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Bounds on unambiguous discrimination between quantum states

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In this paper, we present a general upper bound on the success probability for unambiguous discrimination among arbitrary m mixed quantum states with given *a priori* probability. We further analyze how this upper bound can be achievable by presenting a sufficient condition related to it, and we compare this upper bound with three other upper bounds. Moreover, for the issue of the unambiguous identification of m unknown multicopy qudit pure states, we also get a bound.

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I. INTRODUCTION

Quantum state discrimination is an important issue in quantum information science. This problem may be roughly described by the connection between quantum communication and quantum state discrimination in this manner [1–4]. Suppose that a transmitter, Alice, wants to convey classical information to a receiver, Bob, using a quantum channel, and Alice represents the message conveyed as a mixed quantum state that, with given *a priori* probabilities, belongs to a finite set of quantum states, say $\{\rho_1,\rho_2,\ldots,\rho_m\}$; then Bob identifies the state by a measurement.

As is well known, the nonorthogonal states cannot be perfectly discriminated [1,5,6]. However, if the states are linearly independent and a nonzero probability of inconclusive answer is allowed, we can discriminate them. This approach is the so-called *unambiguous discrimination*, first suggested by Ivanovic [7], Dieks [8], and Peres [9] for the discrimination of two equally probable nonorthogonal pure states. Analytical solutions for the optimal failure probabilities have been given for distinguishing between two and three pure states [10–12]. Chefles [13] showed that a set of pure states can be unambiguously discriminated if and only if they are linearly independent. The optimal unambiguous discrimination between linearly independent symmetric and equiprobable pure states was solved in [14]. A semidefinite programming approach to unambiguous discrimination between pure states has been investigated in detail by Eldar [15]. Some upper bounds on the optimal success probability for unambiguous discrimination between pure states have also been presented [16–19].

Recently, the problem of unambiguous discrimination between mixed states has been considered. Rudolph *et al.* [20] derived a lower bound and an upper bound on the maximal probability of successful discrimination of two mixed states. Raynal *et al.* [21] presented two reduction theorems to reduce the optimal unambiguous discrimination of two mixed states to that of two other mixed states which have the same rank. The analytical results for the optimal unambiguous discrimination between two mixed quantum states have been derived in [22]. In the general case of m-mixed-state discrimination, Fiurasek and Jezek [23] and Eldar [24] gave some sufficient and necessary conditions for the optimal unambiguous discrimination, and some numerical methods were discussed. Feng *et al.* [25] derived a general lower bound on the inconclusive probability for distinguishing a mixed-state set with *a priori* probability. Recently, Li [26] also presented a general upper bound on the success probability for unambiguously discriminating among arbitrary m mixed quantum states with given *a priori* probability, which can be directly used for the unambiguous communication setting. For more work regarding unambiguous discrimination, we may refer to [27].

Unambiguous identification of unknown states is a variant of the ordinary unambiguous discrimination problem and is also called unambiguous discrimination of the universal programmable state, which can be described in the following manner [28]: Assume that a probe system of a d-dimensional Hilbert space is prepared in a pure state chosen from a set of m linearly independent states but we do not know the states in the set. Instead, we are given m reference systems, with each being prepared in one of the m unknown pure states. The problem is to determine the optimum measurement for unambiguously identifying the states of the probe system with the state of one of the reference systems. The identification of unknown pure states has been shown to be equivalent to the discrimination of known mixed states [28–30].

The problem of the unambiguous identification of two unknown pure states was introduced by Bergou and Hillery [31], who first solved for optimum unambiguous identification of two unknown qubits. A number of schemes for implementing the unambiguous identification of two unknown qubit states have been proposed [32–34]. The problem of unambiguously identifying two unknown pure states has been extended to the case where the states are each encoded into a certain number of copies in each reference system [29,30,35–38]. Optimum unambiguous identification of two unknown pure qudit states was solved in [35,39]. Optimum unambiguous identification of d unknown pure qudit states was solved by Herzog and Bergou [28]. However, optimum unambiguous identification of m unknown multicopy qudit pure states ($m < d$) has remained unsolved until now.

In this paper, we present a general upper bound on the success probability for unambiguous discrimination among arbitrary m mixed quantum states with given *a priori* probability. Applying the bound of unambiguous discrimination, we also get a bound for the issue of the unambiguous identification of m unknown qudit pure states.

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The remainder of this paper is organized as follows. In Sec. II, we review some basic lemmas and notations. In Sec. III, we derive an upper bound on the success probability for unambiguous discrimination between arbitrary m mixed states, and we present a sufficient condition for which this upper bound can be achievable. In Sec. IV, we present a proposition and construct examples to clarify the relationship between our bound and the bounds presented by other authors. In Sec. V, we investigate the issue of quantum pure-state unambiguous identification, and we also get a bound. Finally, some concluding remarks are made in Sec. VI.

