First-order corrections to the rotating-wave approximation in the Jaynes-Cummings model

Shu He,[1,2] Chen Wang,[1] Qing-Hu Chen,[3,1,*] Xue-Zao Ren,[2] Tao Liu,[2] and Ke-Lin Wang[4]

[1]*Department of Physics, Zhejiang University, Hangzhou 310027, People's Republic of China*

[2]*School of Science, Southwest University of Science and Technology, Mianyang 621010, People's Republic of China*

[3]*Center for Statistical and Theoretical Condensed Matter Physics, Zhejiang Normal University, Jinhua 321004, People's Republic of China*

[4]*Department of Modern Physics, University of Science and Technology of China, Hefei 230026, People's Republic of China*

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The Jaynes-Cummings model without the rotating-wave approximation can be solved exactly by an extended Swain ansatz with conserved parity. Analytical approximations are then performed at different levels. The well-known rotating-wave approximation (RWA) is naturally covered in the present zeroth- and first-order approximations. A first-order correction to the RWA can be obtained in a second-order approximation, by which the effect of the counter-rotating-wave term emerges clearly. Concise analytical expressions are given explicitly and can be applicable up to the ultrastrong-coupling regime. A preliminary application to vacuum Rabi splitting is shown to be very successful.

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I. INTRODUCTION

The interaction of light with matter is a fundamental concern in optical physics. The simplest paradigm is a two-level atom coupled to the electromagnetic mode of a cavity. In the strong-coupling regime where the coupling strength g/ω (ω is the cavity frequency) between the atom and the cavity mode exceeds the loss rates, the atom and the cavity can repeatedly exchange excitations before coherence is lost. Rabi oscillations can be observed in this strong-coupling atom-cavity system, which is usually called cavity quantum electrodynamics (QED) [1]. Typically, the coupling strength in cavity QED reaches $g/\omega \sim 10^{-6}$. It can be described by the well-known Jaynes-Cummings (JC) model [2].

Recently, for superconducting qubits, a one-dimensional transmission-line resonator [3] or an *LC* circuit [4–6] has been shown to play the role of the cavity; this is known today as circuit QED. More recently, an *LC* resonator inductively coupled to a superconducting qubit [7–9] has been realized experimentally. The qubit-resonator coupling has been strengthened from $g/\omega \sim 10^{-3}$ in the earlier realization [3] to a few percentage later [6,10] and most recently to 10% [7–9]. In the cavity-QED system, the rotating-wave approximation (RWA) is usually made; however, in circuit QED, due to the ultrastrong coupling $g/\omega \sim 0.1$ [7–9], evidence for the breakdown of the RWA has been provided [7]. Therefore, counter-rotating-wave terms (CRTs) should be included in the JC model. Recently, many works have been devoted to this qubit-oscillator system in the ultrastrong-coupling regime [11–15].

Actually, the JC model without the RWA has been studied extensively for more than 40 years. A incomplete list of references is given by Refs. [16–31]. Two main schemes are employed. One is based on photonic Fock states [16–21] following the pioneering work by Swain [16]. Continued fractions are present and the solution then becomes very intricate. The other is on the basis of various polaronlike transformations or shifted operators, which are approaches based on photonic coherent states [22–30]. A very accurate solution can be obtained readily, but infinite photonic Fock states are involved.

The RWA eigenstates include only two bare states, which facilitated earlier clear investigations of various quantum phenomena in quantum optics. Surprisingly, one or a few dominant terms in the eigenstates of the JC Hamiltonian beyond the RWA are still lacking or have not been given explicitly until now, to the best of our knowledge. Simple corrections to the RWA may be very useful to analyze the effect of CRTs on various phenomena at the microscopic level. In this sense, a few dominant terms are more helpful than the exact solution including infinite bare states. The purpose of this paper is not to find a more accurate spectrum, but to give concise first-order corrections to the RWA.

By using the conserved parity, we extend Swain's wave function to the JC model without the RWA. We do not follow the usual exact diagonalization routine. As an alternative, we derive a polynomial equation with only a single variable, just the eigenvalue that we seek. The solutions to this polynomial equation can give exactly all eigenfunctions and eigenvalues for arbitrary parameters. Moveover, we can perform approximations step by step with the help of these exact solutions. The zeroth- and first-order approximations will exactly recover the RWA results. The dominant effect of the CRTs emerges in the second-order approximation.

Without the RWA, the Hamiltonian of a two-level atom (qubit) with transition frequency Δ interacting with a single-mode quantized cavity of frequency ω is

$$H = \frac{\Delta}{2}\sigma_z + \omega a^\dagger a + g(a^\dagger + a)\sigma_x, \tag{1}$$

where g is the coupling strength, σ_x and σ_z are Pauli spin-1/2 operators, and $a^\dagger$ and a are the creation and annihilation operators for the quantized field. Here, $\delta = \Delta - \omega$ is defined as the dimensionless detuning parameter. The energy scale is set as $\omega = 1$ here.

