive macroscopic manifestations of quantum effects: Due to the quantum fluctuations of the electromagnetic field, two macroscopic dielectric bodies standing in a vacuum experience an attractive long-range measurable force [1]. The Casimir effect is a consequence of the zero-point energy of the vacuum, which does not vanish even when the system is in the ground state at zero temperature.

In 2004, Feigel predicted a Casimir-like quantum effect, such that the vacuum fluctuations in crossed electric and magnetic static fields would result in the motion of dielectric liquids [2]. Feigel's prediction raised some criticism and is not consensual [3], but the possibility of a momentum transfer from the quantum vacuum to magnetoelectric matter is generally supported by several studies, at least for the case of Casimir-type geometries, e.g., Refs. [4–6]. One of the unsatisfactory aspects of Feigel's theory is that it is semiclassical, and requires the introduction of a high-frequency cutoff to yield a finite value for the momentum imparted to matter. It has been shown that this problem can be solved with field regularization techniques [4]. A recent paper reported that the estimates of the momentum obtained with Feigel's theory are inconsistent with experimental data [7], but was unable to rule out Feigel's effect based on other alternative theories [4]. A correction to Feigel's theory was also proposed [8].

The objective of this paper is to investigate the possible exchange of momentum between moving matter mediated by the quantum vacuum in Casimir-type geometries. Such possibility was discussed in some works [9,10] but to our best knowledge a complete theory or detailed study of this effect is still needed. Here we develop a fully quantum-mechanical theory of the zero-point momentum in a system with moving components in thermodynamical equilibrium at zero temperature. Our theory is based on a macroscopic quantization of the electromagnetic field, such that the moving matter is modeled as continuous nondispersive lossless medium. We obtain an explicit formula for the regularized additional momentum imparted to the moving matter by the quantum fluctuations of the electromagnetic field, which enables us to numerically quantify the momentum transfer.

II. QUANTIZATION OF THE ELECTROMAGNETIC FIELD AND POLARIZATION WAVES IN MOVING MEDIA

In this paper, we are interested in the characterization of the momentum induced by quantum fluctuations in a scenario wherein a collection of polarizable moving particles interacts with the electromagnetic vacuum. Even though our theory is based on a continuous medium approximation, for convenience, whenever pertinent, we will picture the moving matter as moving electric dipoles formed by two charges with opposite signs. To begin with, we identify the degrees of freedom and the Hamiltonian of the system (a representative geometry is shown in Fig. 1).

As is well known, the total energy of two charges with opposite signs (an electric dipole) can be decomposed into two components: one which determines the oscillations of the charges with respect to the center of mass (manifested macroscopically in the form of "polarization waves"), and another which is associated with the dynamics of the center of mass. In macroscopic nondispersive media, the energy associated with the "polarization waves" (kinetic and potential energy associated with vibrations of the dipoles) and with the electromagnetic field is described by the energy density $W_{\mathrm{EM},P} = \frac{1}{2}(\mathbf{B}\cdot\mathbf{H} + \mathbf{D}\cdot\mathbf{E})$, where $(\mathbf{E},\mathbf{H})$ and $(\mathbf{B},\mathbf{D})$ are the usual macroscopic fields that appear in the context of macroscopic electrodynamics [11]. On the other hand, it is possible to show that the component of the energy associated with the center of mass of a single dipole is given by $P_{\mathrm{can}}^2/(2M)$, where P_{can} is the total *canonical momentum* of the two charges, and M is the total mass. Notice that for charged particles the canonical momentum may differ from the kinetic momentum [11]. The momentum vector $\mathbf{P}_{\mathrm{can}}$ is the canonical conjugate of the center-of-mass coordinates $(\mathbf{r}_0)$. Thus, the total energy of the system may be written as

$$\varepsilon_{\mathrm{tot}} = \sum_{\substack{\text{all dipoles}\\(l)}} \frac{P_{\mathrm{can},l}^2}{2M_l} + \int d^3\mathbf{r}\, W_{\mathrm{EM},P}. \tag{1}$$

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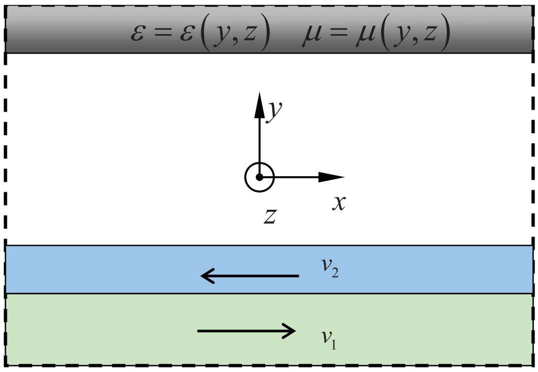

FIG. 1. (Color online) Sketch of a representative geometry of the system under study. In our model, the cavity may contain nondispersive moving dielectric bodies with a nonuniform permittivity and permeability along the y and z directions. All the bodies within the cavity should be invariant to translations along the direction of movement (in the figure, the moving components are the slabs with velocities v_1 and v_2). The cavity is electromagnetically closed and is terminated with periodic boundary conditions.

In the continuous medium approximation, each individual body may be assumed rigid, so that the relative distance between the microscopic dipoles that form the material is fixed. In such a case, the dynamics of the centers of mass of the dipoles associated with the ith body can be simply described by the canonical conjugate coordinates $\mathbf{r}_{0,i}$ and $\mathbf{P}_{\mathrm{can},i}$ (2×3 degrees of freedom per macroscopic body), where $\mathbf{r}_{0,i}$ and $\mathbf{P}_{\mathrm{can},i}$ is the center of mass and total canonical momentum of the ith body, respectively. Within that approximation, the Hamiltonian of the system may be rewritten as

$$H_{\mathrm{tot}} = \sum_{\substack{\text{all bodies}\\(i)}} \frac{P^2_{\mathrm{can},i}}{2M_i} + H_{\mathrm{EM},P}, \tag{2}$$

where M_i is the total mass of the ith body and $H_{\mathrm{EM},P} = \int d^3\mathbf{r}\, W_{\mathrm{EM},P}$. This discussion shows that the degrees of freedom of the system are those of the centers of mass of the moving bodies ($\mathbf{r}_{0,i}$ and $\mathbf{P}_{\mathrm{can},i}$), and those of the polarization waves and the electromagnetic field (associated with $H_{\mathrm{EM},P}$). In our formalism, the coordinates associated with the center of mass are not quantized. This means that the coordinates $\mathbf{r}_{0,i}$ and $\mathbf{P}_{\mathrm{can},i}$ are treated semiclassically, and only the coordinates associated with the electromagnetic field and the oscillations of the dipoles are quantized. Hence, $H_{\mathrm{EM},P}$ is promoted to a quantum operator $\hat{H}_{\mathrm{EM},P}$.

Within a semiclassical description the dynamics of the coordinates $(\mathbf{r}_{0,i},\mathbf{P}_{\mathrm{can},i})$ associated with the centers of mass is thus determined by $\frac{d\mathbf{r}_{0,i}}{dt} = \frac{\mathbf{P}_{\mathrm{can},i}}{M} + \langle \frac{\partial \hat{H}_{\mathrm{EM},P}}{\partial \mathbf{P}_{\mathrm{can},i}} \rangle$ and $\frac{d\mathbf{P}_{\mathrm{can},i}}{dt} = -\langle \frac{\partial \hat{H}_{\mathrm{EM},P}}{\partial \mathbf{r}_{0,i}} \rangle + \mathbf{F}_{\mathrm{ext}}$, where $\mathbf{F}_{\mathrm{ext}}$ represents the action of possible (classical) external forces acting on the bodies, $\langle \frac{\partial \hat{H}_{\mathrm{EM},P}}{\partial \mathbf{P}_{\mathrm{can},i}} \rangle = \langle \psi | \frac{\partial \hat{H}_{\mathrm{EM},P}}{\partial \mathbf{P}_{\mathrm{can},i}} | \psi \rangle$ is the quantum expectation of $\frac{\partial \hat{H}_{\mathrm{EM},P}}{\partial \mathbf{P}_{\mathrm{can},i}}$, and $|\psi(t)\rangle$ represents the state of the electromagnetic field and polarization waves. In the framework of macroscopic electrodynamics of moving media, $H_{\mathrm{EM},P}$ is expressed in terms of $(\mathbf{r}_{0,i},\mathbf{v}_i)$ rather than as a function of $(\mathbf{r}_{0,i},\mathbf{P}_{\mathrm{can},i})$, where $\mathbf{v}_i = d\mathbf{r}_{0,i}/dt$ is the velocity of the bodies. This follows from the fact that the effective parameters (permittivity, etc.) of a moving body are written in terms of the velocity and of its electric susceptibility in the comoving frame (see below). However, it can be checked that within the same degree of approximation used to derive Eq. (2), the operator $\frac{\partial \hat{H}_{\mathrm{EM},P}}{\partial \mathbf{P}_{\mathrm{can},i}}$ can be replaced by $\frac{1}{M}\frac{\partial \hat{H}_{\mathrm{EM},P}}{\partial \mathbf{v}_i}$. Thus, we can write the following equations that characterize the dynamics of the centers of mass of the moving bodies:

$$M\frac{d\mathbf{r}_{0,i}}{dt} = \mathbf{P}_{\mathrm{can},i} + \left\langle \frac{\partial \hat{H}_{\mathrm{EM},P}}{\partial \mathbf{v}_i} \right\rangle, \tag{3a}$$

$$\frac{d\mathbf{P}_{\mathrm{can},i}}{dt} = -\left\langle \frac{\partial \hat{H}_{\mathrm{EM},P}}{\partial \mathbf{r}_{0,i}} \right\rangle + \mathbf{F}_{\mathrm{ext}}. \tag{3b}$$

In the rest of this section, we discuss the quantization of $H_{\mathrm{EM},P}$, i.e., of the electromagnetic field and the polarization waves, in an electromagnetically closed cavity (e.g., a box) filled with dielectric bodies. The dielectric bodies are allowed to move with a constant velocity along the x direction (Fig. 1). In general, $H_{\mathrm{EM},P}$ depends explicitly on $(\mathbf{r}_{0,i},\mathbf{v}_i)$, and hence, as usual, the quantization is parametric so that $(\mathbf{r}_{0,i},\mathbf{v}_i)$ are treated as constant parameters in the quantization process.

The quantization of the electromagnetic field in moving media has been discussed previously in the literature (e.g., Refs. [12–15]), but here, unlike previous works, we admit a general scenario wherein the bodies in the cavity may be nonuniform, particularly the geometry along both the y and z directions can be completely arbitrary. To quantize the macroscopic fields it is assumed that the cavity is terminated with periodic boundary conditions, consistent with the hypothesis that, even if there is a flow of matter, $\hat{H}_{\mathrm{EM},P}$ can have stationary states.

We are particularly interested in the scenario where all the bodies in the cavity are invariant to translations along the direction of movement (Fig. 1). Evidently, in a realistic system the moving slabs must have a finite width along the x direction. Our theory ignores any effects due to the finite width of the slabs.

All the materials are isotropic nondispersive dielectrics in their own comoving frames. It is also supposed that the velocity of all the moving components of the system is below the Cherenkov threshold $|v| < c/n$, where n is the index of refraction of the pertinent moving body in the respective comoving frame. As discussed in Appendix A and also in Ref. [14], only in such conditions the Hamiltonian of the system is positive definite. Above the Cherenkov threshold the system becomes potentially unstable due to the appearance of "negative quanta" and cannot be quantized within the approach of Appendix A. It is clear that in a system above the Cherenkov threshold the state with no photons is no longer the state of minimal energy, and, thus, such a system has no ground state that could be the equilibrium state at zero temperature. In dispersive media, it is likely that this property is related to the phenomenon of quantum friction whose existence recently raised a heated debate [13,16,17]. We do not expect this controversy to affect the findings of this paper, because here we consider nondispersive lossless dielectrics, whereas the friction theory developed in Refs. [16,17] predicts friction in the case of dispersive (and hence, because of Kramers-Kronig's formulas, necessarily lossy) media.