II. PRELIMINARY SETUP

In this section, we first recollect the notation related to unambiguous discrimination. For the unambiguous discrimination of m quantum mixed states ρ_i $(i = 1,2,\ldots,m)$, we need to design a measurement consisting of $m+1$ positive operator-valued measures (POVM), say Π_i, $0 \leqslant i \leqslant m$, satisfying the following conditions:

$$\Pi_0 + \sum_{i=1}^{m} \Pi_i = I, \tag{1}$$

$$\mathrm{Tr}(\Pi_i \rho_j) = \delta_{ij} q_i, \tag{2}$$

where $i,j = 1,2,\ldots,m$; if $i = j$, then $\delta_{ij} = 1$, and otherwise, $\delta_{ij} = 0$, $0 < q_i \leqslant 1$. I denotes the identity operator on H. Condition (2) is also equal to

$$\Pi_i \rho_j = 0 \tag{3}$$

for $i \neq j$, $i,j = 1,2,\ldots,m$.

By means of the measurement Π_i, if the system has been prepared by ρ_i, then $\mathrm{Tr}(\rho_i \Pi_i)$ is the probability of deducing the system being in state ρ_i and $\mathrm{Tr}(\rho_i \Pi_0)$ is the inconclusive probability. Therefore, the average success probability $\mathrm{Pr}[u]$ is as follows:

$$\mathrm{Pr}[u] = \sum_{i=1}^{m} p_i \mathrm{Tr}(\rho_i \Pi_i). \tag{4}$$

One of our main objective is to design an optimum measurement that maximizes the average success probability.

In general, the notations used in this paper will be explained as new symbols appear. Here, we first give some symbols that will be used in the later sections. For any Hilbert subspace H_i, we use $\sum_i H_i$ to denote the Hilbert space that $\{\sum_k |\psi_k\rangle : |\psi_k\rangle \in H_k\}$. For any two linear operators T_1 and T_2 on the same Hilbert space $\mathcal{H}$, we use $T_1 \perp T_2$ to denote that the supports of T_1 and T_2 are orthogonal. The support of a linear operator T is the subspace spanned by all eigenvectors corresponding to all nonzero eigenvalues of T, and we denote the support of T by $\mathrm{Supp}(T)$. We denote the supplementary space of $\mathrm{Supp}(T)$ as $\mathrm{Ker}(T)$. $||\cdot||$ denotes spectral normal, $||\cdot||_{\mathrm{Tr}}$ denotes trace normal, e.g., $||A|| = \max_{|u\rangle}\{||A|u\rangle|| : \langle u|u\rangle = 1\} = \max_i\{S_i(A)\}$, $||A||_{\mathrm{Tr}} = \mathrm{Tr}(\sqrt{A^\dagger A}) = \sum_i S_i(A)$, and $S_i(A)$ denotes a singular value of the operator A.

Next, we recollect some lemmas that are useful in the later sections of this paper.

Lemma 1 [26]. Suppose that $\rho_1, \rho_2, \ldots, \rho_m$ are quantum states, with all their eigenvectors corresponding to nonzero eigenvalues that span d-dimensional complex Hilbert space H, and $r = \sum_{i=1}^{m} \mathrm{rank}(\rho_i)$. If there are POVM $\{\Pi_0, \Pi_1, \ldots, \Pi_m\}$ such that Eqs. (1) and (3) are held, then

$$\mathrm{Tr}(\Pi_0) \geqslant \frac{r-d}{m-1}, \tag{5}$$

and the equality holds when $Y = Y_i$ and $X_i \perp X_j$ $(i \neq j)$, where $Y_i = \mathrm{Supp}(\rho_i) \bigcap \sum_{j\neq i} \mathrm{Supp}(\rho_j)$, where X_i is defined as $X_i \oplus Y_i = \mathrm{Supp}(\rho_i)$ and $Y = \sum_{i=1}^{m} Y_i$;

Lemma 2 [38]. For a pure state $|\psi\rangle$ in d-dimensional Hilbert space H,

$$\int d\mu(\psi)[\psi^{\otimes n}] = \frac{1}{c^{[n]}} P_{\mathrm{sym}}^{[n]}, \tag{6}$$

where $c^{[n]} = C_{n+d-1}^{d-1} = \frac{(n+d-1)!}{n!(d-1)!}$ is the dimension of the fully symmetric space $H^{[n]}$, $\psi = |\psi\rangle\langle\psi|$ and $P_{\mathrm{sym}}^{[n]}$ is the projector onto this space.

