Quantum metrology with entangled spin-coherent states of two modes

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Recently, Gerry *et al.* [Phys. Rev. A **79**, 022111 (2009)] studied the violation of the Bell-Clauser-Horne-Shimony-Holt inequality for two-spin systems, prepared in an entanglement of spin-coherent states, the so-called entangled spin-coherent states (ESCSs), and found maximal violations for a large class of states. In this paper, using the Holstein-Primakoff realization (HPR) of angular momentum algebra, we present an improved phase estimation scheme employing ESCSs and demonstrate that increasing the spin number gives the smallest variance in the phase parameter in comparison to NOON states under perfect and lossy conditions. The phase sensitivity of this interferometric scheme with parity detection on one of the output states is discussed.

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One of the most exciting recent applications of quantum-information theory has been to the field of metrology. There is a great deal of work on optimal phase estimation addressing the practical problems of state generation, loss, and decoherence [1–6]. Caves [1] showed that quantum-mechanical systems can in principle produce greater sensitivity over classical methods, and many quantum parameter estimation protocols have been proposed since then [7]. Correlated quantum states can be used to achieve a resolution in metrology that surpasses the precision limits achievable with uncorrelated probes, while producing a significant result of both fundamental and practical relevance as first put forward by Caves [1]. The potential usefulness of entangled states in overcoming the shot-noise limit in precision spectroscopy was proposed in Ref. [8], and the first experimental results concerning precision measurements using entangled input states have been presented recently [9]. Given a quantum state, the ultimate limit on the attainable precision is provided by the quantum Cramér-Rao bound (QCRB) via the quantum Fisher information (QFI) [10], an abstract quantity that measures the maximum information about a parameter ϕ that can be extracted from a given measurement procedure. Early theoretical efforts in quantum metrology centered around designing quantum states that saturate this bound. Since the mathematical treatment of the lower bound in physical problems has been clarified, the best resource for phase estimation has been discussed [5,11].

The way in which a state of N photons is prepared is closely related to the uncertainty of the parameter estimation: if prepared in a disentangled state then the phase estimate scales as $1/\sqrt{N}$ [12], which is usually referred to as a standard quantum limit or shot-noise limit. However, this limit can be surpassed by exploiting signature quantum properties such as entanglement, as demonstrated in recent experiments [13,14]. In idealized cases, the minimal uncertainty achievable scales with the Heisenberg limit $1/N$, an enhancement of a factor of $1/\sqrt{N}$ [15], which can be achieved by making use of NOON states [16]. Achieving a sub-shot-noise limit or the Heisenberg limit depends on the nature of the input states and the detection strategy of the output measurement.

The Heisenberg limit is to be the ultimate precision in optical phase estimation; however, it is yet an unsolved problem if this limit can be reached in the presence of noise. Many efforts have been made to improve the robustness against particle loss because the resultant mixed state loses phase information rapidly. Recent implementations have provided the potential advantages of nonlinearities and the importance of the query complexity for quantum metrology [17], where the appropriate resource count in different states is needed for the same phase operation [18]. In this paper, we illustrate the attractive feature of spin-coherent states of a single-mode field for the phase estimation problem in the absence and presence of photon losses. We show, surprisingly, that at an increasing spin number, these states lead possibly to a lower precision than the NOON state, but they are more robust with respect to loss rate. Here, in loss regime, both arms of the interferometer are subject to photon losses which can be modeled by fictitious beam splitters (BSs) inserted at arbitrary locations in both channels. Furthermore, we discuss the phase sensitivity of the interferometric scheme with a realistic measurement approach on one mode of the output state.