Consistent with the relativistic constitutive relations in moving media [18], we assume that in the lab frame the classical **D** and **B** fields are linked to the classical **E** and **H** fields as follows:

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \varepsilon_0 \overline{\overline{\varepsilon}} & \frac{1}{c}\overline{\overline{\vartheta}} \\ \frac{1}{c}\overline{\overline{\zeta}} & \mu_0 \overline{\overline{\mu}} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \equiv \mathbf{M} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \tag{4}$$

For a moving body (invariant to translations along the direction of movement), with velocity $\mathbf{v} = v\hat{\mathbf{x}}$ with respect to the lab frame, the dimensionless parameters $\overline{\overline{\varepsilon}}$, $\overline{\overline{\mu}}$, $\overline{\overline{\vartheta}}$, and $\overline{\overline{\zeta}}$ are such that

$$\overline{\overline{\varepsilon}} = \varepsilon_t(\overline{\overline{\mathbf{I}}} - \hat{\mathbf{x}}\hat{\mathbf{x}}) + \varepsilon \hat{\mathbf{x}}\hat{\mathbf{x}}, \quad \varepsilon_t = \varepsilon \frac{1-\beta^2}{1-n^2\beta^2}, \tag{5a}$$

$$\overline{\overline{\mu}} = \mu_t(\overline{\overline{\mathbf{I}}} - \hat{\mathbf{x}}\hat{\mathbf{x}}) + \mu \hat{\mathbf{x}}\hat{\mathbf{x}}, \quad \mu_t = \mu \frac{1-\beta^2}{1-n^2\beta^2}, \tag{5b}$$

$$\overline{\overline{\zeta}} = -\overline{\overline{\vartheta}} = -a\hat{\mathbf{x}} \times \overline{\overline{\mathbf{I}}}, \quad a = \beta \frac{n^2-1}{1-n^2\beta^2}, \tag{5c}$$

where $\beta = v/c$, $n^2 = \varepsilon\mu$, ε, and μ are the material parameters in the respective comoving frame. Since the bodies are assumed invariant to translations along the direction of movement, ε and μ can only depend on y and z. The cavity can have several moving parts with possibly different velocities. For simplicity, in all the examples considered in this work it is assumed that $\mu = 1$, so that in the comoving frame the pertinent materials can be pictured as a collection of electric dipoles with no magnetic response. Note that even though the discussion of the first part of this section—and in particular the derivation of Eq. (3)—is based on a nonrelativistic approximation, there is no particular difficulty in taking into account the relativistic effects in the quantization of $\hat{H}_{\mathrm{EM},P}$, and hence this is done here.

The electromagnetic field satisfies the Maxwell's equations,

$$\begin{pmatrix} 0 & i\nabla\times \\ -i\nabla\times & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}, \tag{6}$$

which, denoting $\mathbf{F} = \binom{\mathbf{E}}{\mathbf{H}}$, can be written in a compact form as

$$\hat{N}\mathbf{F} = i\mathbf{M} \cdot \frac{\partial \mathbf{F}}{\partial t}, \tag{7}$$

where $\hat{N} = \begin{pmatrix} 0 & i\nabla\times \\ -i\nabla\times & 0 \end{pmatrix}$ and $\mathbf{M} = \mathbf{M}(\mathbf{r})$ characterizes the material parameters of the system. From Eq. (5) it is evident that $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ are symmetric and real valued tensors, whereas $\overline{\overline{\vartheta}}$ is real valued and satisfies $\overline{\overline{\vartheta}} = \overline{\overline{\zeta}}^T$ (the superscript T represents the transpose matrix). This implies that the material matrix is Hermitian (and real valued), $\mathbf{M} = \mathbf{M}^\dagger$.

In Appendix A, we present the details of the quantization of $H_{\mathrm{EM},P}$. It is proven that the Hamiltonian can be written in terms of the eigenfrequencies ω_n of the transverse natural modes of the cavity as follows [Eq. (A11)]:

$$\hat{H}_{\mathrm{EM},P} = \sum_{\omega_n>0} \hbar\omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2}\right). \tag{8}$$

The quantized electromagnetic field is such that (in the Schrödinger representation) [Eq. (A12)]

$$\hat{\mathbf{F}} = \begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \end{pmatrix} = \sum_{\omega_n>0} \sqrt{\frac{\hbar\omega_n}{2}} [\hat{a}_n \mathbf{F}_n(\mathbf{r}) + \hat{a}_n^\dagger \mathbf{F}_n^*(\mathbf{r})], \tag{9}$$

where $\mathbf{F}_n$ is the eigenmode associated with ω_n, and hence satisfies $\hat{N}\mathbf{F}_n = \omega_n \mathbf{M} \cdot \mathbf{F}_n$. The modes $\mathbf{F}_n$ should be normalized according to Eq. (A6), and the creation and annihilation operators satisfy the standard commutation relations [Eq. (A13)]. It can be proven that the quantized fields satisfy the equal-time commutation relations $[\hat{\mathbf{G}}(\mathbf{r}),\hat{\mathbf{G}}(\mathbf{r}')] = \hbar\hat{N} \cdot \{\mathbf{I}_{6\times6}\delta(\mathbf{r}-\mathbf{r}')\}$, where $[\hat{\mathbf{G}}(\mathbf{r}),\hat{\mathbf{G}}(\mathbf{r}')]$ should be understood as a tensor (with elements $[\hat{G}_m(\mathbf{r}),\hat{G}_n(\mathbf{r}')]$, $m,n = 1,\ldots 6$), $\hat{\mathbf{G}} = \mathbf{M} \cdot \hat{\mathbf{F}}$, and $\mathbf{I}_{6\times6}$ represents the identity tensor. In particular, one can write $[\hat{\mathbf{D}}(\mathbf{r}),\hat{\mathbf{B}}(\mathbf{r}')] = i\hbar\nabla \times \{\mathbf{I}_{3\times3}\delta(\mathbf{r}-\mathbf{r}')\}$. These commutation relations ensure that the time evolution of the quantum operators in the Heisenberg picture is consistent with the Maxwell's equations, $i\partial_t\hat{\mathbf{G}} = \hat{N}\hat{\mathbf{F}}$. We would like to note that the commutation relations for the macroscopic field operators differ from the commutation relations in vacuum written in terms of the microscopic field operators $\hat{\mathbf{e}}$ and $\hat{\mathbf{b}}$. Indeed, in the presence of matter the microscopic electric field can gain an extra longitudinal component (with nonzero divergence) due to the quantum-induced fluctuations of the polarization charges. The macroscopic electric field is the result of spatial averaging the *total* microscopic electric field, and thus cannot be simply identified with the spatial average of the (transverse) microscopic electric field operator. Moreover, it is known (see, for instance, Ref. [14]) that for plane waves in nondispersive moving media the vectors **D**, **B**, and **k** (the wave vector) always form a triplet of mutually orthogonal vectors, and, thus, are analogous in this sense to the transverse fields **e** and **b** in a vacuum.

From a mathematical point of view there is no difficulty in extending this theory to a more general scenario wherein the bodies *at rest* in the lab frame (i.e., the reference frame where the walls of the cavity are at rest) have an arbitrary geometry. However, the granularity of the structure along the x direction may be incompatible with the hypothesis that the electromagnetic field in the cavity is in equilibrium, because the surface roughness assisted with the local charge density fluctuations in the moving matter may induce some form of friction related to the Smith-Purcell effect [19] or the "washboard effect" [16].

We stress that $\hat{H}_{\mathrm{EM},P}$ has both electromagnetic and matter components, and describes the dynamics of the electromagnetic fields and polarization waves. This "blend" of radiation and matter is well known in macroscopic quantum electrodynamics [20,21]. In a quantum-mechanical model of a dielectric medium the normal modes (i.e., the stationary states) are neither eigenstates of the electromagnetic field energy operator (the associated quanta are photons) nor of the energy operator associated with the dipoles induced in matter (the associated quanta are designated here by polaritons), but rather clothed excitations or quasiparticles [20,21]. Clearly, the quanta of the Hamiltonian $\hat{H}_{\mathrm{EM},P}$ [Eq. (8)] are these quasiparticles.

An important observation is that for systems invariant to translations along the x direction $\hat{H}_{\mathrm{EM},P}$ is completely independent of $x_{0,i}$. Therefore, in a closed system ($\mathbf{F}_{\text{ext}} = 0$), it follows that $\partial H_{\text{tot}}/\partial x_{0,i} = 0 = -\hat{\mathbf{x}} \cdot \partial_t \mathbf{P}_{\text{can},i}$ [Eq. (3a)], and hence the x component of the canonical momentum of each individual slab is conserved.

III. DIFFERENT KINDS OF MOMENTUM

The characterization of the momentum of the electromagnetic field in macroscopic media is surrounded by many controversies, which date back to the beginning of the previous century when Abraham and Minkowski proposed two different formulas for the electromagnetic momentum density in a dielectric [22–26] (for a recent review of the history of this topic, see Ref. [27]). Here, we follow closely Refs. [26–28], and assume that the total momentum density is

$$\mathbf{g}_{\text{tot}} = \mathbf{g}_{\text{kin}} + \mathbf{g}_{\text{EM}}, \tag{10}$$

where $\mathbf{g}_{\text{kin}} = \rho_0 \gamma^2 \mathbf{v}$ may be regarded as the *density of kinetic momentum* (in the nonrelativistic limit this should be replaced by $\mathbf{g}_{\text{kin}} \approx \rho_0 \mathbf{v}$), $\gamma = 1/\sqrt{1 - v^2/c^2}$, ρ_0 is the matter density in the local rest frame,$\mathbf{v}$ is the matter velocity, and $\mathbf{g}_{\text{EM}} = \frac{1}{c^2}\mathbf{S} = \frac{1}{c^2}\mathbf{E} \times \mathbf{H}$. As compared to Refs. [27,28], we have neglected any contributions to the momentum which are neither of an electromagnetic nor of a kinetic nature. The momentum density $\mathbf{g}_{\text{kin}}$ is the material part of the momentum density in the Abraham framework, whereas $\mathbf{g}_{\text{EM}}$ is regarded as the momentum of the electromagnetic part [27].

Our theory is developed under the hypothesis that Eq. (10) represents the total momentum density of the system, and that the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ describes the *energy density flux* of the quanta of the electromagnetic field, i.e., of photons. Notice that if the Poynting vector describes the energy density flux of photons then $\mathbf{g}_{\text{EM}} = \frac{1}{c^2}\mathbf{S}$ must be the corresponding electromagnetic momentum. This follows from the relativistic relation between energy and momentum, $\mathbf{p} = \frac{1}{c^2}\mathbf{v}\varepsilon$ (which is valid for both massive and massless fundamental particles if ε is understood as the total relativistic energy).[1] In the framework of a quantum description $\mathbf{g}_{\text{EM}}$ is promoted to an operator obtained by symmetrization of the classical formula.

