Optimality for indecomposable entanglement witnesses

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We examine various notions related to the optimality for entanglement witnesses arising from Choi-type positive linear maps. We found examples of optimal entanglement witnesses which are nondecomposable but are not "nondecomposable optimal entanglement witnesses" in the sense of Lewenstein, Kraus, Cirac, and Horodecki [Phys. Rev. A **62**, 052310 (2000)]. We suggest using the terms *PPTES witness* and *optimal PPTES witness* in place of "nondecomposable entanglement witness" and "nondecomposable optimal entanglement witnesses" in order to avoid possible confusion. Here, PPTES refers entangled states with positive partial transposes. We also found examples of nonextremal optimal entanglement witnesses which are indecomposable.

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I. INTRODUCTION

Quantum entanglement is now considered the main key resource for applications to quantum information and quantum computation theory. One of the major research topics in the theory of entanglement is, of course, how to distinguish entanglement from separable states. For this purpose, positive linear maps are known to be the most complete tools [1] among various criteria. This criterion for separability using positive maps is equivalent to the duality theory [2] between the positivity of linear maps and the separability of block matrices, through the Jamiołkowski-Choi isomorphism [3,4]. In this sense, we need a positive linear map to detect entanglement. This is formulated as the notion of an entanglement witness [5] that is just a positive linear map which is not completely positive under the isomorphism. We refer to Refs. [6,7] for systematic approaches to the duality using the Jamiołkowski-Choi isomorphism.

An entanglement witness which detects a maximal set of entanglement is said to be optimal, as was introduced in Ref. [8]. The notion of optimality may be explained in terms of the facial structures of the convex cone $\mathbb{P}_1$ consisting of all positive linear maps between matrix algebras. In fact, it was shown [9] that a positive map ϕ is an optimal entanglement witness if and only if the smallest face of $\mathbb{P}_1$ containing ϕ has no completely positive linear map. See also Ref. [10]. Therefore, the most natural candidates for optimal entanglement witnesses are extremal positive maps which are not completely positive. In spite of its importance, the facial structure of cone $\mathbb{P}_1$ is very far from being understood even in the low-dimensional cases. For the case where both the domain and the range have the 2×2 matrix algebra, all extreme points of the convex set consisting of unital positive maps were found in the 1960s [11]. All the facial structures of this convex set are completely understood [12]. See also Ref. [13]. Another sufficient condition for optimality is the notion of the spanning property, as was introduced in Ref. [8]. This is very useful because the spanning property is much easier to verify than the optimality itself. It turns out [14] that a positive map ϕ has the spanning property if and only if the smallest exposed face of cone $\mathbb{P}_1$ containing ϕ has no completely positive map.

Recall that a convex subset F of a convex set C is said to be a face if the following condition holds: If a convex combination of two points $x, y \in C$ belongs to F, then x and y themselves belong to F. A face F of C is said to be an exposed face if it is the intersection of C and a hyperplane. We will give an example of a face which is not exposed through the discussion. See Fig. 1.

For the decomposable case, several necessary and/or sufficient conditions for optimality are known, and there are processes to characterize optimal decomposable entanglement witnesses. See Refs. [9,15,16] for examples. In the case of indecomposable entanglement witnesses, a condition for optimality has been found recently [17], and examples of optimal entanglement witnesses without the spanning property were given. Nevertheless, we still have a few kinds of examples for optimal entanglement witnesses arising from indecomposable maps. We note that the Choi-type positive maps are one of the main resources for indecomposable positive maps. The primary purpose of this Brief Report is to analyze those maps between 3×3 matrix algebras and examine the relations between extremeness, spanning property, and optimality.