In the realm of optics, there has recently been much interest in the use of two-mode maximally entangled number states, sometimes called NOON states, given by

$$|\Psi_N\rangle_{ab} = \frac{1}{\sqrt{2}}\left(|N\rangle_a|0\rangle_b + |0\rangle_a|N\rangle_b\right). \tag{1}$$

These maximal entangled states are very useful in quantum metrology with a small number of photons. However, it remains a challenge to obtain a practical high NOON state in linear (or even nonlinear) optics. Even if high NOON states become achievable, a critical consideration is that these states are extremely fragile to particle loss because the resultant mixed state loses phase information rapidly.

The phase optimization is related to QFI by the QCRB for the output states as

$$\delta\phi \geqslant \frac{1}{\sqrt{F_Q}}. \tag{2}$$

The QFI has been employed in many physical applications [19] and is defined as

$$F_Q = \mathrm{Tr}[\rho(\phi)L^2], \tag{3}$$

where $\rho(\phi)$ is the density matrix of the system, ϕ is the parameter to be measured, and L is the quantum score (symmetric logarithmic derivative) which is defined by

$$\frac{\partial\rho(\phi)}{\partial\phi} = \frac{1}{2}\left[L\rho(\phi) + \rho(\phi)L\right]. \tag{4}$$

We focus on input states as the ESCS introduced recently by Gerry *et al.* [20] in the context of the optical fields using HPR. The spin-coherent states given in terms of a set of single-mode Bose annihilation and creation operators are associated with the HPR form of the spin Lie algebra. This HPR is given by the operators

$$\hat{J}_+ = \hat{a}^\dagger\sqrt{(2j-\hat{n})},\quad \hat{J}_- = \sqrt{(2j-\hat{n})}\hat{a},\quad \hat{J}_z = \hat{n} - j, \tag{5}$$

satisfying the commutation relations

$$[\hat{J}_+,\hat{J}_-] = \hat{J}_z,\quad [\hat{J}_\pm,\hat{J}_z] = \mp\hat{J}_\pm, \tag{6}$$

where it is further assumed that $\hat{a}$ and $\hat{a}^\dagger$ satisfy the Bose algebra $\left[\hat{a},\hat{a}^\dagger\right] = \mathbb{1}$. The number j is, of course, the total angular momentum quantum number, and the angular momentum states $|j,m\rangle$ are related to the Bose number states $|n\rangle$ according to $|j,m\rangle \sim |n\rangle$, $n = j+m$. Note that only number states for $n = 0,1,\ldots,2j$ participate in forming representations for the spin algebra for a fixed j. The Hilbert space of the Bose operator is truncated in this sense.

Now, we aim to find the input state that allows performing phase estimation with the best precision possible, i.e., yielding the highest value of the QFI. In particular, we consider the pure two-mode input state, superposition of macroscopic spin-coherent states which can be understood as a superposition of NOON states, given by

$$\begin{aligned}\left|\Psi_E^{\mathrm{int}}\right\rangle_{ab} &= \frac{\mathcal{N}_{\xi,j}}{(1+|\xi|^2)^j}\sum_{n=0}^{\infty}\frac{\xi^n}{n!}[(\hat{J}_+^a)^n + (\hat{J}_+^b)^n]|0\rangle_a|0\rangle_b\\ &= \mathcal{N}_{\xi,j}[|\xi,j\rangle_a|0\rangle_b + |0\rangle_a|\xi,j\rangle_b],\end{aligned} \tag{7}$$

where the normalization factor $\mathcal{N}_{\xi,j}$ is given by

$$\mathcal{N}_{\xi,j} = \left[2 + \frac{2}{(1+|\xi|^2)^{2j}}\right]^{-\frac{1}{2}}. \tag{8}$$