The RWA is made by neglecting the CRTs $a^\dagger\sigma_+ + a\sigma_-$; then one can easily diagonalize the Hamiltonian and obtain the eigenfunctions as [32]

$$|\,\rangle^{\mathrm{RWA}} = \begin{pmatrix} c_n\,|n\rangle \\ d_n\,|n+1\rangle \end{pmatrix}, \quad n = 0,1,2,\ldots. \tag{2}$$

*qhchen@zju.edu.cn

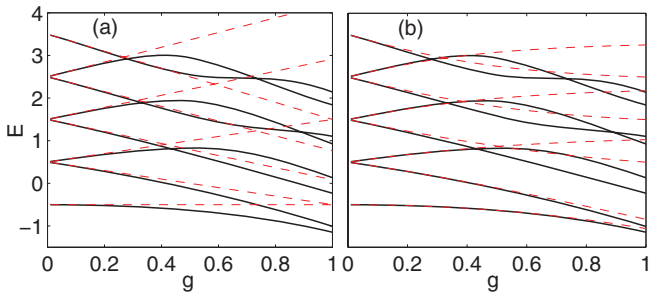

FIG. 1. (Color online) The first eight energy levels as a function of coupling constant g at resonance. The exact results without the RWA are presented by black solid lines. (a) The RWA results (red dashed line) and (b) the analytical results in Eqs. (26), (27), (30), and (32) of the second-order approximation (red dashed line).

For later use, we also list the relevant eigenvalues

$$E_{1n}^{\mathrm{RWA}} = n + \tfrac{1}{2} - \tfrac{1}{2}R_n, \tag{3}$$

$$E_{2n}^{\mathrm{RWA}} = n + \tfrac{1}{2} + \tfrac{1}{2}R_n, \tag{4}$$

where $R_n = \sqrt{\delta^2 + 4g^2(n+1)}$. In the ground state (GS), the qubit is in the GS and the photon is in a vacuum state. The GS energy is $E^0 = -\frac{\Delta}{2}$.

Associated with the JC Hamiltonian with and without the RWA is conserved parity Π, such that $[H,\Pi] = 0$, which is given by

$$\Pi = -\sigma_z \exp(i\pi \widehat{N}), \tag{5}$$

where $\widehat{N} = a^\dagger a$ is the bosonic number operator. Π has two eigenvalues ± 1, depending on whether the excitation number is even or odd. The above state (2) with even n is of odd parity and with odd n even parity. The ground state is of even parity. The RWA results for the first eight energy levels at resonance, $E_{1(2)n} = n + \frac{1}{2} \pm g\sqrt{n+1}$, are given in Fig. 1(a) for later comparison.

II. EXACT SOLUTION WITHOUT THE RWA

First we introduce a scheme to obtain the exact solution to the JC model without the RWA. For convenience, we can write a transformed Hamiltonian with a rotation around the y axis by an angle $\frac{\pi}{2}$. In the matrix form it is

$$H = \begin{pmatrix} a^\dagger a + g(a^\dagger + a) & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & a^\dagger a - g(a^\dagger + a) \end{pmatrix}. \tag{6}$$

About 40 years ago, Swain first proposed an ansatz for the wave function in the photonic Fock states [16], which is also a starting point of the standard numerically exact diagonalization scheme. Since then, various methods have been developed along this line [17–20], but the conserved parity has not been considered, to our knowledge. Therefore continued fractions are unavoidably present in the expressions for the eigensolutions.

We also proceed along this line, but the parity is incorporated in Swain's ansatz, which is then given by

$$|\rangle = \begin{pmatrix} \sum_{n=0}^{M} c_n |n\rangle \\ \pm \sum_{n=0}^{M} (-1)^n c_n |n\rangle \end{pmatrix} \tag{7}$$

where $+(-)$ stands for even (odd) parity, and M is the truncated number. The Schrödinger equation gives

$$\sum_{n=0} a^\dagger a c_n |n\rangle + g \sum_{n=0} c_n (\sqrt{n}|n-1\rangle + \sqrt{n+1}|n+1\rangle) \mp \frac{\Delta}{2} \sum_{n=0} (-1)^n c_n |n\rangle = E c_n |n\rangle. \tag{8}$$

Left-multiplying the photonic states $\langle m|$ gives

$$m c_m + g\sqrt{m+1} c_{m+1} + g\sqrt{m} c_{m-1} \mp \frac{\Delta}{2}(-1)^m c_m = E c_m;$$

then we have the recurrence relation

$$c_{m+1} = \frac{1}{g\sqrt{m+1}} \left[E - m \pm \frac{\Delta}{2}(-1)^m \right] c_m - \sqrt{\frac{m}{m+1}} c_{m-1}. \tag{9}$$

By careful inspection of Eq. (7), one can find that c_0 is flexible in the Schrödinger equation where normalization for the eigenfunction is not necessary, so we select $c_0 = 1.0$. Then we have

$$c_1 = \frac{1}{g} \left[E \pm \frac{\Delta}{2} \right].$$

Once the first two terms are fixed, the coefficients of the other terms higher than $a^\dagger$ should be determined by the recurrence relation Eq. (9).