Now, we present our main results.

III. AN UPPER BOUND ON UNAMBIGUOUS DISCRIMINATION BETWEEN MIXED STATES

To unambiguously discriminate any m quantum mixed states in d-dimensional Hilbert space, we have results as follows.

Theorem 1. Suppose that $\rho_1, \rho_2, \ldots, \rho_m$ are quantum states, with all their eigenvectors corresponding to nonzero eigenvalues that span d-dimensional complex Hilbert space H, and $r = \sum_{i=1}^{m} \mathrm{rank}(\rho_i)$. Let the *a priori* probabilities of the mixed states $\rho_1, \rho_2, \ldots, \rho_m$ be $p_1, p_2, \ldots, p_m$, respectively; then we have the following results.

(1) The success probability $\mathrm{Pr}_{[u]}$ for unambiguously discriminating ρ_i satisfies $\mathrm{Pr}_{[u]} \leqslant L$, where

$$L = \frac{dm-r}{m-1} \max_i\{||p_i \rho_i||\} \tag{7}$$

and $||\cdot||$ denotes spectral normal.

(2) Equality is reached in the bound $\mathrm{Pr}_{[u]} \leqslant L_0$ if the mixed states $\rho_1, \rho_2, \ldots, \rho_m$ satisfy the following conditions:

(I) $||p_1\rho_1|| = ||p_2\rho_2|| = \cdots = ||p_m\rho_m||$; (II) $X_i \perp X_j$ $(i \neq j)$ and $Y = Y_i$, where $Y_i = \mathrm{Supp}(\rho_i) \bigcap \sum_{j\neq i} \mathrm{Supp}(\rho_j)$, with X_i defined as $X_i \oplus Y_i = \mathrm{Supp}(\rho_i)$ and $Y = \sum_{i=1}^{m} Y_i$; and (III) $S_1(p_i\rho_i P_{X_i}) = S_2(p_i\rho_i P_{X_i}) = \cdots = ||p_i\rho_i||$, where P_{X_i} denotes the projector into the subspace X_i and $S_i(A)$ denotes the ith singular value of operator A.

Proof. (1) For unambiguously discriminating ρ_i, suppose that Π_i $(0 \leqslant i \leqslant m)$ are any positive semidefinite operators satisfying Eqs. (1) and (3). Let $\mathrm{Pr}_{[u]}$ denote the success probability; then we have

$$\mathrm{Pr}_{[u]} = \sum_{i=1}^{m} \mathrm{Tr}(p_i \rho_i \Pi_i). \tag{8}$$

Moreover, we have

$$\mathrm{Tr}(AB) \leqslant ||A||\,||B||_{\mathrm{Tr}}, \tag{9}$$

where A,B denote positive semidefinite operators, $||\cdot||$ denotes spectral normal, and $||\cdot||_{\mathrm{Tr}}$ denotes trace normal. If B is an operator in the subspace spanned by all eigenvectors corresponding to the maximum eigenvalue of A (in other words, if $S_1(AP_B) = S_2(AP_B) = \cdots = ||A||$, where P_B denotes the projector into the subspaces spanned by all eigenvectors corresponding to the nonzero eigenvalue of B), then Eq. (9) holds.

Therefore, we have that

$$\mathrm{Pr}_{[u]} \leqslant \sum_{i=1}^{m} ||p_i\rho_i||\,||\Pi_i||_{\mathrm{Tr}} \tag{10}$$

$$\leqslant \max_i\{||p_i\rho_i||\}\sum_{i=1}^{m}||\Pi_i||_{\mathrm{Tr}}. \tag{11}$$

Because Π_i are positive semidefinite operators, satisfying Eq. (1), we have that

$$\mathrm{Pr}_{[u]} \leqslant \max_i\{||p_i\rho_i||\}\sum_{i=1}^{m}\mathrm{Tr}(\Pi_i) \tag{12}$$

$$= \max_i\{||p_i\rho_i||\}\mathrm{Tr}(I-\Pi_0). \tag{13}$$

Because $\rho_1, \rho_2, \ldots, \rho_m$ are quantum states, with all their eigenvectors corresponding to nonzero eigenvalues that span d-dimensional complex Hilbert space H, and $r = \sum_{i=1}^{m}\mathrm{rank}(\rho_i)$, according to Lemma 1, we have

$$\mathrm{Tr}(\Pi_0) \geqslant \frac{r-d}{m-1}. \tag{14}$$

So we can get that

$$\mathrm{Pr}_{[u]} \leqslant \max_i\{||p_i\rho_i||\}\left(d-\frac{r-d}{m-1}\right) \tag{15}$$