The spin-coherent state in mode i can be rewritten in terms of the number states per the HPR as

$$|\xi,j\rangle_i = \frac{1}{\sqrt{{}_1F_0^{(-)}(2j,|\xi|^2)}}\sum_{n=0}^{2j}\left[\frac{(2j)_{-n}}{n!}\right]^{\frac{1}{2}}\xi^n|n\rangle_i, \tag{9}$$

where $(A)_{-n}$ is a negative Pochammer symbol: $(A)_{-n} = A(A-1)\cdots(A-n+1)$; $(A)_0 = 1$, and ${}_1F_0^{(-)}(A,x) = \sum_{n=0}^{A}(A)_{-n}x^n/n!$ is the "negative" hypergeometric function. The parameter $\xi = e^{-i\varphi}\tan(\theta/2)$ ranges over the entire complex plain; $0 \leqslant |\xi| < \infty$. These spin-coherent states have been considered by Markham and Vedral [21] for generating entanglement from a beam splitter and are essentially those built upon the HPR of the angular momentum operators given in terms of Bose operators for a single-mode field. The corresponding spin-coherent states of the form originally considered by Radcliff turn out to be just the single-mode binomial states [22]. The number of Fock states included in the superposition is finite but tends to infinity in the limit that the spin goes to infinity.

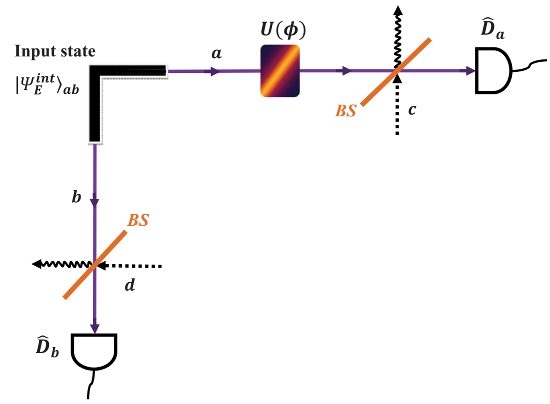

FIG. 1. (Color online) Interferometric phase estimation scheme for the ESCS. Channel a acquires a phase ϕ relative to channel b. After applying a shift phase $U(\phi)$ in mode a, the parity measurement is performed.

Let us consider an interferometer with two arms a and b, as shown in Fig. 1. An initial ESCS, $|\Psi_E^{\mathrm{int}}\rangle$, is prepared in modes a and b and acquires a phase ϕ in the channel b relative to the channel a by a unitary operation, $U(\phi) = \exp(i\phi\hat{b}^\dagger\hat{b})$, where $\hat{b}$ is the creation operator in mode b. The unitary operator applying in the input state leads to the following output state:

$$\begin{aligned}\left|\Psi_E^{\mathrm{out}}\right\rangle_{ab} &= [\mathbb{1}_a \otimes U(\phi)]\left|\Psi_E^{\mathrm{int}}\right\rangle_{ab}\\ &= \mathcal{N}_{\xi,j}[|\xi,j\rangle_a|0\rangle_b + |0\rangle_a|\xi e^{i\phi},j\rangle_b].\end{aligned} \tag{10}$$

The QFI for the pure state $|\Psi_E^{\mathrm{out}}\rangle_{ab}$ is given by

$$F_Q = 4\left[\langle\Phi|\Phi\rangle_{ab} - \left|\langle\Phi|\Psi_E^{\mathrm{out}}\rangle_{ab}\right|^2\right], \tag{11}$$

where $|\Phi\rangle_{ab} = \partial|\Psi_E^{\mathrm{out}}\rangle_{ab}/\partial\phi$.

Let us first consider the situation with no loss of photons, the optimal phase estimation of the pure state is analytically achieved. For the state NOON, we find that $\delta\phi_N \geqslant \frac{1}{N}$, and for the ESCS,

$$\delta\phi_E \geqslant \frac{1+|\xi|^2}{2\sqrt{2j}\mathcal{N}_{\xi,j}|\xi|\{1+2j|\xi|^2[1-(\mathcal{N}_{\xi,j})^2]\}^{\frac{1}{2}}}. \tag{12}$$

To compare the phase uncertainty for the ESCS with NOON, we take into account an equivalent resource case for the states [2]. Considering the same average photon number for mode a given by

$$\overline{n}_N = \overline{n}_E = \frac{N}{2} = (\mathcal{N}_{\xi,j})^2\overline{n}_s, \tag{13}$$

where $\overline{n}_s = 2j|\xi|^2/(1+|\xi|^2) = j(1-\cos\theta)$ is the average photon number of the spin-coherent state.