For $m = M$, the terms higher then $(a^\dagger)^M$ are neglected, and we may set $c_{M+1} = 0$; then we have

$$\left[E - M \pm \frac{\Delta}{2}(-1)^M \right] c_M - g\sqrt{M} c_{M-1} = 0. \tag{10}$$

Note that this is actually a one-variable polynomial equation of degree M. The variable is just the eigenvalue we want to obtain. It is expected that the roots of Eq. (10) will give the exact solutions to the JC model without the RWA if M is large enough.

To obtain the true exact results, in principle, the truncated number M should be taken to infinity. Fortunately, it is not necessary. It is found that finite terms in the state (7) are sufficient to give exact results in the whole coupling range. The usual numerical exact diagonalization can readily give the energy levels and their wave functions in the JC model, which can be regarded as a benchmark. We will increase the truncated number M until the relative difference of the energies obtained from the roots and the standard value is just less than 10^{-7}, which sets the criterion for the convergence achieved. Interestingly, for coupling constant $g \leqslant 0.1$ for three typical atom frequencies $\Delta = 0.1$, 1, and 1.5, the truncated number $M = 15$ in the polynomial equation can give exactly 20 energy levels by the above criterion for convergence. For $g = 1.0$ and 2.0, the polynomial equations with $M = 25$ and 40 can yield about 20 energy levels exactly. In fact, all the above calculations can be immediately done on an ordinary PC.

For later use, the first eight exact energy levels as a function of the coupling constant g obtained by the above polynomial equations with $M = 25$ at resonance ($\Delta = 1$) are presented in Figs. 1(a) and 1(b) by black solid lines. The parity is not changed after the level crossing. We then try to follow the

energy curves to get some analytical approximate results in the next section.

III. ANALYTICAL RESULTS WITHOUT THE RWA

The recurrence relation Eq. (9) can be simplified to a tridiagonal form:

$$\begin{pmatrix} \Omega_0(E) & 1 & 0 & \cdots & 0 \\ 1 & \Omega_1(E) & \sqrt{2} & \cdots & 0 \\ 0 & \sqrt{2} & \Omega_2(E) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \sqrt{M} & \Omega_M(E) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \cdots \\ c_M \end{pmatrix} = 0,$$

where

$$\Omega_m(E) = \frac{1}{g}\left[m - E \mp \frac{\Delta}{2}(-1)^m\right]. \tag{11}$$

The eigensolutions can be obtained from the zeros of the determinant of the matrix. By increasing M, we can get the eigenvalues to any desired accuracy. Alternatively, analytical approximations can be performed systematically by increasing the order of the matrix step by step. In this way, the results are improved gradually by considering more off-diagonal matrix elements.

Neglecting all the off-diagonal terms, the zeroth-order approximation $\Omega_m(E) = 0$ gives

$$E^m = m \mp (-1)^m \frac{\Delta}{2}.$$

The lowest energy is

$$E^0 = -\frac{\Delta}{2}.$$

The ground state is of even parity, so the state is

$$|0\rangle' = \begin{pmatrix} c_0 |0\rangle \\ c_0 |0\rangle \end{pmatrix}. \tag{12}$$

Transforming back to the original frame gives

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_0 |0\rangle \\ c_0 |0\rangle \end{pmatrix} \propto |g,0\rangle .$$

Interestingly, the first element in the zeroth-order approximation actually gives exactly the ground state in the RWA. The other solutions are g irrelevant and therefore omitted.

A. The first-order approximation

The first-order approximation is made by selecting the matrices with order 2 along the diagonal line,

$$\begin{pmatrix} \Omega_m(E) & \sqrt{m+1} \\ \sqrt{m+1} & \Omega_{m+1}(E) \end{pmatrix}\begin{pmatrix} c_m \\ c_{m+1} \end{pmatrix} = 0. \tag{13}$$

It is expected that the mth second-order determinant would contain the information about two levels with the same parity. Comparing with the RWA results, it can be inferred that even m corresponds to odd parity and odd m to even parity. Fortunately, we have the same $\Omega(E)$ for any value of m:

$$\Omega_m(E) = \frac{1}{g}\left[m - E + \frac{\Delta}{2}\right].$$

Then we have the following quadratic equation:

$$\left(m - E + \frac{\Delta}{2}\right)\left(m + 1 - E - \frac{\Delta}{2}\right) - (m+1)\,g^2 = 0,$$

which yields the eigenvalues

$$E = m + \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{(1-\Delta)^2 + 4g^2\,(m+1)}. \tag{14}$$

These are just the RWA eigenvalues in Eqs. (3) and (4).