Similarly, the kinetic momentum density of the moving matter $\mathbf{g}_{\text{kin}}$ also depends on the quantum fluctuations, and thus also needs to be promoted to a quantum operator. Indeed, as discussed in Sec. II, a moving dipole may be characterized in terms of the degrees of freedom of the center of mass and those of the "polarization waves." Thus, the kinetic momentum of a slab is the sum of the canonical momentum of the center of mass and the momentum of the polarization waves. Therefore, classically we may write $\mathbf{g}_{\text{kin}} = \mathbf{g}_{\text{can}} + \mathbf{g}_{\text{ps}}$, where $\mathbf{g}_{\text{can}}$ is the density of momentum associated with canonical ($\mathbf{P}_{\text{can}}$) momentum of the slabs and $\mathbf{g}_{\text{ps}}$ is the density of momentum of the "polarization waves." We note that this decomposition is formally the same as that shown in Eq. (3a), except that here we consider a density of momentum. From the discussion of Sec. II, $\mathbf{g}_{\text{can}}$ is not quantized in our formalism, whereas $\mathbf{g}_{\text{ps}}$ must be promoted to an operator because the dynamics of the polarization waves is related to $\hat{H}_{\text{EM},P}$. Hence, we may write

$$\hat{\mathbf{g}}_{\text{kin}} = \mathbf{g}_{\text{can}} + \hat{\mathbf{g}}_{\text{ps}}. \tag{11}$$

Based on the above equation, it is also useful to consider the following decomposition of the total momentum density [Eq. (10)],

$$\hat{\mathbf{g}}_{\text{tot}} = \mathbf{g}_{\text{can}} + \hat{\mathbf{g}}_{\text{ps}} + \hat{\mathbf{g}}_{\text{EM}} = \mathbf{g}_{\text{can}} + \hat{\mathbf{g}}_{\text{wv}}, \tag{12}$$

where we defined the *wave momentum* density as [24,25]

$$\hat{\mathbf{g}}_{\text{wv}} = \hat{\mathbf{g}}_{\text{ps}} + \hat{\mathbf{g}}_{\text{EM}}. \tag{13}$$

Thus, in agreement with the decomposition (2) for the total energy, the wave momentum can be regarded as the part of the system momentum associated with $\hat{H}_{\text{EM},P}$, i.e., the combined momentum of the electromagnetic field and polarization waves. To make further progress we need an explicit formula for $\hat{\mathbf{g}}_{\text{wv}}$. Following Ref. [26] (see also Refs. [24,25]), we suppose that the x component of the wave momentum is coincident with the Minkowski momentum. Thus, classically we can write

$$\hat{\mathbf{x}} \cdot \mathbf{g}_{\text{wv}} = \hat{\mathbf{x}} \cdot (\mathbf{D} \times \mathbf{B}), \tag{14a}$$

$$\begin{aligned}\hat{\mathbf{x}} \cdot \mathbf{g}_{\text{ps}} &= \hat{\mathbf{x}} \cdot \left(\mathbf{D} \times \mathbf{B} - \frac{1}{c^2}\mathbf{E} \times \mathbf{H}\right) \\ &= \hat{\mathbf{x}} \cdot (\mathbf{P}_e \times \mathbf{B} + \varepsilon_0 \mathbf{E} \times \mathbf{P}_m),\end{aligned} \tag{14b}$$

with $\mathbf{P}_e = \mathbf{D} - \varepsilon_0 \mathbf{E}$ the electric polarization vector, and $\mathbf{P}_m = \mathbf{B} - \mu_0 \mathbf{H}$ a magnetic polarization vector. The density of momentum $\mathbf{g}_{\text{ps}}$ is (in the nonrelativistic and nonmagnetic case) the *pseudomomentum* density of Refs. [24,25], and will also be designated here in the same manner even when $\mathbf{P}_m \neq 0$. The quantum operators $\hat{\mathbf{g}}_{\text{wv}}$ and $\hat{\mathbf{g}}_{\text{ps}}$ are defined by symmetrization of the classical formulas. Ahead, we shall prove that formula (14b) is consistent in the nonrelativistic limit with Eqs. (3a) and (11).

IV. THE CASIMIR-MOMENTUM PROBLEM IN MOVING MEDIA

As mentioned previously, the objective is to compute the expectation of the momentum induced by the quantum fluctuations at zero temperature. We are interested in both the quantum expectation of the momentum of matter and fields.

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[1]It is interesting to note that classically $\nabla \cdot \mathbf{S} + \partial_t W_{\text{EM},P} = 0$, and because $W_{\text{EM},P}$ has both a matter and electromagnetic component (see Sec. II), it may be tempting to think that $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ also has a matter component, rather than being purely electromagnetic. Is it possible to reconcile the law $\nabla \cdot \mathbf{S} + \partial_t W_{\text{EM},P} = 0$ with the assumption that $\mathbf{S}$ is the macroscopic energy density flux of the quanta of the electromagnetic field? To do this let us denote W_{EM} as the "true" macroscopic energy density of the electromagnetic field, i.e., the energy density associated exclusively with photons. Its exact *macroscopic* formula is of no concern to us. Then, we may rewrite the energy conservation law as $\nabla \cdot \mathbf{S} + \partial_t W_{\text{EM}} = R$, where we put $R = \partial_t (W_{\text{EM}} - W_{\text{EM},P})$. This parameter may be regarded as a volumetric power density due to the continuous creation and annihilation of photons in the medium. This is an obvious consequence of the matter-radiation interactions. In other words, the modified energy conservation law is fully consistent with the understanding that in a material the number of quanta of the field in a certain volume can vary with time because of the continuous transfer of energy between the fields and matter.

Let $\hat{\mathbf{P}}_{\text{kin}} = \int \hat{\mathbf{g}}_{\text{kin}} d^3\mathbf{r}$, $\hat{\mathbf{P}}_{\text{EM}} = \int \hat{\mathbf{g}}_{\text{EM}} d^3\mathbf{r}$, and $\hat{\mathbf{P}}_{\text{wv}} = \int \hat{\mathbf{g}}_{\text{wv}} d^3\mathbf{r}$ represent the operators of the total kinetic, electromagnetic, and wave momentum in the considered cavity. Moreover, we define $\mathbf{P}_{\text{can}} = \int \mathbf{g}_{\text{can}} d^3\mathbf{r}$ as the total canonical momentum. Let us assume that our system is prepared so that for given $(\mathbf{r}_{0,i},\mathbf{v}_i)$ the initial state is the ground state of $\hat{H}_{\text{EM},P}$, denoted by $|0\rangle$. Because $\hat{H}_{\text{EM},P} = \hat{H}_{\text{EM},P}(\mathbf{r}_{0,i},\mathbf{v}_i)$, one may write more rigorously $|0\rangle = |0_{\mathbf{r}_{0,i},\mathbf{v}_i}\rangle$. We will omit the labels $(\mathbf{r}_{0,i},\mathbf{v}_i)$ unless they are relevant for the discussion. In general, a completely isolated system will not remain in this initial state because $\frac{d\mathbf{P}_{\text{can},i}}{dt} = -\langle \frac{\partial \hat{H}_{\text{EM},P}}{\partial \mathbf{r}_{0,i}}\rangle$ is usually different from zero [see Eq. (3b)]. This is true even if all the bodies are at rest, and is a consequence of the usual Casimir effect. To avoid this, we assume that an external force $\mathbf{F}_{\text{ext}}$ is applied to counterbalance the standard Casimir force so that $d\mathbf{P}_{\text{can},i}/dt = 0$. It is interesting to mention that because $|0\rangle = |0_{\mathbf{r}_{0,i},\mathbf{v}_i}\rangle$ is an eigenstate of $\hat{H}_{\text{EM},P}$, one has $\langle \frac{\partial \hat{H}_{\text{EM},P}}{\partial \mathbf{r}_{0,i}}\rangle = \langle 0|\frac{\partial \hat{H}_{\text{EM},P}}{\partial \mathbf{r}_{0,i}}|0\rangle = \frac{\partial E_0}{\partial \mathbf{r}_{0,i}}$, where $E_0 = \sum_{\omega_n>0} \frac{\hbar\omega_n}{2}$ is the zero-point energy of the system. Note that because the Hamiltonian $\hat{H}_{\text{EM},P}$ is independent of $x_{0,i}$ (because of the assumed translational invariance along x) it follows that $-\partial E_0/\partial x_{0,i} = 0$, i.e., within the hypothesis that the slabs are dispersionless lossless dielectrics there is no "frictional"-type force acting on the slabs. In case of dispersive dielectrics a friction force may appear.

Under the action of the external force (such that $d\mathbf{P}_{\text{can},i}/dt = 0$), the state $|0\rangle = |0_{\mathbf{r}_{0,i},\mathbf{v}_i}\rangle$ can thus be regarded as a stationary state of the total system. Next, we obtain the quantum expectation of the total momentum of the system in this stationary state. From Eq. (12) we can write

$$\langle 0|\hat{\mathbf{P}}_{\text{tot}}|0\rangle = \mathbf{P}_{\text{can}} + \langle 0|\hat{\mathbf{P}}_{\text{wv}}|0\rangle. \tag{15}$$

In the absence of quantum fluctuations (either associated with the electromagnetic field or with polarization waves), i.e., in the classical limit, we have $\mathbf{g}_{\text{wv}} = 0$ in the ground state. Therefore, the canonical momentum $\mathbf{P}_{\text{can}}$ may be seen as the total momentum of the system in the classical limit. On the other hand, the quantum fluctuations cannot generate a net momentum, because otherwise it would be possible to extract a net momentum from the vacuum fluctuations, which is physically absurd. This indicates that

$$\langle 0|\hat{\mathbf{P}}_{\text{wv}}|0\rangle = 0. \tag{16}$$

In Appendix B, we present further evidence based on the definition of $\hat{\mathbf{P}}_{\text{wv}}$ that indeed $\hat{\mathbf{x}}\cdot\langle 0|\hat{\mathbf{P}}_{\text{wv}}|0\rangle = 0$. In particular, these results imply that $\hat{\mathbf{x}}\cdot\langle\hat{\mathbf{P}}_{\text{tot}}\rangle = \hat{\mathbf{x}}\cdot\mathbf{P}_{\text{can}}$. This property also ensures that for systems invariant to translations along the x axis the quantum expectation of $\hat{\mathbf{S}}_{\text{tot}}$, i.e., the total relativistic energy density flux (of matter and radiation), has the same value as in the classical case. In other words, the quantum fluctuations also cannot create a net flow of energy in the system. This is a trivial consequence of the relativistic relation between momentum and energy which implies that $\hat{\mathbf{S}}_{\text{tot}} = c^2\hat{\mathbf{g}}_{\text{tot}}$.

On the other hand, let $\hat{\mathbf{P}}_{\text{kin},V_i} = \int_{V_i} \hat{\mathbf{g}}_{\text{kin}} d^3\mathbf{r}$ represent the kinetic momentum operator associated with the body V_i, and $\hat{\mathbf{P}}_{\text{ps},V_i} = \int_{V_i} \hat{\mathbf{g}}_{\text{ps}} d^3\mathbf{r}$ represent the corresponding pseudomomentum. From (11) it is evident that

$$\langle 0|\hat{\mathbf{P}}_{\text{kin},V_i}|0\rangle = \mathbf{P}_{\text{can},V_i} + \langle 0|\hat{\mathbf{P}}_{\text{ps},V_i}|0\rangle. \tag{17}$$

Therefore, it is seen that $\langle 0|\hat{\mathbf{P}}_{\text{ps},V_i}|0\rangle$ is the additional kinetic momentum acquired by matter due to the quantum fluctuations, which agrees with the theory of Feigel [Eq. [14] of Ref. [2]. In the classical limit, i.e., provided the zero-point fluctuations of the electromagnetic field are neglected, the kinetic momentum reduces to the canonical momentum, $\mathbf{P}_{\text{kin}} = \mathbf{P}_{\text{can}}$.