Then, the phase uncertainty for the ESCS can be compared with respect to N for the state NOON as displaced in Fig. 2. The solid green line is for the state NOON, the dotted blue line ($j=1$), the dashed-dotted red line ($j=3$), and the solid black line ($j=20$) are for the ESCS. Two interesting features appear. The first is that the phase uncertainty provides a different order as functions of N for small numbers of spin j, with $\delta\phi_E$ initially smaller than $\delta\phi_N$ for smaller photon

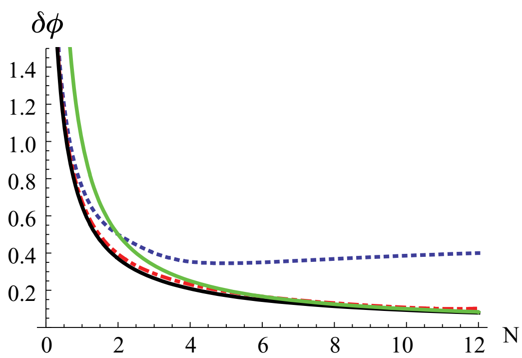

FIG. 2. (Color online) The lower bound on the uncertainty of phase, given by QFI in Eq. (12), as function of average photon number. Blue (dotted line): ESCS for $j=1$; red (dotted-dashed line): ESCS for $j=3$; black (solid line): ESCS for $j=20$; green (solid line): NOON state. Surprisingly, the ESCS is better than the NOON state for a wide range of photon numbers when the spin number becomes significantly larger.

number N; but for the other ranges of significantly larger N, $\delta\phi_E$ becomes larger than $\delta\phi_N$. In addition, the critical values N_c for which $\delta\phi_E = \delta\phi_N$ increase with increasing spin j. The second feature of interest is that when the spin number becomes significantly larger, we can see that $\delta\phi_E \approx \delta\phi_N$ for large values of N, which means that the ESCS becomes approximately equivalent to the NOON state, being dominated by the NOON amplitude at $N = \overline{n}_s$. On the other hand, we find that $\delta\phi_E$ is less than $\delta\phi_N$ for small values of N due to the superposition property of the input state, where $|\Psi_E^{\text{int}}\rangle$ contains $2j+1$ NOON states including N values exceeding $\overline{n}_s$. From these results, the superposition property provides an advantage for the ESCS at large spin values with small $\overline{n}_s$. In this way, the addition of more particles increases the total spin number and makes then the system with better sensitivity of phase estimation. For a detailed example, considering $N=5$ for the NOON state with $\overline{n}_N = 2.50$ and $\overline{n}_s = 5$ with $\overline{n}_E \approx 2.50$ for the ESCS state in the large spin limit case, so the values of the corresponding optimal phase are equal to $\delta\phi_N = 0.20$ and $\delta\phi_E \approx 0.17$. We then demonstrate that increasing the spin number actually improves the phase uncertainty of the ESCS and can beat the Heisenberg limit given by NOON, which could be of significant utility in quantum metrology.