According to the wave function Eq. (7), the eigenstate is then obtained as

$$|m\rangle' = \begin{pmatrix} c_m |m\rangle + c_{m+1} |m+1\rangle \\ \pm(-1)^m \left[c_m |m\rangle - c_{m+1} |m+1\rangle\right] \end{pmatrix}. \tag{15}$$

By the above relation between m and parity, we always have $\pm(-1)^m = -1$. Transforming back to the original frame gives

$$|m\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}|m\rangle' \propto \begin{pmatrix} c_m |m\rangle \\ c_{m+1} |m+1\rangle \end{pmatrix}, \tag{16}$$

which are just the eigenstates under the RWA in Eq. (2) for excited states.

So in the first approximation, we cannot obtained results beyond the RWA ones for all excited states. The effect of the CRTs emerges only beyond the first-order approximation.

B. The second-order approximation

Naturally, the second-order approximation is performed by reducing the problem to the mth third-order determinant as

$$\begin{vmatrix} \Omega_m(E) & \sqrt{m+1} & 0 \\ \sqrt{m+1} & \Omega_{m+1}(E) & \sqrt{m+2} \\ 0 & \sqrt{m+2} & \Omega_{m+2}(E) \end{vmatrix} = 0. \tag{17}$$

To be more concise, we consider only the resonant case $\delta = 0$. It is straightforward to extend the result to finite detunings.

Two univariate cubic equations for even and odd parity can be explicitly derived for any mth third-order determinant. Three roots for each univariate cubic equation can be obtained easily by the formula presented in the Appendix. Comparing with the exact results obtained above, one can see that the second roots y_2 are the true solutions. In particular, the first root y_1 for $m = 0$ determinant with even (odd) parity gives the GS energy (the energy of the first excited state). With these results in mind, the general solutions can be grouped into the following two cases, and all eigenvalues and eigenfunctions can be given analytically in a unified way.

1. $m = 2k$ with even parity and $m = 2k+1$ with odd parity

For both $m = 2k$ with even parity and $m = 2k+1$ with odd parity ($k = 0,1,2,\ldots$), we have the same univariate cubic equation in the following form:

$$E^3 - \left(3m + \tfrac{5}{2}\right)E^2 + \left[\left(m + \tfrac{3}{2}\right)\left(3m + \tfrac{1}{2}\right) - (2m+3)g^2\right]E - \left(m - \tfrac{1}{2}\right)\left(m + \tfrac{3}{2}\right)^2 + \left[2m^2 + 4m + \tfrac{1}{2}\right]g^2 = 0. \tag{18}$$

According to the Appendix, we have

$$\begin{aligned} A &= 4 + (6m+9)\,g^2, \\ B &= -8m - 12 - (12m^2 + 22m - 3)g^2, \\ C &= (2m+3)^2\,g^4 + \left(6m^3 + 13m^2 + \tfrac{3}{2}m - \tfrac{3}{4}\right)g^2 \\ &\quad + 4m^2 + 12m + 9. \end{aligned}$$

It can be readily proven that $\Gamma = B^2 - 4AC < 0$ in this case, so there are three different real roots. Note above that the energy level is given by the second root y_2, so

$$E = \frac{\left(3m + \frac{5}{2}\right) + \sqrt{4 + (6m+9)g^2}[\cos\theta - \sqrt{3}\sin\theta]}{3}, \tag{19}$$

with

$$\theta = \frac{1}{3}\arccos\left(\frac{8 - (9m+27)\,g^2}{\sqrt{[4 + (6m+9)\,g^2]^3}}\right).$$

In particular, $m = 0$ with even parity will also give the GS. The GS energy is given by the first root y_1:

$$E_{\rm GS} = \frac{\frac{5}{2} - 2\sqrt{4 + 9g^2}\cos\theta}{3}, \tag{20}$$

with

$$\theta = \frac{1}{3}\arccos\left(\frac{8 - 27g^2}{\sqrt{(4+9g^2)^3}}\right).$$

The states in this case all take the form

$$|\,\rangle \propto \begin{pmatrix} c_{m+1}\,|m+1\rangle \\ c_m\,|m\rangle + c_{m+2}\,|m+2\rangle \end{pmatrix}. \tag{21}$$