Thus, to determine the additional momentum imparted to matter, we need to calculate $\langle 0|\hat{\mathbf{P}}_{\text{ps},V_i}|0\rangle$. This can be done in two ways. The first (nonrelativistic) approach is simply based on Eq. (3a). Comparing Eqs. (3a) and (17), one may readily identify $\langle 0|\hat{\mathbf{P}}_{\text{ps},V_i}|0\rangle = \langle 0|\frac{\partial \hat{H}_{\text{EM},P}}{\partial \mathbf{v}_i}|0\rangle$. But because $|0\rangle = |0_{\mathbf{r}_{0,i},\mathbf{v}_i}\rangle$ is an eigenstate of $\hat{H}_{\text{EM},P}$, it follows immediately that $\langle 0|\hat{\mathbf{P}}_{\text{ps},V_i}|0\rangle = \frac{\partial E_0}{\partial \mathbf{v}_i}$, where E_0 is the zero-point energy. More rigorously, one must write $\langle 0|\hat{\mathbf{x}}\cdot\hat{\mathbf{P}}_{\text{ps},V_i}|0\rangle = \frac{\partial E_0}{\partial v_i}$, where v_i is the velocity of the ith body along the x direction because $\hat{H}_{\text{EM},P}$ can only be defined using a macroscopic theory when the velocity of the bodies is parallel to the direction of translational invariance. The second approach is based on the definition (14b) of the pseudomomentum density. In the next section, we prove that it yields a result consistent with the first approach in the nonrelativistic limit. However, very interestingly, we shall see that this second approach enables formulating a relativistically covariant theory.

V. QUANTUM EXPECTATION OF THE PSEUDOMOMENTUM

In what follows we determine $\langle 0|\hat{P}_{\text{ps},V_0}|0\rangle$, where $\hat{P}_{\text{ps},V_0} = \int_{V_0} \hat{\mathbf{x}}\cdot\hat{\mathbf{g}}_{\text{ps}} d^3\mathbf{r}$ is the pseudomomentum of a generic body (invariant to translations along the x direction) that occupies the region V_0, and $\hat{\mathbf{x}}\cdot\hat{\mathbf{g}}_{\text{ps}} = \frac{1}{2}\hat{\mathbf{x}}\cdot(\hat{\mathbf{D}}\times\hat{\mathbf{B}} - \frac{1}{c^2}\hat{\mathbf{E}}\times\hat{\mathbf{H}}) + \text{H.c.}$ (H.c. stands for Hermitian conjugate).

A. Zero-point pseudomomentum

Let the velocity of the considered body relative to the rest frame be v. Consistent with Sec. II, the material matrix depends on the velocity of the matter within the region V_0. To keep track of this dependence we write $\mathbf{M} = \mathbf{M}(\mathbf{r};v)$ [in general we can write $\mathbf{M} = \mathbf{M}(\mathbf{r};v_1,\ldots,v_N)$ to keep track of the dependence of the material matrix on the velocity of all the bodies in the cavity]. Obviously, the eigenfrequencies of the cavity depend on v. Let us determine the rate of change of the frequency of resonance ω_n of the nth eigenmode with respect to the velocity of the considered body. From the theory of Appendix D [Eq. (D5)], this is given by

$$\frac{\partial\omega_n}{\partial v} = \frac{-\omega_n \int_{V_0} d^3\mathbf{r}\,\mathbf{F}_n^*\cdot\frac{\partial\mathbf{M}}{\partial v}\cdot\mathbf{F}_n}{\int d^3\mathbf{r}\,\mathbf{F}_n^*\cdot\mathbf{M}\cdot\mathbf{F}_n}, \tag{18}$$

where $\mathbf{F}_n = \binom{\mathbf{E}_n}{\mathbf{H}_n}$ represents the fields associated with the nth eigenmode, and the integral in the denominator is over the entire cavity volume. We used the fact that $\frac{\partial\mathbf{M}}{\partial v}$ vanishes everywhere in the cavity, except over V_0, i.e., over the volume of the considered moving region.

In the region V_0 the effective parameters of the macroscopic medium are defined consistently with Eq. (5), and hence it is possible to write:

$$\frac{\partial \mathbf{M}}{\partial v} = \begin{pmatrix} \varepsilon_0\varepsilon \frac{\partial}{\partial v}\left(\frac{1-\beta^2}{1-n^2\beta^2}\right)\bar{\bar{\mathbf{I}}}_t & \frac{\varepsilon\mu-1}{c^2}\frac{\partial}{\partial v}\left(\frac{v}{1-n^2\beta^2}\right)\hat{\mathbf{x}}\times\bar{\bar{\mathbf{I}}} \\ -\frac{\varepsilon\mu-1}{c^2}\frac{\partial}{\partial v}\left(\frac{v}{1-n^2\beta^2}\right)\hat{\mathbf{x}}\times\bar{\bar{\mathbf{I}}} & \mu_0\mu\frac{\partial}{\partial v}\left(\frac{1-\beta^2}{1-n^2\beta^2}\right)\bar{\bar{\mathbf{I}}}_t \end{pmatrix}, \tag{19}$$

where $\bar{\bar{\mathbf{I}}}_t = \bar{\bar{\mathbf{I}}} - \hat{\mathbf{x}}\hat{\mathbf{x}}$, $n^2 = \varepsilon\mu$, $\beta = v/c$, and ε and μ are the effective parameters of the body in the comoving frame. Some lengthy but straightforward calculations show that if we put $\mathbf{F}_1 = \binom{\mathbf{E}_1}{\mathbf{H}_1}$ and $\mathbf{F}_2 = \binom{\mathbf{E}_2}{\mathbf{H}_2}$, and define $\binom{\mathbf{D}_1}{\mathbf{B}_1} = \mathbf{M}\cdot\binom{\mathbf{E}_1}{\mathbf{H}_1}$ and $\binom{\mathbf{D}_2}{\mathbf{B}_2} = \mathbf{M}\cdot\binom{\mathbf{E}_2}{\mathbf{H}_2}$, then the following purely algebraic relation holds:

$$-\mathbf{F}_1\cdot\frac{\partial \mathbf{M}}{\partial v}\cdot\mathbf{F}_2 = \frac{1}{(1-\beta^2)}\hat{\mathbf{x}}\cdot(\mathbf{D}_1\times\mathbf{B}_2 + \mathbf{D}_2\times\mathbf{B}_1 - \varepsilon_0\mu_0\mathbf{E}_1\times\mathbf{H}_2 - \varepsilon_0\mu_0\mathbf{E}_2\times\mathbf{H}_1). \tag{20}$$

In particular, if we pick $\mathbf{F}_1 = \mathbf{F}$ and $\mathbf{F}_2 = \mathbf{F}^*$, it follows that

$$-\frac{1}{2}\mathbf{F}^*\cdot\frac{\partial \mathbf{M}}{\partial v}\cdot\mathbf{F} = \frac{1}{(1-\beta^2)}\hat{\mathbf{x}}\cdot\mathrm{Re}\{\mathbf{D}\times\mathbf{B}^* - \varepsilon_0\mu_0\mathbf{E}\times\mathbf{H}^*\}. \tag{21}$$

Substituting the above formula into Eq. (18), it is found that

$$\frac{\partial\omega_n}{\partial v} = \frac{1}{(1-\beta^2)}\frac{\omega_n\int_{V_0} d^3\mathbf{r}\,\hat{\mathbf{x}}\cdot\mathrm{Re}\{\mathbf{D}_n\times\mathbf{B}_n^* - \varepsilon_0\mu_0\mathbf{E}_n\times\mathbf{H}_n^*\}}{\frac{1}{2}\int d^3\mathbf{r}\,\mathbf{F}_n^*\cdot\mathbf{M}\cdot\mathbf{F}_n}. \tag{22}$$

We are now ready to calculate the ground-state quantum expectation of the operator $\hat{P}_{\mathrm{ps},V_0}$. Using Eqs. (9) and (A14), it is straightforward to verify that

$$P_{\mathrm{ps},V_0} \equiv \langle 0|\hat{P}_{\mathrm{ps},V_0}|0\rangle = \sum_{\omega_n>0}\frac{\hbar\omega_n}{2}\hat{\mathbf{x}}\cdot\int_{V_0}\mathrm{Re}\left\{\mathbf{D}_n\times\mathbf{B}_n^* - \frac{1}{c^2}\mathbf{E}_n\times\mathbf{H}_n^*\right\}d^3\mathbf{r}. \tag{23}$$

But from Eq. (22) and from the normalization condition (A6), it follows that

$$P_{\mathrm{ps},V_0} = (1-\beta^2)\sum_{\omega_n>0}\frac{\hbar}{2}\frac{\partial\omega_n}{\partial v}. \tag{24}$$

Hence, since the zero-point energy of the system ($\hat{H}_{\mathrm{EM},P}$) is $E_0 = \sum_{\omega_n>0}\hbar\omega_n/2$, we finally obtain

$$P_{\mathrm{ps},V_0} = (1-\beta^2)\frac{\partial E_0}{\partial v}. \tag{25}$$

Thus, the quantum expectation of the pseudomomentum transferred to V_0 can be written in terms of the derivative of the zero-point energy with respect to the velocity v of the pertinent body. In the nonrelativistic limit ($\beta \to 0$), this result agrees with that obtained in the end of Sec. IV based on Eq. (3a).

For future reference, it is noted that the total pseudomomentum in the cavity is the sum of the pseudomomenta of the different bodies, and hence it is given by

$$P_{\mathrm{ps,tot}} = \sum_i\left(1-\beta_i^2\right)\frac{\partial E_0}{\partial v_i}, \tag{26}$$

where $\beta_i = v_i/c$, and the summation is over all the bodies in the cavity.

B. Properties of the zero-point pseudomomentum

Equation (25) establishes that the pseudomomentum can be expressed in terms of the zero-point energy of the system. Clearly, only the interacting part of the zero-point energy can depend on the velocity of the moving bodies. Hence, in Eq. (25) we can replace E_0 by $E_{0,\mathrm{int}}$, where $E_{0,\mathrm{int}}$ is the zero-point interaction energy.

To calculate the interaction energy $E_{0,\mathrm{int}}$, we rely on the theory of Ref. [14], which is based on a summation of the zero-point energy of the system using the argument principle [29,30]. Specifically, for a multilayered system of moving parallel dielectric slabs (invariant to translations along x and z) the interaction part of the zero-point energy at zero temperature may be written as a generalized Lifshitz formula Eq. (51) of Ref. [14]

$$\frac{E_{0,\mathrm{int}}}{A} = \frac{\hbar}{(2\pi)^3}\iint dk_x dk_z\int_0^{+\infty} d\xi\,\ln D(i\xi,k_x,k_z), \tag{27}$$

where $D(\omega,k_x,k_z)$ is such that $D(\omega,k_x,k_z) = 0$ determines the characteristic equation of normal Bloch modes associated with the transverse wave vector (k_x,k_z), and $A = L_x \times L_z$ is the cross-sectional area of the cavity parallel to the xoz plane. The dispersion equation can be written explicitly in terms of reflection matrices, and for more details the reader is referred to Ref. [14]. It should be noted that $D(\omega,k_x,k_z)$ depends on the geometry and material parameters of the bodies under analysis, and so we can write $D = D(\omega,k_x,k_z; v_1,\ldots,v_N,d_{12},d_{13},\ldots)$, where v_i represents the velocity of the ith body in the cavity and $d_{i,j}$ represents the relative distance between the ith and jth bodies. Thus, the interaction energy is such that $E_{0,\mathrm{int}} = E_{0,\mathrm{int}}(v_1,\ldots,v_N,d_{12},d_{13},\ldots)$. In the particular case of two interacting bodies, separated by the distance d, the Casimir force acting on the bodies is $F = -\partial E_{0,\mathrm{int}}/\partial d$ [14], which can be shown to be consistent with the quantum expectation of the Maxwell stress tensor.