We now discuss the parity measurement, which detects whether the number of photons in a given output mode is even or odd. The measurement is applied in mode b, the uncertainty in the estimation of the phase shift $\Delta\phi$ upon measurement of the parity operator $\hat{\Pi}_b = (-1)^{\hat{n}}$ is given by Ref. [23]

$$(\Delta\phi)^2 = \frac{(\Delta\Pi_b)^2}{(|\partial\langle\hat{\Pi}_b\rangle/\partial\phi|)^2}, \tag{14}$$

where $(\Delta\Pi_b)^2 = \langle\hat{\Pi}_b^2\rangle - \langle\hat{\Pi}_b\rangle^2 = 1 - \langle\hat{\Pi}_b\rangle^2$ since $\hat{\Pi}_b^2 = \mathbb{1}$. For the input state (7), the expectation value of the parity operator is

$$\langle\hat{\Pi}_b\rangle = \frac{2 + (1+|\xi|^2 e^{i\phi})^{2j} + (1+|\xi|^2 e^{-i\phi})^{2j}}{2 + 2(1+|\xi|^2)^{2j}}, \tag{15}$$

from which we readily evaluate the phase uncertainty. This is plotted in the dotted-dashed line of Fig. 3. We can clearly see that the parity measurement on the ESCS does not saturate the optimal phase uncertainty given by the QCRB for this state, but it still beats the Heisenberg limit given by the NOON state.

In the following, we shall determine a lower bound for the uncertainty of the parameter estimation employing ESCSs in the realistic scenario of the photon loss. In other words, we wish to see how the ESCSs resist to photon loss in comparison with the NOON state for different values of spin j in the presence of loss. To this end, we apply two BS transformations characterized by the transmission rate T, considering the scenario of equal losses in both arms of the interferometer, i.e., $T_1 = T_2 = T$, with loss modes c and d located after the phase operation. After the transformations, the obtained output mixed state ρ_{out}^{ab}, given by performing a partial trace over the modes c and d of the BSs, is evaluated for the estimation of phase uncertainty.

When the output state is a mixed state ρ_E^{out}, the QFI is given by

$$F_Q = \sum_{i,j} \frac{2}{\lambda_i + \lambda_j} \left|\langle\lambda_i|\partial\rho_E^{\text{out}}/\partial\phi|\lambda_j\rangle\right|^2, \tag{16}$$

where λ_i and $|\lambda_i\rangle$ are the eigenvalues and eigenvectors of ρ_E^{out}, respectively.

After applying the BSs, the output state is written by $|\Psi\rangle \equiv \hat{U}_{\text{BS}_{ac}}\hat{U}_{\text{BS}_{bd}}|\Psi_E^{\text{out}}\rangle|0\rangle_c|0\rangle_d$. Tracing over modes c and d, the mixed state can be written in four components:

$$\rho_E^{\text{out}} = \frac{(2j)!(\mathcal{N}_{\xi,j})^2}{(1+|\xi|^2)^{2j}} \sum_{nm}^{1} \rho_{nm}, \tag{17}$$

where

$$\begin{aligned}
\rho_{00}^T &= \sum_{p=0}^{2j}\sum_{p'=0}^{2j}\sum_{m=0}^{\min(2j-p,2j-p')} \frac{1}{\sqrt{(2j-m-p)!\,(2j-m-p')!}} \frac{\xi^p\overline{\xi^{p'}}T^{\frac{p}{2}}T^{\frac{p'}{2}}}{\sqrt{p!p'!}} \frac{|\xi|^{2m}(1-T)^m}{m!}[|p\rangle_1|0\rangle_2\,{}_1\langle p'|\,{}_2\langle 0|],\\
\rho_{01}^T &= \sum_{p=0}^{2j}\sum_{p'=0}^{2j} \frac{e^{ip\phi}}{\sqrt{(2j-p)!\,(2j-p')!}} \frac{\xi^p\overline{\xi^{p'}}T^{\frac{p}{2}}T^{\frac{p'}{2}}}{\sqrt{p!p'!}}[|0\rangle_1|p\rangle_2\,{}_1\langle p'|\,{}_2\langle 0|],\\
\rho_{10}^T &= \sum_{p=0}^{2j}\sum_{p'=0}^{2j} \frac{e^{-ip'\phi}}{\sqrt{(2j-p)!\,(2j-p')!}} \frac{\xi^p\overline{\xi^{p'}}T^{\frac{p}{2}}T^{\frac{p'}{2}}}{\sqrt{p!p'!}}[|p\rangle_1|0\rangle_2\,{}_1\langle 0|\,{}_2\langle p'|],\\
\rho_{11}^T &= \sum_{p=0}^{2j}\sum_{p'=0}^{2j}\sum_{m=0}^{\min(2j-p,2j-p')} \frac{e^{i(p-p')\phi}}{\sqrt{(2j-m-p)!\,(2j-m-p')!}} \frac{\xi^p\overline{\xi^{p'}}T^{\frac{p}{2}}T^{\frac{p'}{2}}}{\sqrt{p!p'!}} \frac{|\xi|^{2m}(1-T)^m}{m!}[|0\rangle_1|p\rangle_2\,{}_1\langle 0|\,{}_2\langle p'|].
\end{aligned} \tag{18}$$