2. $m = 2k+1$ with even parity and $m = 2k$ with odd parity

For both $m = 2k+1$ with even parity and $m = 2k$ with odd parity, we have the same univariate cubic equation in the other form:

$$E^3 - \left(3m + \tfrac{7}{2}\right)E^2 + \left[\left(m + \tfrac{1}{2}\right)\left(3m + \tfrac{11}{2}\right) - (2m+3)\,g^2\right]E - \left(m + \tfrac{1}{2}\right)^2\left(m + \tfrac{5}{2}\right) + \left(2m^2 + 6m + \tfrac{7}{2}\right)g^2 = 0. \tag{22}$$

Similarly, we have

$$\begin{aligned} A &= (6m+9)g^2 + 4, \\ B &= (-12m^2 - 38m - 21)g^2 + (-8m - 4), \\ C &= (-2m-3)^2 g^4 + \left(6m^3 + 29m^2 + \tfrac{83}{2}m + \tfrac{81}{4}\right)g^2 \\ &\quad + 4m^2 + 4m + 1. \end{aligned}$$

One can also readily prove that $\Gamma = B^2 - 4AC < 0$, so there are also three different real roots. The energy level is given by the second root y_2,

$$E = \frac{\left(3m + \frac{7}{2}\right) + \sqrt{(6m+9)g^2 + 4}[\cos\theta - \sqrt{3}\sin\theta]}{3}, \tag{23}$$

with

$$\theta = \frac{1}{3}\arccos\left(\frac{-8 + 9mg^2}{\sqrt{[4 + (6m+9)g^2]^3}}\right).$$

In particular, $m = 0$ with odd parity will yield also the first excited state. The corresponding eigenenergy is given by the first root y_1,

$$E_1 = \frac{\frac{7}{2} - 2\sqrt{9g^2 + 4}\cos\theta}{3}, \tag{24}$$

with

$$\theta = \frac{1}{3}\arccos\left(\frac{-8}{\sqrt{(4+9g^2)^3}}\right).$$

The states in this case all take the form

$$|\,\rangle \propto \begin{pmatrix} c_m\,|m\rangle + c_{m+2}\,|m+2\rangle \\ c_{m+1}\,|m+1\rangle \end{pmatrix}. \tag{25}$$

3. Unified expressions

For future use, we will give unified expressions for the eigenvalues and eigenfucntions and their one-to-one correspondence to those in the RWA in the following. Setting $m = n-1$ in Sec. II B 1 and $m = n$ in Sec. II B 2, the eigenvalues in Eqs. (19) and (23) can be expanded in terms of g as

$$E_{1n} = n + \frac{1}{2} - g\sqrt{n+1} + \frac{n}{4}g^2 + \frac{1}{32}\frac{(3n+4)\,n}{\sqrt{n+1}}g^3 + \cdots, \tag{26}$$

$$E_{2n} = n + \frac{1}{2} + g\sqrt{n+1} - \frac{(n+2)}{4}g^2 - \frac{1}{32}\frac{(n+2)\,(3n+2)}{\sqrt{n+1}}g^3 + \cdots. \tag{27}$$

The corresponding eigenstates take the forms

$$|1n\rangle \propto \begin{pmatrix} c_n\,|n\rangle \\ c_{n-1}\,|n-1\rangle + c_{n+1}\,|n+1\rangle \end{pmatrix}, \tag{28}$$

$$|2n\rangle \propto \begin{pmatrix} c_n\,|n\rangle + c_{n+2}\,|n+2\rangle \\ c_{n+1}\,|n+1\rangle \end{pmatrix}. \tag{29}$$

It should be pointed out that the GS and the first excited state cannot be brought into the above general expression for $n = 0$. The GS energy and the GS are

$$E_{\rm GS} = -\tfrac{1}{2} - \tfrac{1}{2}g^2 - \tfrac{1}{8}g^4 + \cdots, \tag{30}$$

$$|{\rm GS}\rangle \propto \begin{pmatrix} c_1\,|1\rangle \\ c_0\,|0\rangle + c_2\,|2\rangle \end{pmatrix}, \tag{31}$$

and the energy of the first excited state and the state itself are

$$E_1 = \tfrac{1}{2} - g - \tfrac{1}{2}g^2 + \tfrac{1}{8}g^3 + \cdots, \tag{32}$$

$$|1{\rm EX}\rangle \propto \begin{pmatrix} c_0\,|0\rangle + c_2\,|2\rangle \\ c_1\,|1\rangle \end{pmatrix}. \tag{33}$$

In any case, the ratios of coefficients in the second-order approximation are

$$c_m{:}c_{m+1}{:}c_{m+2} = \left[-\frac{\sqrt{m+1}}{\Omega_m(E)}\right]{:}1{:}\left[-\frac{\sqrt{m+2}}{\Omega_{m+2}(E)}\right], \tag{34}$$

where the sign $\mp$ in Eq. (11) for $\Omega_m(E)$ for any eigenstate is only parity dependent.