It should be noted that in Eq. (27) it is implicit that the length of the considered cavity along the x and z directions (L_x and L_z, respectively) is rather large so that (k_x,k_z) vary in a continuum, rather than being quantized (e.g., $k_x = 2\pi m/L_x$, $m = 0,\pm1,\pm2,\ldots$).

Based on Eq. (27), it is possible to conclude that for the ith body the zero-point pseudomomentum per unit of cross-sectional area satisfies

$$\frac{P_{\mathrm{ps},V_i}}{A} = \left(1-\beta_i^2\right)\frac{\partial}{\partial v_i}\left(\frac{E_{0,\mathrm{int}}}{A}\right). \tag{28}$$

Let us discuss some important consequences of our theory. To this end, let us consider the interaction between two parallel slabs, separated by a distance d in a vacuum, and moving with velocities v_1 and v_2, respectively. First of all, we note that by symmetry it is obvious that $E_{0,\mathrm{int}}(v_1,v_2=0)$ [$E_{0,\mathrm{int}}(v_1=0,v_2)$] is an even function of v_1 (v_2), and hence for $v_1 = 0$ and $v_2 = 0$ the zero-point pseudomomentum vanishes for both bodies: $P_{\mathrm{ps},V_i} = 0$. Thus, as expected, the zero-point pseudomomentum can be nonzero only when at least one of the bodies is moving.

Suppose now that $(v_1,v_2) \neq (0,0)$. It was shown in Ref. [14] that $E_{0,\mathrm{int}}/A$ is a relativistic invariant in the sense that it has the same value for all the reference frames that move with a constant x-directed velocity u with respect to the lab frame, provided that the velocities of the slabs and material parameters are transformed in a covariant manner. Such a result assumes that $L_x \to \infty$. This important property implies that $E_{0,\mathrm{int}}(v_1,v_2)/A$ is such that

$$\frac{1}{A}E_{0,\mathrm{int}}\left(\frac{v_1+u}{1+uv_1/c^2},\frac{v_2+u}{1+uv_2/c^2}\right) = \mathrm{const}, \tag{29}$$

where u can be arbitrary and the constant on the right-hand side is independent of u. Differentiating the above formula with respect to u and putting $u=0$, we readily find that $(1-\beta_1^2)\frac{\partial}{\partial v_1}\frac{E_{0,\mathrm{int}}}{A} + (1-\beta_2^2)\frac{\partial}{\partial v_2}\frac{E_{0,\mathrm{int}}}{A} = 0$. Hence, from Eq. (26) it follows that the total pseudomomentum of the system vanishes: $P_{\mathrm{ps,tot}} = 0$. This result can be readily generalized to the case of N different slabs. Therefore, we proved that the total pseudomomentum $P_{\mathrm{ps,tot}}$ generated by quantum fluctuations in the cavity vanishes, and hence, from Eq. (13) ($\hat{\mathbf{P}}_{\mathrm{wv}} = \hat{\mathbf{P}}_{\mathrm{ps}} + \hat{\mathbf{P}}_{\mathrm{EM}}$) and Eq. (16) also the quantum expectation of the total electromagnetic momentum $P_{\mathrm{EM,tot}}$ in the cavity vanishes: $\langle 0|\hat{\mathbf{P}}_{\mathrm{EM}}|0\rangle = 0$. This result is full of physical significance, and makes manifest that both the *total* additional momentum imparted to matter and the *total* momentum imparted to the fields by the quantum fluctuations are zero. In particular, we see that our theory predicts that a bulk uniform moving medium must have $P_{\mathrm{ps,tot}} = 0$.

It can be also proven in a similar manner that the pseudomomentum of a given slab per unit of area ($P_{\mathrm{ps},V_i}/A$) stays invariant in all reference frames that move along the x axis with constant velocity. The proof is given in Appendix E. In light of the results derived above, this shows that if all the bodies in the system move with the *same* velocity, then the individual pseudomomentum of each body vanishes. Therefore, unlike Feigel's theory [2], and in agreement with Ref. [4], our results do not predict an additional momentum imparted to matter for the case of a single *uniform unbounded body*. Moreover, the assumption that the body is uniform and unbounded is unnecessary: All that is required is that the body is invariant to translations along the direction of movement, so that all the particles from which the body is made up move with the same velocity.

Even though $P_{\mathrm{ps,tot}} = 0$, in general the quantum expectation of the pseudomomentum associated with a specific body V_i can be different from zero. However, since $P_{\mathrm{ps,tot}} = 0$, if one body acquires an extra pseudomomentum, this must be done at the expense of the other bodies in the system. Moreover, because we are interested in the processes in which the x-directed canonical momentum of each body is conserved, a change in the pseudomomentum is always accompanied with a compensating change in the kinetic momentum of a body. Thus, the phenomenon under analysis can be pictured simply as an *exchange* of momentum between different bodies moving at *different velocities* induced by the zero-point quantum fluctuations.

One may wonder why $P_{\mathrm{ps,tot}} = 0$ is independent of the geometry of the system (e.g., independent of the distance between two bodies), whereas, for example, the zero-point energy of the system depends on the specific configuration under analysis. To understand the reason, one can imagine that the two interacting rigid bodies, let us say, two slabs, are first infinitely far apart and stand in a vacuum. In such conditions, since the bodies do not interact, the total pseudomomentum of each body must vanish. Let us now suppose that one of the bodies is brought to the vicinity of another by applying an external force $\mathbf{F}_{\mathrm{ext}}$ to the first body. In this process the center-of-mass coordinates of each slab can change, and in particular $(\mathbf{r}_{0,i},\mathbf{v}_i)$ will change. We imagine that the process is sufficiently slow so that at each time instant $d\mathbf{P}_{\mathrm{can},i}/dt = -\langle\partial\hat{H}_{\mathrm{EM},P}/\partial\mathbf{r}_{0,i}\rangle + \mathbf{F}_{\mathrm{ext}} = 0$, i.e., the canonical momentum of each slab is conserved. Moreover, the process should be sufficiently slow so that the state of the electromagnetic field and polarization waves is always $|0_{\mathbf{r}_{0,i},\mathbf{v}_i}\rangle$ (thus, we neglect the dynamic Casimir effect). However $|0_{\mathbf{r}_{0,i},\mathbf{v}_i}\rangle$ changes as the slabs are brought to the vicinity of one another, because $(\mathbf{r}_{0,i},\mathbf{v}_i)$ also changes. Note that even though v_i depend on the relative distance between the slabs, we assume that the velocities vary slowly with time so that the quantization theory of Sec. II still applies. In the enunciated conditions, $\langle\partial\hat{H}_{\mathrm{EM},P}/\partial x_{0,i}\rangle = 0$, and hence the external force is perpendicular to the x direction. Clearly, in the outlined process the external force is required to do some work, to counteract the quantum fluctuation-induced Casimir forces, and hence this explains why the zero-point interaction energy of the system is changed. However, as the external force acts perpendicularly to the direction of initial movement of the pertinent slab it does not change the x component of the total momentum of the system, which must therefore be conserved. Moreover, since the x component of the canonical momentum of each slab is conserved, it follows that the wave momentum also is. Because heuristically one may also expect that the net electromagnetic momentum generated by quantum fluctuations vanishes, the previous discussion suggests that $P_{\mathrm{ps,tot}} = 0$, in agreement with our theory.

It is emphasized that our quantization of $\hat{H}_{\mathrm{EM},P}$ is based on the relativistic constitutive relations in moving media, and that formula (25) for the pseudomomentum is also consistent with relativity. Hence, it seems reasonable to modify Eq. (3a) to take also into account the relativistic effects in the kinetic momentum. This can be easily done by using $\mathbf{g}_{\mathrm{kin},i} = \rho_{0,i}\gamma_i^2\mathbf{v}_i$ [see Eq. (10)], $\rho_{0,i}$ is the density of mass of the ith slab in the comoving frame. Using Eq. (25), one readily finds that the x component of the kinetic, canonical, and pseudomomentum of the ith slab is related by

$$\rho_{0,i}\gamma_i^2 v_i = g_{\mathrm{can},i} + \left(1-\beta_i^2\right)\frac{1}{AL_{y,i}}\frac{\partial E_0}{\partial v_i}, \tag{30}$$

where $L_{y,i}$ is the thickness of the ith slab along the y direction, $\gamma_i = 1/\sqrt{1-v_i^2/c^2}$, and $g_{\mathrm{can},i} = P_{\mathrm{can},i}/V$ is the density of canonical momentum. As outlined previously, in the type of problems we are interested in, $P_{\mathrm{can},i}$ is conserved, and can typically be regarded as the kinetic momentum when the bodies are infinitely far apart. This implies that $g_{\mathrm{can},i}/\gamma_i$ is invariant and equal to $\mathbf{g}_{\mathrm{kin},i,\infty}/\gamma_{i,\infty} = \rho_{0,i}\gamma_{i,\infty}\mathbf{v}_{i,\infty}$, where $v_{i,\infty}$ is the velocity of each slab when they are infinitely far apart and $\gamma_{i,\infty} = 1/\sqrt{1-v_{i,\infty}^2/c^2}$. Thus, we have

$$\rho_{0,i}\gamma_i^2 v_i = \rho_{0,i}\gamma_i\gamma_{i,\infty}v_{i,\infty} + \frac{1}{L_{y,i}}\frac{1}{\gamma_i^2}\left(\frac{1}{A}\frac{\partial E_0}{\partial v_i}\right),$$
$$i = 1,2,\ldots,N. \tag{31}$$

Note that both the left- and right-hand sides of the equation depend on the velocities of the slabs $v_1, v_2, \ldots, v_N$. By solving the above system it is possible to determine how the velocity of each slab varies with the distance between the slabs, provided $v_{i,\infty}$ are specified. It is important to make clear that the velocities v_i depend on the (tiny) effect (i.e., on the expectation) of the quantum fluctuations, and hence need to be determined self-consistently. To an excellent approximation, since typically the quantum corrections on $v_{i,\infty}$ are quite tiny, one can simply evaluate P_{ps,V_i} [the second term in the right-hand side of Eq. (31)] using the velocities $v_{i,\infty}$, rather than v_i.

VI. NUMERICAL EXAMPLES

To quantify the electromagnetic momentum generated by quantum fluctuations in Casimir-type geometries, we made a numerical study that is reported next. In the first example we consider a scenario wherein a pair of extremely thick dielectric slabs (modeled here as semi-infinite layers) are separated by a vacuum gap. With respect to the laboratory frame, the first slab moves with a constant speed v_1 and the second is stationary ($v_2 = 0$). The relative velocity of the layers is kept below the Cherenkov threshold.

Using the theory of Ref. [14], we numerically calculate the Casimir interaction energy in this system (per unit area), and then find the pseudomomenta (per unit area) of the layers from Eq. (28). The results of these calculations are plotted in Fig. 2 as functions of the normalized velocity of the first layer $\beta = v_1/c$. We note that the pseudomomentum of the moving layer P_{ps,V_1} is antiparallel to the velocity of this layer ($\beta P_{\mathrm{ps},V_1} < 0$) and grows in magnitude monotonically with the velocity. Since $P_{\mathrm{kin},V_i} = P_{\mathrm{can},V_i} + P_{\mathrm{ps},V_i}$ this implies that in a closed system in which the canonical momentum is conserved the interaction between the two slabs causes a reduction of the kinetic momentum of slab 1, and an increase of the kinetic momentum of the slab 2, which is the most intuitive result.