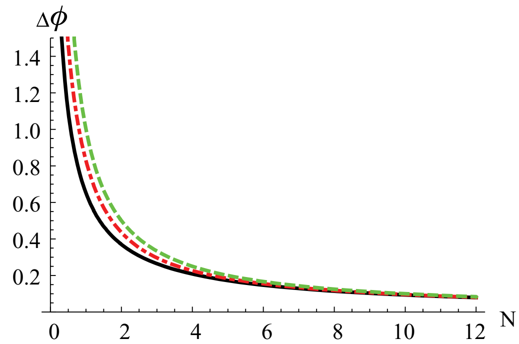

FIG. 3. (Color online) Phase sensitivity with parity detection for ESCS interferometry as function of average photon number. The red dotted-dashed line shows the optimal phase estimation given by Eq. (14) for $j = 20$, while the green dashed line and black solid lines show the optimal phase estimation with QFI for ESCS and NOON, respectively.

Here, the transmission rate parameter in the BSs characterizes the robustness of phase estimation for the input state against the photon loss. To calculate QFI for the mixed state ρ_E^{out}, we need to evaluate the eigenvalues and eigenvectors. Then the QFI can be calculated numerically using Eqs. (16) and (17).

In Fig. 4, the optimal phase estimations for the ESCS and the NOON states are plotted and compared. We find that increasing the spin number actually improves the optimal phase estimation of the ESCS, which would be expected to have strong nonclassical properties including nonlocality [20], over that of NOON for different values of the transmission rate parameter, which could be of significant utility under loss conditions. This effect wins out because the photon losses in both modes do not destroy the superposition effects provided

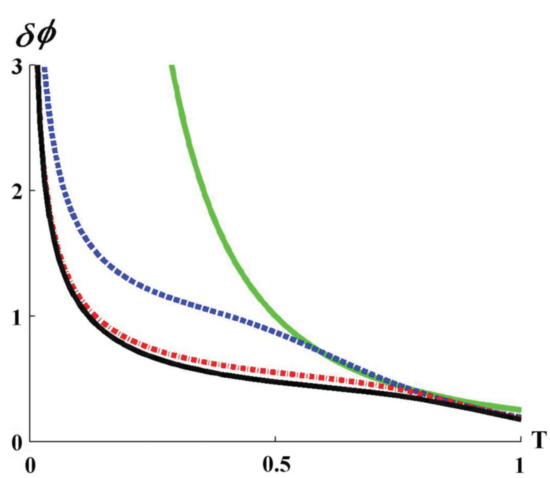

FIG. 4. (Color online) Optimal phase uncertainty achieved with $N = 5$ for equal losses in both arms, i.e., photon losses can be modeled by inserting fictitious beam splitters with transmissivities $T_a = T_b = T$ into both channels of the interferometer which couple the two-mode to an uncorrelated environment. Blue (dashed line): ESCS for $j = 3$; red (dotted-dashed line): ESCS for $j = 10$; black (solid line): ESCS for $j = 20$; green (solid line): NOON state. As the spin number increases, $\delta\phi_N$ is significantly bigger than $\delta\phi_E$ while $\delta\phi_N$ approaches $\delta\phi_E \approx 0.20$ at $T = 1$. The optimal phase estimation for the NOON state is already known (see Refs. [5,6]).