It is interesting to note that the first two terms in Eqs. (26) and (27) are no other than the RWA results of Eqs. (3) and (4) at resonance. The additional terms appear just because of the presence of the CRTs. Also, the eigenfunctions in Eqs. (28) and (29) contain the components of the RWA eigenfunctions in Eq. (2). Another bare state which cannot be generated by the rotating-wave terms also emerges.

The analytical results for the energy levels in the second-order approximation are shown in Fig. 1(b) with red dashed lines. It is shown that for $g \leqslant 0.4$, the present second-order approximation can give reasonably good results. Moreover, it should be emphasized that the analytical expressions are almost exact for a remarkable wide coupling regime, $g \leqslant 0.2$. So it could become a solid and concise platform to discuss the effect of CRTs on various physical phenomena in the presently experimentally accessible systems. Note that the present maximum value for the coupling strength in a superconducting qubit coupled to a resonant circuit [7] has reached $g = 0.12$, to our knowledge. An application to vacuum Rabi splitting is performed in the next section as a preliminary example.

IV. VACUUM RABI SPLITTINGS

If we pump the dressed atom from its ground to an excited state, it will decay to the dressed ground state through spontaneous emissions. Under the RWA, when the atom is excited by the operator $V = |e\rangle\langle g| + |g\rangle\langle e|$ from the ground state $|g,0\rangle$, the emission spectrum has two peaks with equal heights. The separation of the two peaks, $2g$, is just the vacuum Rabi splitting [33,34].

Without the RWA, we have two choices for the initial states. When the CRTs are included, the photon in the ground state is no longer in a vacuum state, as shown in Eq. (31). In the framework of the second-order approximation, we first use V to excite the atom from the ground state at resonance,

$$|VRS\rangle^{(1)} = V\,|GS\rangle \propto \begin{pmatrix} c_0\,|0\rangle + c_2\,|2\rangle \\ c_1\,|1\rangle \end{pmatrix},$$

which can be expanded in terms of the eigenstates with odd parity. Note that only the following four excited states are included:

$$|VRS\rangle^{(1)} = v_1\,|1EX\rangle + v_2\,|2EX\rangle + v_5\,|5EX\rangle + v_6\,|6EX\rangle\,. \tag{35}$$

The probabilities $h_i = |v_i|^2$ can be expressed in g as

$$\begin{aligned} h_1 &= \frac{1}{2} + \frac{1}{4}g + \frac{11}{32}g^3 - \frac{1}{16}g^4 + \cdots, \\ h_2 &= \frac{1}{2} - \frac{1}{4}g - \frac{11}{32}g^3 - \frac{1}{16}g^4 + \cdots, \\ h_3 &= \frac{3}{64}g^6 + \cdots, \\ h_4 &= \frac{1}{16}g^4 + \frac{\sqrt{3}}{48}g^5 + \cdots. \end{aligned}$$

The atom will decay from the initial state to the dressed ground state via an emission spectrum. The heights of the peaks in the spectrum are proportional to the squares of the probabilities of the corresponding eigenstates. Therefore from the above four probabilities one may find two main peaks, and the other two peaks are too small to be visible.

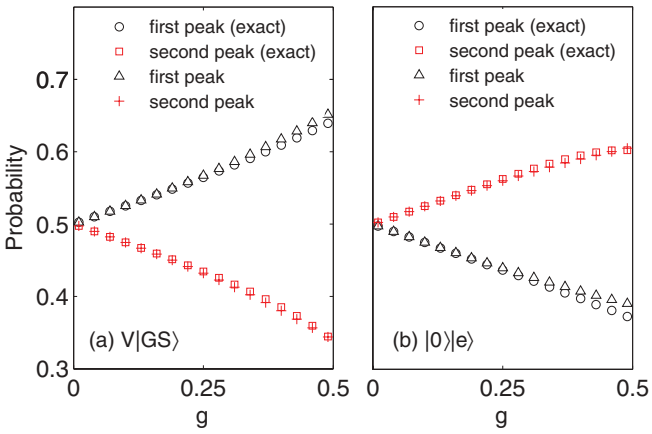

FIG. 2. (Color online) The peak heights in emission spectrum as a function of coupling constant at resonance for the initial states (a) $V|GS\rangle'$ and (b) $|e,0\rangle$. The present analytical results for the first and the second peaks are marked by open triangles and pluses, the exact results by open circles and squares.

In Fig. 2(a), we plot the peak heights from the first two excited states h_1 (open triangles) and h_2 (pluses). The exact numerical results for these heights are also given with open circles and open squares, respectively. It is interesting to note that the present analytical results for the main peak height agree excellently with the exact ones in a wide coupling regime ($0 < g < 0.2$).