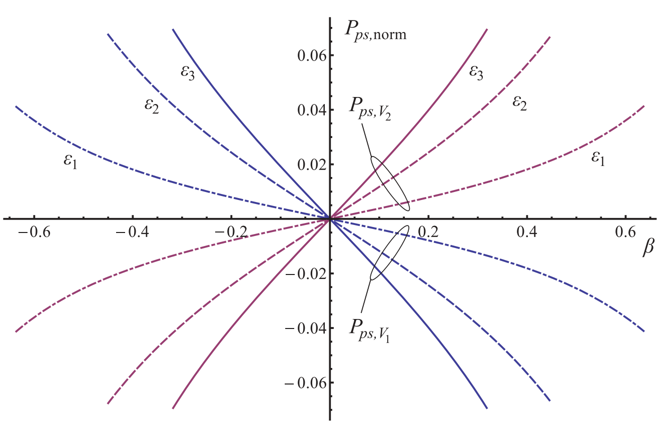

FIG. 2. (Color online) Normalized pseudomomenta $P_{\mathrm{ps},V_1,\mathrm{norm}}$ and $P_{\mathrm{ps},V_2,\mathrm{norm}}$ of two moving nondispersive identical semi-infinite dielectric layers separated by a vacuum gap of width d as functions of the normalized velocity of the first layer $\beta = v_1/c$. The second layer is stationary ($v_2 = 0$). The values of the pseudomomentum per unit area [Eq. (28)] are normalized as $P_{\mathrm{ps}}/A = P_{\mathrm{ps,norm}}\pi^2\hbar/(720d^3)$. The different curves correspond to different values of the dielectric constant of the layers: $\varepsilon_1 = 2$, $\varepsilon_2 = 4$, $\varepsilon_3 = 8$.

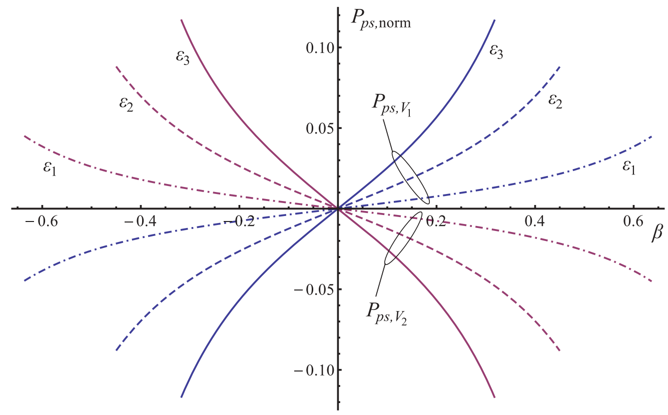

FIG. 3. (Color online) Normalized pseudomomenta $P_{\mathrm{ps},V_1,\mathrm{norm}}$ and $P_{\mathrm{ps},V_2,\mathrm{norm}}$ in a system composed of a moving dielectric slab of thickness d (first layer) on top of a semi-infinite dielectric (second layer) as functions of the normalized velocity of the first layer, $\beta = v_1/c$. The second layer is stationary ($v_2 = 0$). The rest of the legend is as in Fig. 2. Both layers have the same dielectric constant in the respective rest frame.

We remind that the canonical momentum of each slab, P_{can,V_i}, is a constant and is independent of the distance of the slabs. As expected, at any given velocity β of the first layer, the second (stationary) layer acquires exactly the opposite amount of the pseudomomentum, $P_{\mathrm{ps},V_2} = -P_{\mathrm{ps},V_1}$, so that the total pseudomomentum of the system vanishes, which is in full agreement with the theoretical result obtained above.

In the next example we study the zero-point pseudomomentum in a system that comprises a moving dielectric layer with thickness d placed directly on the top of a very thick dielectric "substrate" (modeled as semi-infinite layer). The velocity of the moving layer with respect to the substrate is v_1, and the substrate is assumed stationary with respect to the laboratory frame, $v_2 = 0$. Above the moving slab there is a vacuum. When both layers are of the same permittivity this situation can be visualized as a special case of a nonuniformly moving fluid, where only a thin layer close to the free surface of the fluid is moving.

The numerically calculated pseudomomenta of the moving layer and the substrate for this case are plotted in Fig. 3. Contrary to the previous example, in this setup the pseudomomentum of the moving layer is collinear with its velocity, $\beta P_{\mathrm{ps},V_1} > 0$. As before, the total pseudomomentum of the system vanishes at arbitrary velocity β: $P_{\mathrm{ps},V_2} + P_{\mathrm{ps},V_1} = 0$. In both examples, the pseudomomentum acquired by the slabs is higher for higher values of the slab permittivities. One may speculate that the sign of P_{ps,V_1} may be somehow related to the difference of the probabilities of transmission from medium 2 to medium 1, of two virtual quasiparticles (which, loosely speaking, we may identify with plane-wave modes) with transverse momentum $\hbar k_x/2$ and $-\hbar k_x/2$, respectively. Further studies are required to understand if this explanation is physically correct.

The result $\beta P_{\mathrm{ps},V_1} > 0$ may look incompatible with the general behavior predicted by Fig. 2. However, it must be noted that the scenario of Fig. 3 cannot be regarded as the limit case $d \to 0$ of the configuration of Fig. 2. Indeed, as $d \to 0$ in the configuration of Fig. 2, the pseudomomentum [which for a fixed velocity is proportional to $\pi^2\hbar/(720d^3)$] diverges to ∞ unless the velocity of the slabs is progressively closer. Thus, to keep the canonical momentum of the two slabs invariant (as it should be), it is necessary that in the limit $d \to 0$ the two slabs move with the same velocity, which is very different from the case of Fig. 3.

Simple estimations based on Eq. (31) show that in the scenario of Fig. 2 an initially stationary 10-nm-thick slab of a material with mass density $\rho = 10^3$ kg m^{-3} and the dielectric constant $\varepsilon_r = 4$ will acquire a velocity of about $v_1 = 5.6$ nm/s when brought from infinity to a close distance of 10 nm from a massive slab of the same material moving at a speed $v_2 = 0.3c$. This is a quite encouraging result since for magnetoelectric materials the theory of Tiggelen [4] predicts velocities of the order 10^{-17} nm/s. Thus, the scenarios considered in this paper may be instrumental for detecting a momentum exchange originated by the quantum fluctuations. On the other hand, at the separation of 1 Å the same slab will acquire a velocity on the order of 5.6 mm/s. Therefore, the limiting case discussed above (i.e., $v_1 \to v_2$ in the limit $d \to 0$) is purely theoretical as it necessarily happens at unphysically small separations wherein the description of matter as an effective continuous medium is obviously inapplicable.

So far we have considered dielectric layers with the same permittivity, however, and especially for the second example, it is interesting to check how a small difference in the dielectric permittivity of the layers affects the behavior of the pseudomomentum. In this regard, it is instructive to compare the dependence of the Casimir energy on the layer velocities in a few representative cases. In order to do this, we let the permittivity of the moving layer be $\varepsilon_1 = \varepsilon_2 + \Delta\varepsilon$, where ε_2 is the permittivity of the stationary layer. In the configuration of the second example (thin layer on top of a thick substrate) one may expect $E_{0,\text{int}} > 0$ when $\Delta\varepsilon < 0$ (which is the case of the Casimir repulsion for nondispersive dielectrics with $\varepsilon_2 > \varepsilon_1 > 1$), and $E_{0,\text{int}} < 0$ when $\Delta\varepsilon > 0$

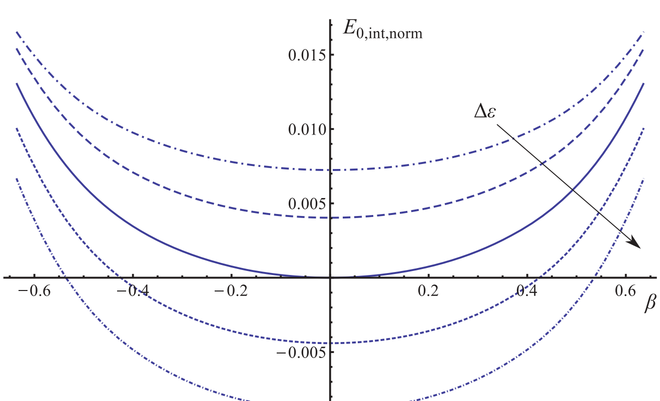

FIG. 4. (Color online) Normalized Casimir interaction energy $E_{0,\text{int,norm}}$ in a system composed of a dielectric layer of thickness d above a semi-infinite dielectric layer as a function of the normalized velocity of the top layer $\beta = v_1/c$. The bottom layer is stationary ($v_2 = 0$) and has permittivity $\varepsilon_2 = 2$. The permittivity of the moving layer is $\varepsilon_1 = \varepsilon_2 + \Delta\varepsilon$, with varying $\Delta\varepsilon = -0.2, -0.1, 0, 0.1, 0.2$, as indicated by the arrow. The energy per unit area is normalized to $\pi^2\hbar c/(720d^3)$.

(the Casimir attraction case) at zero relative velocity $\beta = 0$. Thus, the Casimir interaction energy changes sign when $\Delta\varepsilon$ crosses zero, but how does this affect the pseudomomentum? This may be understood from Fig. 4 where we show the velocity dependence of the Casimir interaction energy for different values of $\Delta\varepsilon$. It is seen that when $\Delta\varepsilon > 0$ the Casimir attraction may shift to Casimir repulsion at sufficiently large velocities (and at some intermediate velocity the Casimir force is fully compensated). This result by itself is quite interesting and complements the findings of earlier work on Casimir forces in moving media (see Ref. [14]), where this specific case was not studied. However, when the pseudomomentum in this structure is considered, it can be inferred from Fig. 4 and Eq. (28) that independently of the sign of $\Delta\varepsilon$ the behavior of the pseudomomentum with the velocity β remains qualitatively the same as in Fig. 3.

VII. CONCLUSION

In this paper, we proposed a macroscopic quantum theory to characterize the exchange of momentum in moving matter induced by the quantum vacuum at zero temperature. In our approach, both the electromagnetic field and the polarization waves (related to the oscillations of electric dipoles associated with the moving matter) are quantized, but the canonical momentum associated with the center of mass of the moving particles is treated classically. For lossless nondispersive systems invariant to translations along the direction of movement, the canonical momentum is conserved. It was argued that the total net (wave) momentum generated by quantum fluctuations (i.e., the momentum of the clothed polariton photons) vanishes. In general, the quantum expectation of the additional kinetic momentum imparted to a specific moving body may be nonzero, and can be expressed in terms of the pseudomomentum. We have demonstrated that the zero-point pseudomomentum at zero temperature can be written in terms of the zero-point interaction energy of the system, and we numerically calculated it for several illustrative examples. Since the total pseudomomentum of the system vanishes, there is no net total momentum generated by quantum fluctuations neither in matter components of the system nor in the electromagnetic field.

ACKNOWLEDGMENT

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APPENDIX A: QUANTIZATION OF THE ELECTROMAGNETIC FIELD

Here, we develop a theory that characterizes the quantum fluctuations in macroscopic moving media. The geometry of the system is the one described in Sec. II, and it is assumed without loss of generality that the pertinent cavity is terminated with periodic boundary conditions.