We note that a positive map ϕ detects entanglement with positive partial transposes if and only if it is indecomposable. An indecomposable positive map ϕ is said to be a nondecomposable optimal entanglement witness (nd-OEW) in Ref. [8] if it detects a maximal set of PPTES. But it is not clear at all that an optimal entanglement witness which is nondecomposable is really nd-OEW in the sense of Ref. [8]. We found that this is not the case. In order to avoid such confusion, we use the following terminology in this Brief Report. A positive linear map ϕ is said (i) to be *co-optimal* if the smallest face of $\mathbb{P}_1$ containing ϕ has no completely copositive map, (ii) to be *bi-optimal* if it is optimal and co-optimal, (iii) to have the *cospanning property* if the smallest exposed face of $\mathbb{P}_1$ containing ϕ has no completely copositive map, and (iv) to have the *bispanning property* if it has both the spanning and cospanning properties. It is clear that ϕ is co-optimal (has the cospanning property) if and only if the composition $\phi \circ t$ with the transpose map t is optimal (has the spanning property). If we use the Jamiołkowski-Choi isomorphism, then a self-adjoint block matrix W is co-optimal (has the co-spanning property) if and only if the partial transpose W^Γ is optimal (has the spanning property). It is also clear that ϕ is bi-optimal (has the bispanning property) if and only if the smallest face (the smallest exposed face) of $\mathbb{P}_1$ containing ϕ has no decomposable map. Therefore,

ϕ is an nd-OEW in the sense of Ref. [8] if and only if it is bi-optimal. We note that if ϕ is bi-optimal then it is automatically indecomposable. We will present examples of indecomposable optimal positive linear maps which are not bi-optimal. Since an optimal decomposable entanglement witness is completely copositive, it is never co-optimal. Therefore, the notions of co-optimality and cospanning are useful only for indecomposable entanglement witnesses.

For nonnegative real numbers a,b, and c, the Choi-type map is given by

$$\Phi[a,b,c](X) = \begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11}+ax_{22}+bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11}+cx_{22}+ax_{33} \end{pmatrix}$$

for $X = [x_{ij}] \in M_3$, where M_3 denotes the C^* algebra of all 3×3 matrices over the complex field $\mathbb{C}$. Choi [18] showed that the map $\Phi[1,2,2]$ is a two-positive linear map which is not completely positive. This is the first known example to distinguish n-positivities for different $n = 2,3,\dots$ The map $\Phi[1,0,\mu]$ with $\mu \geqslant 1$ is also the first example of an indecomposable positive linear map [19] in the literature, and the map $\Phi[1,0,1]$ is extremal [20], that is, generates an extremal ray of the cone $\mathbb{P}_1$. Later, it was shown [21] that this map $\Phi[1,0,1]$ is not the sum of a two-positive map and a two-copositive map. See also Ref. [22]. The map $\Phi[1,0,1]$ is usually called the Choi map. The maps $\Phi[a,b,c]$ have been considered in Ref. [23] to distinguish various notions of positivity. See also [17,21,24–33] for another variation of the Choi map. It is known [23] that the map $\Phi[a,b,c]$ is positive if and only if the condition

$$a+b+c \geqslant 2, \quad 0 \leqslant a \leqslant 1 \Longrightarrow bc \geqslant (1-a)^2 \qquad (1)$$

holds. Note that $\Phi[1,0,1]$ is optimal by the extremeness. It is also well known that $\Phi[1,0,1]$ does not have the spanning property, as was observed in Ref. [34]. See also Ref. [35]. It is also known to have the cospanning property [36]. Recently, we [37] have shown that if $0 < a < 1$ and the equalities hold in the both inequalities in Eq. (1), then $\Phi[a,b,c]$ has the bispanning property. We note that the Choi matrix $C_\Phi = \sum_{i,j=0}^{2} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)$ of the map $\Phi[a,b,c]$ is given by

$$W[a,b,c] = \begin{pmatrix} a & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a \end{pmatrix}. \qquad (2)$$

In the next section, we examine the above-mentioned properties for boundary points of the convex body determined by condition (1), and we discuss the result in the final section.

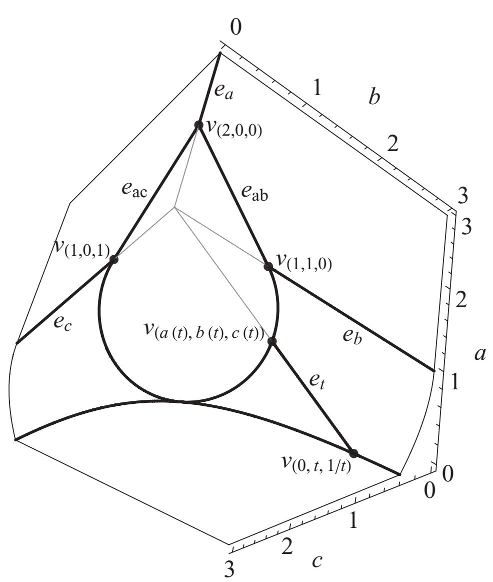

FIG. 1. Part of the convex body determined by Eq. (1). The smallest face containing $v_{(1,1,0)}$ is itself, but the smallest exposed face containing it is e_{ab}. The straight lines containing faces e_a, e_b, e_c, and e_t meet each other at the point (1,0,0), which is not in the convex body.