With increase of the coupling strength, the third peak from the sixth excited state (h_4) becomes visible. Recently a full numerically exact study [35] showed three peaks (not four peaks) at $g = 0.8$ in Fig. 2(b) of that paper. Our analytical results are qualitatively consistent with this exact study. If the third peak is visible, the coupling constant should exceed 0.5; the present second-order approximation can describe it only qualitatively.

The other initial state is the usual one $|e,0\rangle$ which includes only the first and second excited states in the framework of the second-order approximation:

$$|VRS\rangle^{(2)} = |e,0\rangle = v_1\,|1EX\rangle + v_2\,|2EX\rangle\,. \tag{36}$$

We can derive two peaks up to $o(g^6)$:

$$\begin{aligned} h_1 &= \tfrac{1}{2} - \tfrac{1}{4}g + \tfrac{5}{32}g^3 - \tfrac{1}{16}g^4 + \cdots, \\ h_2 &= \tfrac{1}{2} + \tfrac{1}{4}g - \tfrac{5}{32}g^3 - \tfrac{1}{16}g^4 + \cdots. \end{aligned}$$

Those peak heights are also listed in Fig. 2(b) with open triangles and pluses, respectively. It is also shown that the analytical results in this case are perfectly consistent with the exact ones in a wide coupling regime ($0 < g < 0.2$).

In both initial states, the level difference for the first two excited states will give the vacuum Rabi splitting:

$$E_{2EX} - E_{1EX} = 2g - \tfrac{1}{4}g^3 + O(g^4),$$

which is smaller than the RWA value by a small amount, $\frac{1}{4}g^3$.

For the recently experimentally accessible ultrastrong-coupling constant $g = 0.1$, the effect of CRTs on the vacuum Rabi splitting only results only in a tiny shift around $0.0025(\omega)$,

which is too small to be distinguished in the experimental data. The ratios of the two heights h_1/h_2 for the first and the second initial states can be evaluated as 1.107 and 0.905, respectively, which are however large enough to be seen experimentally.

Finally, we would like to give some remarks. As shown in Fig. 1, in a wide coupling regime ($0 < g < 0.2$), the difference between the RWA energy and the present second-order approximate energy is very subtle, but the states are essentially different. Some bare states in the latter are absent in the former. This is also the reason that the difference in the vacuum Rabi splitting is invisible, but is evident in the peak heights. In the JC system, the components in the eigenstates are very important and play the dominant role in many physical processes.

V. SUMMARY

In this paper, the JC model without the RWA is mapped to a polynomial equation with a single variable, the eigenvalue, via the bosonic Fock space and parity symmetry. The solutions to this polynomial equation recover exactly all eigenvalues and eigenfunctions of the model for all coupling strengths and detunings. Furthermore, the analytical results are presented in different stages. The first approximation in the present formalism reproduces exactly the RWA results. The effect of the CRTs emerges clearly just in the second-order approximation. All eigenvalues and eigenfunctions are derived analytically. It is shown that they play a dominant role up to the remarkable coupling strength of $g = 0.2$, suggesting that they could be convincingly applied to recent circuit quantum electrodynamic systems operating in the ultrastrong-coupling regime up to $g = 0.12$. Application of the analytical results to the vacuum Rabi splitting is performed. The different heights of the two main peaks are given explicitly and agree well with the exact ones in a wide coupling regime. The concise analytical results including only three bare states will be very useful for the exploration of the effects of CRTs on various phenomena in the ultrastrong-coupling regime.

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APPENDIX: SOLUTIONS TO THE UNIVARIATE CUBIC EQUATION

The univariate cubic equation can be generally expressed as

$$x^3 + bx^2 + cx + d = 0.$$

Its solutions can be found in any mathematics manual. If

$$\Gamma = B^2 - 4AC < 0$$

with

$$A = b^2 - 3c, \quad B = bc - 9d, \quad C = c^2 - 3bd,$$

there are three different real roots,

$$y_1 = \frac{-b - 2\sqrt{A}\cos\theta}{3}, \tag{A1}$$

$$y_2 = \frac{-b + \sqrt{A}[\cos\theta - \sqrt{3}\sin\theta]}{3}, \tag{A2}$$

$$y_3 = \frac{-b + \sqrt{A}[\cos\theta + \sqrt{3}\sin\theta]}{3}, \tag{A3}$$

where

$$\theta = \frac{1}{3}\arccos\left(\frac{2Ab - 3B}{2\sqrt{A^3}}\right). \tag{A4}$$