Let us first characterize the normal modes of the system from a classical point of view. Within a macroscopic approach, the electromagnetic field satisfies the Maxwell's equations

[Eq. (7)]

$$\hat{N}\mathbf{F} = i\mathbf{M}\cdot\frac{\partial\mathbf{F}}{\partial t}. \tag{A1}$$

As discussed in Sec. II, the material matrix $\mathbf{M}$ is Hermitian and real valued, $\mathbf{M} = \mathbf{M}^\dagger$. In addition, it can be easily checked from Eq. (5) that, provided the velocity of each moving component of the system is below the Cherenkov threshold, i.e., if $|v| < c/n$, then $\mathbf{M}$ is a positive definite matrix at every point $\mathbf{r}$. This condition is assumed to hold throughout this work. An important consequence of $\mathbf{M}$ being positive definite is that the stored energy,

$$H_{\mathrm{EM},P} = \frac{1}{2}\int d^3\mathbf{r}\mathbf{B}\cdot\mathbf{H} + \mathbf{D}\cdot\mathbf{E} = \frac{1}{2}\int d^3\mathbf{r}\mathbf{F}\cdot\mathbf{M}\cdot\mathbf{F}, \tag{A2}$$

is always positive in the lab frame, i.e., $H_{\mathrm{EM},P} > 0$. Such a property will be of key importance in what follows.

Let us introduce an inner product $\langle\,|\,\rangle$ such that for two six-component vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ we have

$$\langle\mathbf{F}_2|\mathbf{F}_1\rangle = \frac{1}{2}\int d^3\mathbf{r}\,\mathbf{F}_2^*\cdot\mathbf{M}(\mathbf{r})\cdot\mathbf{F}_1. \tag{A3}$$

Since $\mathbf{M}$ is positive definite, it is evident that $\langle\mathbf{F}|\mathbf{F}\rangle > 0$ for $\mathbf{F} \neq 0$, and thus $\langle\,|\,\rangle$ really defines an inner product. It is straightforward to check that, provided $\mathbf{F}_1$ and $\mathbf{F}_2$ satisfy periodic boundary conditions at the walls of the cavity, one has

$$\langle\mathbf{F}_2|\mathbf{M}^{-1}\hat{\mathbf{N}}\mathbf{F}_1\rangle = \langle\mathbf{M}^{-1}\hat{\mathbf{N}}\mathbf{F}_2|\mathbf{F}_1\rangle, \tag{A4}$$

i.e., $\mathbf{M}^{-1}\cdot\hat{\mathbf{N}}$ is an Hermitian operator in the Hilbert space of six vectors that satisfy periodic boundary conditions with an inner product determined by $\langle\,|\,\rangle$. In particular, it follows that $\mathbf{M}^{-1}\cdot\hat{\mathbf{N}}$ has a complete set of eigenfunctions $\mathbf{F}_n = \binom{\mathbf{E}_n}{\mathbf{H}_n}$, such that

$$\mathbf{M}^{-1}\cdot\begin{pmatrix} 0 & i\nabla\times \\ -i\nabla\times & 0\end{pmatrix}\mathbf{F}_n = \omega_n\mathbf{F}_n, \tag{A5}$$

where ω_n are the eigenfrequencies of the cavity, and

$$\delta_{m,n} = \langle\mathbf{F}_m \mid \mathbf{F}_n\rangle = \frac{1}{2}\int d^3\mathbf{r}\,\mathbf{F}_m^*\cdot\mathbf{M}(\mathbf{r})\cdot\mathbf{F}_n. \tag{A6}$$

It is interesting to note that $\mathbf{F}_n$ is necessarily complex valued for $\omega_n \neq 0$. In fact, if $\mathbf{F}_n$ were purely real valued, the left-hand side of Eq. (A5) would be an imaginary pure vector, whereas the right-hand side of the same equation would be a purely real vector, and this is impossible for a nontrivial $\mathbf{F}_n$. Therefore $\mathbf{F}_n$ is complex valued for $\omega_n \neq 0$. In particular, it should be clear (because $\mathbf{M}$ is real valued) that $\mathbf{F}_n^*$ is an eigenfunction associated with $-\omega_n$. Hence, despite the nonreciprocal response of the materials, the spectrum of the cavity is always symmetric with respect to $\omega = 0$.

It is evident that all the eigenmodes associated with an $\omega_n \neq 0$ are transverse fields, i.e., they satisfy $\nabla\cdot\mathbf{D} = 0 = \nabla\cdot\mathbf{B}$, i.e., the density of electric charge vanishes. In particular, any six-vector field $\mathbf{F}$ that satisfies the periodic boundary conditions and $\nabla\cdot\mathbf{D} = 0 = \nabla\cdot\mathbf{B}$ can be expanded in terms of transverse eigenmodes, as follows:

$$\mathbf{F} = \sum_{\omega_n>0}[b_n\mathbf{F}_n(\mathbf{r}) + \tilde{b}_n\mathbf{F}_n^*(\mathbf{r})], \tag{A7}$$

where $b_n = \langle\mathbf{F}_n|\mathbf{F}\rangle$, etc. Notice that the summation is restricted to eigenvalues $\omega_n > 0$, because the eigenfunction associated with $-\omega_n$ is $\mathbf{F}_n^*$, as discussed previously. In particular, when $\mathbf{F}$ is real valued we may write

$$\mathbf{F} = \sum_{\omega_n>0}[b_n\mathbf{F}_n(\mathbf{r}) + b_n^*\mathbf{F}_n^*(\mathbf{r})]. \tag{A8}$$

The stored energy associated with the field $\mathbf{F}$ is obviously equal to $\langle\mathbf{F}|\mathbf{F}\rangle$, and hence using the normalization of the modes [Eq. (A6)] it satisfies

$$H_{\mathrm{EM},P} = 2\sum_{\omega_n>0}|b_n|^2 = 2\sum_{\omega_n>0}\left(b_n'^2 + b_n''^2\right), \tag{A9}$$

where $b_n = b_n' + i\,b_n''$. In order that $\mathbf{F}$ satisfies the Maxwell's equations (A1), it is necessary that $\omega_n b_n = i\dot{b}_n$ with $\dot{b}_n = db_n/dt$ [this implies $b_n(t) = b_n(0)e^{-i\omega_n t}$]. Thus, the real and imaginary parts of b_n must satisfy $\dot{b}_n' = \omega_n b_n''$ and $\dot{b}_n'' = -\omega_n b_n'$.

Let us now define the variables q_n and p_n such that $b_n' = \frac{\omega_n}{2}\sqrt{m}q_n$ and $b_n'' = \frac{1}{2\sqrt{m}}p_n$, where m has dimensions of mass (and can be chosen arbitrarily). Then, from the previous results it is clear that the stored energy can be written as

$$H_{\mathrm{EM},P} = \sum_{\omega_n>0}\left(\frac{1}{2}m\omega_n^2 q_n^2 + \frac{1}{2m}p_n^2\right), \tag{A10}$$

with $\dot{q}_n = \frac{1}{m}p_n = \frac{\partial H_{\mathrm{EM},P}}{\partial p_n}$ and $\dot{p}_n = -m\omega_n^2 q_n = -\frac{\partial H_{\mathrm{EM},P}}{\partial q_n}$. Therefore, the classical system is described by an infinite set of decoupled harmonic oscillators.

We are now ready to quantize the electromagnetic field. To this end, we only need to promote q_n and p_n to operators that satisfy canonical commutation relations, $[\hat{q}_n,\hat{p}_n] = i\hbar$, etc. By doing this, introducing the annihilation operator $\hat{a}_n = \sqrt{\frac{m\omega_n}{2\hbar}}\hat{q}_n + i\sqrt{\frac{1}{2\hbar m\omega_n}}\hat{p}_n$, it is straightforward to verify that the Hamiltonian of the quantized system is given by

$$\hat{H}_{\mathrm{EM},P} = \sum_{\omega_n>0}\hbar\omega_n\left(\hat{a}_n^\dagger\hat{a}_n + \frac{1}{2}\right), \tag{A11}$$

whereas the quantized field operator becomes (in the Schrödinger representation)

$$\hat{\mathbf{F}} = \begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix} = \sum_{\omega_n>0}\sqrt{\frac{\hbar\omega_n}{2}}[\hat{a}_n\mathbf{F}_n(\mathbf{r}) + \hat{a}_n^\dagger\mathbf{F}_n^*(\mathbf{r})]. \tag{A12}$$

Note that the modes $\mathbf{F}_n$ should be normalized according to Eq. (A6). This result agrees with the quantized electromagnetic field in a standard dielectric cavity (with no moving components and assuming that the materials are not dispersive) [31]. The creation and annihilation operators satisfy the usual commutation relations

$$[a_n,a_m] = [a_n^\dagger,a_m^\dagger] = 0, \quad [a_n,a_m^\dagger] = \delta_{n,m}. \tag{A13}$$

The operators associated with the electric displacement and with the magnetic induction are defined by

$$\hat{\mathbf{G}} \equiv \begin{pmatrix}\hat{\mathbf{D}}\\ \hat{\mathbf{B}}\end{pmatrix} = \mathbf{M}\cdot\begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix} = \sum_{\omega_n>0}\sqrt{\frac{\hbar\omega_n}{2}}[\hat{a}_n\mathbf{G}_n(\mathbf{r}) + \hat{a}_n^\dagger\mathbf{G}_n^*(\mathbf{r})], \tag{A14}$$

where $\mathbf{G}_n = \mathbf{M}\cdot\mathbf{F}_n$.

APPENDIX B: THE QUANTUM EXPECTATION OF THE WAVE MOMENTUM

In this Appendix, we provide evidence that $\langle 0|\hat{\mathbf{x}}\cdot\hat{\mathbf{P}}_{\rm wv}|0\rangle = 0$, where $|0\rangle$ denotes the ground state of $\hat{H}_{{\rm EM},P}$, $\hat{\mathbf{P}}_{\rm wv} = \int \hat{\mathbf{g}}_{\rm wv} d^3\mathbf{r}$ is the total wave momentum in the cavity and the integration is over the volume of the cavity. To begin with, we note that based on Eqs. (14) and (A14) it is possible to write

$$\hat{\mathbf{x}}\cdot\langle 0|\hat{\mathbf{g}}_{wv}|0\rangle = \hat{\mathbf{x}}\cdot\sum_{\omega_n>0}\frac{\hbar\omega_n}{2}\frac{1}{2}(\mathbf{D}_n\times\mathbf{B}_n^* + \mathbf{D}_n^*\times\mathbf{B}_n). \quad \text{(B1)}$$

Since by hypothesis our system is invariant to translations along the x direction, the dependence on x of a generic mode is necessarily of the form $e^{ik_x x}$, where k_x depends on the considered mode. The allowed values of k_x are determined by the length L_x of the cavity along x: $k_x = 2\pi m/L_x$, $m = 0,\pm1,\pm2,\ldots$. In Appendix C, we demonstrate that a generic eigenmode $\mathbf{F}_n$ associated with the eigenfrequency ω_n and with the wave number k_x satisfies [Eq. (C1)]

$$\int d^3\mathbf{r}\,\hat{\mathbf{x}}\cdot\frac{1}{2}(\mathbf{D}_n\times\mathbf{B}_n^* + \mathbf{D}_n^*\times\mathbf{B}_n) = \frac{k_x}{\omega_n}\left\{\frac{1}{2}\int d^3\mathbf{r}(\mathbf{D}_n^*\cdot\mathbf{E}_n + \mathbf{B}_n^*\cdot\mathbf{H}_n)\right\} = \frac{k_x}{\omega_n}, \quad \text{(B2)}$$

where the second identity is a consequence of the normalization condition (A6). A similar relation between the energy and the wave momentum associated with a plane-wave mode in a uniform moving medium was also derived in Ref. [14]. Hence, substituting the above result into Eq. (B1), it follows that

$$\langle 0|\hat{\mathbf{x}}\cdot\hat{\mathbf{P}}_{wv}|0\rangle = \sum_{\omega_n>0}\frac{\hbar(k_x)_n}{2}, \quad \text{(B3)}$$

where $(k_x)_n$ represents the wave number associated with the nth mode. This formula shows that the "zero-point momentum" of each quantum oscillator is $\hbar k_x/2$. The relation between the wave momentum and $N\hbar\mathbf{k}$ was also pointed out in Ref. [24] for the case of a uniform medium.