II. FACIAL STRUCTURES AND OPTIMALITY

Before going further, we note that the six properties, optimal, co-optimal, bi-optimal, spanning, cospanning, and bispanning, are properties depending on the faces: If ϕ_1 and ϕ_2 determine the same smallest face containing them, then they are interior points of a common face and share each property because the properties are described in terms of faces. Therefore, we can say that a face itself has one of six properties without confusion, and this means that every interior point of the face satisfies the property. It is also clear that if a face has a property, then every subface also has the same property. Hence, if a point ϕ does not have a property, then every interior point in the face containing ϕ does not have the property. Therefore, we need to clarify the facial structures of the three-dimensional convex body determined by Eq. (1). It should be noted that the face of the convex body need not give rise to a real face of the convex cone $\mathbb{P}_1$. Nevertheless, an interior point of a face of the convex body gives rise to an interior point of the face of the cone $\mathbb{P}_1$ determined by the corresponding map.

First of all, the convex body has the following four two-dimensional faces: (i) $f_{ab} = \{(a,b,c) : c = 0,\ a+b \geqslant 2,\ a \geqslant 1\}$, (ii) $f_{ac} = \{(a,b,c) : b = 0,\ a+c \geqslant 2,\ a \geqslant 1\}$, (iii) $f_{bc} = \{(a,b,c) : a = 0,\ bc \geqslant 1\}$, and (iv) $f_{abc} = \{(a,b,c) : a+b+c = 2, 0 \leqslant a \leqslant 1 \Longrightarrow bc \geqslant (1-a)^2\}$. We note [23] that $\Phi[a,b,c]$ is completely positive if and only if $a \geqslant 2$, and it is completely copositive if and only if $bc \geqslant 1$. Therefore, face $f_{\rm abc}$ has the completely positive map $\Phi[2,0,0]$ and the completely copositive map $\Phi[0,1,1]$, and so $f_{\rm abc}$ is neither optimal nor co-optimal. It is also easy to examine the optimality for the first three cases. For example, if $a > 2$, then the map $\Phi[a,0,0]$ is written by

$$\Phi[a,0,0] = \Phi[2,0,0] + (a-2)D,$$

where D is the diagonal map which sends $[x_{ij}]$ to the diagonal matrix with the diagonal entries (x_{11},x_{22},x_{33}). Map D is both completely positive and completely copositive. This means that the map $\Phi[a,0,0]$ never satisfies optimality and co-optimality. Therefore, no interior point in the two-dimensional faces $f_{\rm ac}$ and $f_{\rm ab}$ ever satisfy the above properties. By the same argument, this is also the case for face $f_{\rm bc}$.

We note that the convex body has also the following five one-dimensional faces which are on the a axis, ab plane, or ac plane: (i) $e_{\rm a} = \{(a,0,0) : a \geqslant 2\}$, (ii) $e_{\rm b} = \{(1,b,0) : b \geqslant 1\}$, (iii) $e_{\rm c} = \{(1,0,c) : c \geqslant 1\}$, (iv) $e_{\rm ab} = \{(a,b,0) : a + b = 2,\ 1 \leqslant a \leqslant 2\}$, and (v) $e_{\rm ac} = \{(a,0,c) : a + c = 2,\ 1 \leqslant a \leqslant 2\}$. Among them, we have already seen that face $e_{\rm a}$ is neither optimal nor co-optimal. This is also the case for $e_{\rm b}$ and $e_{\rm c}$ since it is possible to subtract a map which is both completely positive and completely copositive. It is also clear that neither $e_{\rm ab}$ nor $e_{\rm ac}$ is optimal. In order to find other one-dimensional faces, we note that the parametrization

$$(a(t),\, b(t),\, c(t)) = \frac{1}{1-t+t^2}((1-t)^2, t^2, 1), \quad 0 < t < \infty$$

satisfies the condition

$$a(t) + b(t) + c(t) = 2, \quad 0 \leqslant a(t) \leqslant 1,$$
$$b(t)c(t) = [1 - a(t)]^2,$$

as was considered in Ref. [37]. For each fixed positive number $t > 0$ with $t \neq 1$, the line segment given by

$$e_t = \{(1-s, st, s/t) : t/(t^2 - t + 1) \leqslant s \leqslant 1\}$$

lies on the surface $bc = (1-a)^2$ for $0 \leqslant a < 1$ and connects the point $(a(t),\, b(t),\, c(t))$ to the point $(0,t,1/t)$. This gives us one-dimensional faces e_t for each $t > 0$ with $t \neq 1$. Note that $\Phi[0,t,1/t]$ is completely copositive for each $t > 0$, and so it is clear that e_t is not co-optimal.