[1] J. M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. **73**, 565 (2001); H. Mabuchi and A. C. Doherty, Science **298**, 1372 (2002).
[2] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).
[3] A. Wallraff *et al.*, Nature (London) **431**, 162 (2004).
[4] I. Chiorescu *et al.*, Nature (London) **431**, 159 (2004); J. Johansson, S. Saito, T. Meno, H. Nakano, M. Ueda, K. Semba, and H.Takayanagi, Phys. Rev. Lett. **96**, 127006 (2006).
[5] H. Wang *et al.*, Phys. Rev. Lett. **101**, 240401 (2008); M. Hofheinz *et al.*, Nature (London) **459**, 546 (2009).
[6] F. Deppe *et al.*, Nat. Phys. **4**, 686 (2008).
[7] T. Niemczyk *et al.*, Nat. Phys. **6**, 772 (2010).
[8] P. Forn-Díaz, J. Lisenfeld, D. Marcos, J. J. Garcia-Ripoll, E. Solano, C. J. P. M. Harmans, and J. E. Mooij, Phys. Rev. Lett. **105**, 237001 (2010).
[9] A. Fedorov, A. K. Feofanov, P. Macha, P. Forn-Díaz, C. J. P. M. Harmans, and J. E. Mooij, Phys. Rev. Lett. **105**, 060503 (2010).
[10] J. Fink *et al.*, Nature (London) **454**, 315 (2008).
[11] T. Werlang, A. V. Dodonov, E. I. Duzzioni, and C. J. Villas-Boas, Phys. Rev. A **78**, 053805 (2008).
[12] D. Zueco, G. M. Reuther, S. Kohler, and P. Hänggi, Phys. Rev. A **80**, 033846 (2009).
[13] S. Ashhab and F. Nori, Phys. Rev. A **81**, 042311 (2010).
[14] J. Hausinger and M. Grifoni, Phys. Rev. A **82**, 062320 (2010).
[15] Q. H. Chen, L. Li, T. Liu, and K. L. Wang, Chin. Phys. Lett. **29**, 014208 (2012).
[16] S. Swain, J. Phys. A **6**, 1919 (1973).
[17] M. Kus, J. Math. Phys. **26**, 2792 (1985); M. Kus and M. Lewenstein, J. Phys. A **19**, 305 (1986).
[18] C. Durstt, E. Sigmundt, P. Reineke, and A. Scheuing, J. Phys. C **19**, 2701 (1986).
[19] E. A. Tur, Opt. Spectrosc. **89**, 574 (2000).
[20] R. F. Bishop, N. J. Davidson, R. M. Quick, and D. M. van der Walt, Phys. Rev. A **54**, 4657R (1996); R. F. Bishop and C. Emary, J. Phys. A **34**, 5635 (2001).
[21] I. D. Feranchuk, L. I. Komarov, and A. P. Ulyanenkov, J. Phys. A **29**, 4035 (1996).

[22] T. Liu, K. L. Wang, and M. Feng, J. Phys. B **40**, 1967 (2007).
[23] E. K. Irish, Phys. Rev. Lett. **99**, 173601 (2007).
[24] Q. H. Chen, Y. Y. Zhang, T. Liu, and K. L. Wang, Phys. Rev. A **78**, 051801(R) (2008).
[25] T. Liu, K. L. Wang, and M. Feng, Europhys. Lett. **86**, 54003 (2009).
[26] Q. H. Chen, Y. Yang, T. Liu, and K. L. Wang, Phys. Rev. A **82**, 052306 (2010).
[27] F. Pan, X. Guan, Y. Wang, and J. P. Draayer, J. Phys. B **43**, 175501 (2010).
[28] J. Casanova, G. Romero, I. Lizuain, J. J. García-Ripoll, and E. Solano, Phys. Rev. Lett. **105**, 263603 (2010).
[29] Q. H. Chen, T. Liu, Y. Y. Zhang, and K. L. Wang, Europhys. Lett. **96**, 14003 (2011).
[30] D. Braak, Phys. Rev. Lett. **107**, 100401 (2011).
[31] J. Larson, Phys. Rev. Lett. **108**, 033601 (2012).
[32] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997); M. Orszag, *Quantum Optics Including Noise Reduction, Trapped Ions, Quantum Trajectories, and Decoherence* (Science Publishers, London, 2007).
[33] J. J. Sanchez-Mondragon, N. B. Narozhny, and J. H. Eberly, Phys. Rev. Lett. **51**, 550 (1983); G. S. Agarwal, *ibid*. **53**, 1732 (1984).
[34] R. J. Thompson, G. Rempe, and H. J. Kimble, Phys. Rev. Lett. **68**, 1132 (1992); A. Boca, R. Miller, K. M. Birnbaum, A. D. Boozer, J. McKeever, and H. J. Kimble, *ibid*. **93**, 233603 (2004); A. Fedorov, A. K. Feofanov, P. Macha, P. Forn-Díaz, C. J. P. M. Harmans, and J. E. Mooij, *ibid*. **105**, 060503 (2010).
[35] Y. Y. Zhang, Q. H. Chen, and S. Y. Zhu, arXiv:1106.2191v1 (to be published).