But since the allowed values for k_x are symmetric with respect to the origin ($k_x = 2\pi m/L_x$, $m = 0,\pm1,\pm2,\ldots$), it follows that if we consider the same number of modes for each k_x the series (B3) has a sum equal to zero:

$$\langle 0|\hat{\mathbf{x}}\cdot\hat{\mathbf{P}}_{wv}|0\rangle = 0. \quad \text{(B4)}$$

In other words, the quantum expectation of the total wave momentum in the ground state of the system is zero. It should be mentioned that our argument is only partially rigorous because, strictly speaking, the series (B3) is not absolutely convergent, and thus the sum of the series may depend on the order of summation.

APPENDIX C: THE WAVE MOMENTUM FOR A NATURAL MODE

Here, we prove that if $\mathbf{F}_\omega = \binom{\mathbf{E}_\omega}{\mathbf{H}_\omega}$ is a eigenmode [i.e., it satisfies $\hat{N}\mathbf{F}_\omega = \omega\mathbf{M}\cdot\mathbf{F}_\omega$: See Eq. (A5)] associated with the frequency ω and with the wave number k_x (the system is invariant for translations along x) then

$$\int d^3\mathbf{r}\,\hat{\mathbf{x}}\cdot{\rm Re}(\mathbf{D}_\omega\times\mathbf{B}_\omega^*) = \frac{k_x}{\omega}\left\{\frac{1}{2}\int d^3\mathbf{r}(\mathbf{D}_\omega^*\cdot\mathbf{E}_\omega + \mathbf{B}_\omega^*\cdot\mathbf{H}_\omega)\right\}. \quad \text{(C1)}$$

To begin with, we note that the following identity holds for arbitrary vectors:

$$\hat{\mathbf{x}}\cdot(\mathbf{E}\nabla\cdot\mathbf{D} - \mathbf{D}\times\nabla\times\mathbf{E}) = \nabla\cdot(\mathbf{D}\,E_x) - \mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial x}. \quad \text{(C2)}$$

Replacing $\mathbf{D}\to\mathbf{D}_\omega^*$ and $\mathbf{E}\to\mathbf{E}_\omega$ in the above equation, and using Eq. (A5) and $\nabla\cdot\mathbf{D}_\omega = 0$, it follows that

$$\hat{\mathbf{x}}\cdot(-\mathbf{D}_\omega^*\times i\omega\mathbf{B}_\omega) = \nabla\cdot(\mathbf{D}_\omega^*\,E_{\omega,x}) - \mathbf{D}_\omega^*\cdot\frac{\partial\mathbf{E}_\omega}{\partial x}. \quad \text{(C3)}$$

Hence, integrating over the entire cavity we obtain

$$\int d^3\mathbf{r}\,\hat{\mathbf{x}}\cdot(-\mathbf{D}_\omega^*\times i\omega\mathbf{B}_\omega) = \int d^3\mathbf{r}\left(-\mathbf{D}_\omega^*\cdot\frac{\partial\mathbf{E}_\omega}{\partial x}\right). \quad \text{(C4)}$$

Likewise, it is also possible to show that

$$\int d^3\mathbf{r}\,\hat{\mathbf{x}}\cdot[-\mathbf{B}_\omega^*\times(-i\omega\mathbf{D}_\omega)] = \int d^3\mathbf{r}\left(-\mathbf{B}_\omega^*\cdot\frac{\partial\mathbf{H}_\omega}{\partial x}\right). \quad \text{(C5)}$$

This implies that

$$-i\omega\int d^3\mathbf{r}\,\hat{\mathbf{x}}\cdot(\mathbf{D}_\omega^*\times\mathbf{B}_\omega + \mathbf{D}_\omega\times\mathbf{B}_\omega^*) = -\int d^3\mathbf{r}\left(\mathbf{D}_\omega^*\cdot\frac{\partial\mathbf{E}_\omega}{\partial x} + \mathbf{B}_\omega^*\cdot\frac{\partial\mathbf{H}_\omega}{\partial x}\right). \quad \text{(C6)}$$

Using now the fact that the system is invariant to translations along the x direction, and hence $\frac{\partial}{\partial x} = ik_x$, we readily obtain Eq. (C1). It is interesting to mention that Eq. (C1) can be rewritten as

$$\mathbf{g}_{\rm wv,av}\cdot\hat{\mathbf{x}} = W_{\rm av}\frac{k_x}{\omega}, \quad \text{(C7)}$$

where $\mathbf{g}_{\rm wv,av} = \frac{1}{V}\int d^3\mathbf{r}\frac{1}{2}{\rm Re}(\mathbf{D}_\omega\times\mathbf{B}_\omega^*)$ is the time- and volume-averaged wave momentum density, $W_{\rm av} = \frac{1}{4V}\int d^3\mathbf{r}(\mathbf{D}_\omega^*\cdot\mathbf{E}_\omega + \mathbf{B}_\omega^*\cdot\mathbf{H}_\omega)$ is time- and volume-averaged stored energy density, and V is the volume of the region of interest.

APPENDIX D: PERTURBATION OF THE EIGENFREQUENCIES OF AN ELECTROMAGNETIC CAVITY UNDER AN INFINITESIMAL VARIATION OF THE MATERIAL PARAMETERS

Here, we consider a generic cavity filled with a nonuniform, nondispersive, bianisotropic material described by the material matrix $\mathbf{M} = \mathbf{M}(\mathbf{r})$, consistent with the geometry of Fig. 1. We suppose that the material matrix depends on a continuous parameter u, which controls, for example, either the geometry of the system or the velocity of one of the moving components of the system, so that $\mathbf{M} = \mathbf{M}(\mathbf{r},u)$. The objective is to determine the perturbation of the eigenfrequencies of the cavity under an infinitesimal variation of u. To begin with, let us consider an eigenmode $\mathbf{F}_n = \begin{pmatrix} \mathbf{E}_n \\ \mathbf{H}_n \end{pmatrix}$ such that [Eq. (A5)]

$$\hat{\mathbf{N}} \cdot \mathbf{F}_n = \omega_n \mathbf{M} \cdot \mathbf{F}_n, \tag{D1}$$

where ω_n is the eigenfrequency (for simplicity, in what follows the subscript n will be dropped). Since $\mathbf{M}$ depends on the parameter u, it is clear that $\mathbf{F}$ and ω also do. Hence, we may write

$$(\hat{\mathbf{N}} - \omega\,\mathbf{M}) \cdot \frac{\partial \mathbf{F}}{\partial u} = \frac{\partial \omega}{\partial u}\mathbf{M} \cdot \mathbf{F} + \omega \frac{\partial \mathbf{M}}{\partial u} \cdot \mathbf{F}. \tag{D2}$$

Calculating the scalar product of both sides with $\mathbf{F}^*$, and integrating over the volume of the cavity, we find that

$$\int d^3\mathbf{r}\,\mathbf{F}^* \cdot (\hat{\mathbf{N}} - \omega \mathbf{M}) \cdot \frac{\partial \mathbf{F}}{\partial u} = \int d^3\mathbf{r} \left(\frac{\partial \omega}{\partial u}\mathbf{F}^* \cdot \mathbf{M} \cdot \mathbf{F} + \omega\,\mathbf{F}^* \cdot \frac{\partial \mathbf{M}}{\partial u} \cdot \mathbf{F} \right). \tag{D3}$$

But straightforward calculations show that

$$\int d^3\mathbf{r}\,\mathbf{F}^* \cdot (\hat{\mathbf{N}} - \omega \mathbf{M}) \cdot \frac{\partial \mathbf{F}}{\partial u} = \int d^3\mathbf{r}[(\hat{\mathbf{N}} - \omega \mathbf{M}) \cdot \mathbf{F}]^* \cdot \frac{\partial \mathbf{F}}{\partial u} = 0, \tag{D4}$$

where we used the fact that $\hat{\mathbf{N}} - \omega \mathbf{M}$ is Hermitian with respect to the standard canonical scalar product, and Eq. (D1) (notice that the boundary conditions satisfied by $\frac{\partial \mathbf{F}}{\partial u}$ at the walls of the cavity are the same as those satisfied by $\mathbf{F}$, e.g., periodic boundary conditions). Feeding this result into Eq. (D3), we obtain the desired formula for the perturbation of the eigenfrequency in terms of the variation in the material matrix:

$$\frac{\partial \omega}{\partial u} = \frac{-\omega \int d^3\mathbf{r}\,\mathbf{F}^* \cdot \frac{\partial \mathbf{M}}{\partial u} \cdot \mathbf{F}}{\int d^3\mathbf{r}\,\mathbf{F}^* \cdot \mathbf{M} \cdot \mathbf{F}}. \tag{D5}$$

APPENDIX E: THE RELATIVISTIC INVARIANCE OF THE PSEUDOMOMENTUM

Let us consider a reference frame that moves with a constant velocity u with respect to the laboratory frame. We assume that the velocity is directed along the x axis. In this frame, the pseudomomentum P'_{ps,V_i} of a given slab reads [Eq. (28)]

$$\begin{aligned}\frac{P'_{\text{ps},V_i}}{A'} &= \left(1 - \beta_i'^2\right) \frac{\partial}{\partial v_i'} \left(\frac{E_{0,\text{int}}(v_1', \ldots, v_i', \ldots, v_n')}{A'} \right) \\ &= \left(1 - \beta_i'^2\right) \frac{\partial}{\partial v_i'} \left(\frac{E_{0,\text{int}}(v_1, \ldots, v_i, \ldots, v_n)}{A} \right),\end{aligned} \tag{E1}$$

where the primed quantities are with respect to the moving frame, and we have used the invariance of $\frac{E_{0,\text{int}}}{A}$ with respect to movements along the x direction, as discussed in Sec. V B. The slab velocities in the laboratory frame and in the moving frame are related by the relativistic velocity addition law,

$$v_i = \frac{v_i' + u}{1 + u v_i'/c^2}, \quad v_i' = \frac{v_i - u}{1 - u v_i/c^2}, \tag{E2}$$

therefore, the pseudomomentum as seen from the moving frame is

$$\frac{P'_{\text{ps},V_i}}{A'} = \left(1 - \beta_i'^2\right) \left(\frac{\partial v_i}{\partial v_i'} \right) \frac{\partial}{\partial v_i} \left(\frac{E_{0,\text{int}}}{A} \right). \tag{E3}$$

But straightforward calculations show that

$$\left(1 - \beta_i'^2\right) \frac{\partial v_i}{\partial v_i'} = 1 - \beta_i^2, \tag{E4}$$

and thus Eqs. (28) and (E3) imply that $P'_{\text{ps},V_i}/A' = P_{\text{ps},V_i}/A$, i.e., that the pseudomomentum density is the same in both frames and is invariant with respect to relative motion along the x axis.

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