It remains to list the zero-dimensional faces as follows: (i) $v_{(2,0,0)}$, $v_{(1,0,1)}$, $v_{(1,1,0)}$, (ii) $v_{(a(t),b(t),c(t))}$ for $t > 0$ and $t \neq 1$, and (iii) $v_{(0,t,1/t)}$ for $t > 0$.

So far, we have seen that faces $f_{\rm ab}$, $f_{\rm ac}$, $f_{\rm bc}$, $f_{\rm abc}$, $e_{\rm a}$, $e_{\rm b}$, and $e_{\rm c}$ are neither optimal nor co-optimal. Therefore, they have neither the spanning property nor the cospanning property. We test the other faces. First of all, we show that e_t and $v_{(0,t,1/t)}$ have the spanning properties. To do this, it suffices to consider the case when (a,b,c) satisfies the condition

$$0 \leqslant a < 1, \qquad bc = (1-a)^2, \qquad a+b+c > 2. \quad (3)$$

We recall [14] (see also Ref. [8]) that $\phi \in \mathbb{P}_1$ has the spanning property if and only if the set

$$P[\phi] := \{\xi \otimes \eta \in \mathbb{C}^m \otimes \mathbb{C}^n : \langle \xi \otimes \eta | C_\phi | \xi \otimes \eta \rangle = 0\}$$

spans the whole space $\mathbb{C}^m \otimes \mathbb{C}^n$, where C_ϕ is the Choi matrix of ϕ. We define vectors in $\mathbb{C}^3$ as follows:

$$\begin{aligned}
|\xi^0_{\theta,\sigma}\rangle &= e^{i\theta} b^{1/4}|1\rangle + e^{i\sigma} c^{1/4}|2\rangle,\\
|\xi^1_{\theta,\sigma}\rangle &= e^{i\theta} b^{1/4}|2\rangle + e^{i\sigma} c^{1/4}|0\rangle,\\
|\xi^2_{\theta,\sigma}\rangle &= e^{i\theta} b^{1/4}|0\rangle + e^{i\sigma} c^{1/4}|1\rangle,\\
|\eta^0_{\theta,\sigma}\rangle &= e^{-i\theta} (bc)^{1/4}|1\rangle + e^{-i\sigma} b^{1/2}|2\rangle,\\
|\eta^1_{\theta,\sigma}\rangle &= e^{-i\theta} (bc)^{1/4}|2\rangle + e^{-i\sigma} b^{1/2}|0\rangle,\\
|\eta^2_{\theta,\sigma}\rangle &= e^{-i\theta} (bc)^{1/4}|0\rangle + e^{-i\sigma} b^{1/2}|1\rangle.
\end{aligned} \quad (4)$$

Then, it is easy to check that

$$\begin{aligned}
&\langle \xi^k_{\theta,\sigma} \otimes \eta^k_{\theta,\sigma} | C_\Phi | \xi^k_{\theta,\sigma} \otimes \eta^k_{\theta,\sigma} \rangle\\
&\quad = \langle \xi^k_{\theta,\sigma} \otimes \eta^k_{\theta,\sigma} | W[a,b,c] | \xi^k_{\theta,\sigma} \otimes \eta^k_{\theta,\sigma} \rangle\\
&\quad = -2(1-a)bc^{1/2} + 2b^{3/2}c
\end{aligned}$$