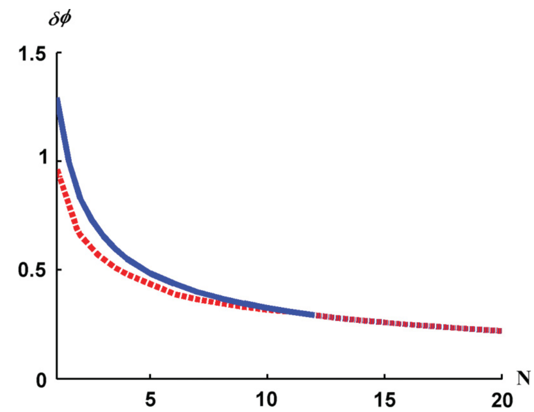

FIG. 5. (Color online) Phase uncertainty for symmetric ($T_a = T_b = 0.6$) losses for ESCSs and optimal N-photon states. Red (dashed line): ESCS for $j = 20$; blue (solid line): optimal N-photon states. As N increases, $\delta\phi_E$ becomes approximately equivalent to $\delta\phi_{\text{optimal}}$.

by spin-coherent states, while for the NOON state a loss of photons renders the state useless for phase estimation. We note that when T is very close to unity, the density matrix is approximately equal to $\rho_E^{\text{out}} \approx \rho_{01}^1 + \rho_{10}^1$, and the minimal phase uncertainty value of the ESCS follows that of the NOON state. Moreover, we find that the ESCS contains phase estimation even in the large loss rate $T \ll 1$. In addition, we find that this state is more robust in the presence of losses than other optimal states for the whole range of transmission rate parameters [5,24].

Finally, we compare the phase uncertainty for the ESCSs to the rigorous asymptotic bound in Ref. [25] in the realistic scenario of particle loss. The results are illustrated in Fig. 5. As we can see, in the limit of large photon numbers N, the ESCSs become approximately equivalent to the optimal N-photon states. However, interestingly, $\delta\phi_{\text{optimal}}$ is significantly bigger than $\delta\phi_E$ for small N. From these results, the ESCSs effectively offer an advantage over the optimal states.

In summary, we have studied the phase uncertainty of the ESCS under perfect and lossy conditions. By analytical and numerical analyses, we have shown that while increasing the spin number, our system provides an improvement in the phase estimation compared to NOON and other states, which could be of significant utility in quantum metrology. We have shown that even the parity detection does not saturate the QCRB of the interferometric scheme, and the ESCS still beats the Heisenberg limit provided by the NOON state. This should offer experimentalists, looking to implement interferometry with ESCS, more options with a better sensitivity. The ESCS states allow us to beat the Heisenberg limit with a fixed and finite mean number of particles. This possibility arises because of the combination of two effects. First, performance estimators, such as QFI, may be a nonlinear function of the photon number for nonclassical states, even if we are dealing exclusively with linear processes. Second, the use of probes in states (ESCS), including coherent superpositions of NOON states with different photon numbers, shifts the number distribution to larger photon numbers. Interestingly, beating the split-resource Heisenberg limit is very appropriate when the number of repetitions is large because the mean number of photons of the probe is small, as is usually the case for quantum states of interest in metrology. These results may provide a new perspective on quantum metrology, possibly replacing previously acclaimed performance limits.

Realistic quantum systems are not closed, therefore it is important to study the robustness of phase uncertainty when the system loses its coherence due to interactions with the environment. An important future investigation will be the study of the effects of finite-temperature Markovian and non-Markovian environments on the dynamics of phase estimation.

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