for all $k = 1,2,3$ and $\langle \xi^k_{\theta,\sigma} \otimes \eta^k_{\theta,\sigma} | C_\Phi | \xi^k_{\theta,\sigma} \otimes \eta^k_{\theta,\sigma} \rangle = 0$ whenever condition (3) holds. Therefore, the vectors $|\xi^k_{\theta,\sigma} \otimes \eta^k_{\theta,\sigma}\rangle$ belong to $P[\Phi[a,b,c]]$ for all $k = 1,2,3$ whenever condition (3) holds. We take $\sigma_1 = 0$, $\sigma_2 = \pi/2$, and $\sigma_3 = \pi$ and consider the 9×9 matrix whose columns are nine vectors $|\xi^k_{0,\sigma_\ell} \otimes \eta^k_{0,\sigma_\ell}\rangle$ for $k, \ell = 1,2,3$. Then the determinant of M is given by

$$|\det M| = 128\, b^{\frac{9}{2}} c^{\frac{9}{4}}$$

which is nonzero. This shows that e_t and $v_{(0,t,1/t)}$ have the spanning properties.

Next, we consider the zero-dimensional face $v_{(2,0,0)}$. We see that the smallest exposed face F containing $v_{(1,0,1)}$ already contains $v_{(2,0,0)}$ in Fig. 1 (see Ref. [14] for a more general approach). We have seen [36] that $\Phi[1,0,1]$ has the cospanning property, and so F has no completely copositive map. This show that $v_{(2,0,0)}$ has the cospanning property, and so $e_{\rm ab}$ and $e_{\rm ac}$ also have the cospanning properties.

TABLE I. Summary of (co-)optimality and (co)spanning property for faces of the convex body illustrated in Fig. 1. Span., spanning; Opt., optimality.

	(Co)spanning property			(Co-)optimality		
Faces	Span.	Co-span.	Bi-span.	Opt.	Co-opt.	Bi-opt.
$f_{\rm ab}, f_{\rm ac}, f_{\rm bc}, f_{\rm abc}, e_{\rm a}, e_{\rm b}, e_{\rm c}$	N	N	N	N	N	N
$e_{\rm ab}, e_{\rm ac}, v_{(2,0,0)}$	N	Y	N	N	Y	N
$e_t, v_{(0,t,1/t)}$	Y	N	N	Y	N	N
$v_{(1,0,1)}, v_{(1,1,0)}$	N	Y	N	Y	Y	Y
$v_{(a(t),b(t),c(t))}$	Y	Y	Y	Y	Y	Y

We summarize the result in Table I. We note [23] that the map $\Phi[a,b,c]$ is decomposable if and only if the condition

$$0 \leqslant a \leqslant 2 \Longrightarrow bc \geqslant \left(\frac{2-a}{2}\right)^2$$

holds. Therefore, we see that interior points of the faces $e_{\rm b}$, $e_{\rm c}$, $e_{\rm ab}$, $e_{\rm ac}$, e_t, $v_{(1,0,1)}$, $v_{(1,1,0)}$, $v_{(a(t),b(t),c(t))}$ give rise to indecomposable positive maps. We note that every interior point of face e_t gives rise to an example of an indecomposable optimal entanglement witness which is not bi-optimal. So this is not nd-OEW in the sense of Ref. [8]. If we consider the composition of the transpose map, then faces $e_{\rm ab}$ and $e_{\rm ac}$ play the exact same role. They also provide us examples of nonextremal entanglement witnesses with the spanning property. On the other hand, the Choi maps $v_{(1,0,1)}$ and $v_{(1,1,0)}$ are extremal entanglement witnesses without the spanning property. Therefore, we see that two sufficient conditions, extremeness and spanning property, for the optimality are logically independent.

III. CONCLUSIONS

In this Brief Report, we considered Choi-type positive maps between 3×3 matrices and determined their optimality, co-optimality, spanning property, and cospanning property. We have seen that even though a nondecomposable entanglement witness is optimal, it need not to be a nondecomposable optimal entanglement witness in the sense of Ref. [8]. Because a positive map detects a PPTES if and only if it is indecomposable, we suggest using the term *PPTES witness* in place of nondecomposable entanglement witness and using the term *optimal PPTES witness* in place of nd-OEW. In other words, we say that a positive map is an optimal PPTES witness when it is bi-optimal. This is very natural since a positive map detects a maximal set of PPTES if and only if it is bi-optimal.

Optimality is not very easy to determine for a given positive linear map because we do not know all the facial structures of the convex cone $\mathbb{P}_1$ consisting of all positive maps. The spanning property is stronger than optimality and relatively easy to check. Another sufficient condition for optimality is extremeness. We also showed that the spanning property and extremeness are logically independent.

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