$$= \frac{dm-r}{m-1}\max_i\{||p_i\rho_i||\}. \tag{16}$$

(2) From the above proof and Lemma 1, if $X_i \perp X_j$ $(i \neq j)$ and $Y = Y_i$, we can take $\Pi_0 = P_Y$ and $\Pi_i = P_{X_i}$; then the equality in the bound (15) holds, where P_Y and P_{X_i} denote the projectors into subspace Y and X_i, respectively. If $||p_1\rho_1|| = ||p_2\rho_2|| = \cdots = ||p_m\rho_m||$, the equality in (11) holds. If $S_1(\rho_i P_{X_i}) = S_2(\rho_i P_{X_i}) = \cdots = ||\rho_i||$ and $X_i \perp X_j$ $(i \neq j)$, we take $\Pi_i = P_{X_i}$, and then the equality in (10) holds.

So if the mixed states $\rho_1, \rho_2, \ldots, \rho_m$ satisfy conditions (I)–(III), equality is reached in the bound $\mathrm{Pr}_{[u]} \leqslant L$. We complete the proof. ■

A natural question is what the relationship between the new upper bound L and the other existing bounds is.

IV. COMPARISONS BETWEEN DIFFERENT BOUNDS

Suppose that the *a priori* probabilities of the mixed states $\rho_1, \rho_2, \ldots, \rho_m$ are $p_1, p_2, \ldots, p_m$, respectively. All eigenvectors corresponding to nonzero eigenvalues of these states span d-dimensional complex Hilbert space H, and $r = \sum_{i=1}^{m}\mathrm{rank}(\rho_i)$. To unambiguously discriminate among them, there are some bounds on the success probability.

For the case of $m = 2$, Rudolph *et al.* [20] proved that a lower bound on the optimal inconclusive probability Q_U for unambiguously discriminating ρ_1,ρ_2, with given prior probabilities p_1,p_2, respectively, is expressed as

$$Q_U \geqslant 2\sqrt{p_1p_2}F(\rho_1,\rho_2), \tag{17}$$

where $F(\rho_1,\rho_2) = \mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}$. Let us denote its corresponding success probability upper bound as $L_R = 1 - 2\sqrt{p_1p_2}F(\rho_1,\rho_2)$.

A generalization to the case of m states has been given by Feng *et al.* [25]:

$$Q_U \geqslant \sqrt{\frac{m}{m-1}\sum_{i\neq j}p_ip_jF(\rho_i,\rho_j)^2}. \tag{18}$$

We denote its corresponding success probability upper bound as $L_F = 1 - \sqrt{\frac{m}{m-1}\sum_{i\neq j}p_ip_jF(\rho_i,\rho_j)^2}$. Recently, Li [26] also presented a general upper bound L_0 on the success probability:

$$L_0 = \mathrm{Pr}\left(\{p_i\}, \frac{dm-r}{m-1}\max_i\{||\rho_i||\}\right), \tag{19}$$

where $\mathrm{Pr}(\{p_i\},x)$ denotes the probability of the x most likely states in the set $\{\rho_i : i = 1,\ldots,m\}$ and $||\cdot||$ denotes spectral normal.

As is well known, when $m = 2$, $L_R = L_F$. So we can only consider the relationship between L_F, L_0, and L. We have the following result.

Proposition 1. By comparing the three bounds L_F, L_0, and L, we have the following.

(1) If $p_1 = p_2 = \cdots = p_m$, then $L_0 = L$.

(2) If p_i are not all equal and $||\rho_i|| = \lambda$ $(i = 1,2,\ldots,m)$, then $L_0 < L$.

(3) If p_i are not all equal and $||p_i\rho_i|| = \lambda$ $(i = 1,2,\ldots,m)$, then $L_0 > L$.

(4) If there is no restrict condition, the three upper bounds, L_F, L_0, and L, have no strict bigger or smaller relationship.

Proof. (1) If $p_1 = p_2 = \cdots = p_m$, we can easily get $L_0 = L = \frac{dm-r}{m(m-1)}\max_i\{||\rho_i||\}$.

(2) If $||\rho_i|| = \lambda$ $(i = 1,2,\ldots,m)$, then $L_0 = \mathrm{Pr}(\{p_i\},\frac{dm-r}{m-1}\lambda)$, and $L = \frac{dm-r}{m-1}\lambda\max_i\{p_i\}$. If p_i are not all equal, then $L_0 < L$.

(3) If p_i are not all equal and $||p_i\rho_i|| = \lambda$ $(i = 1,2,\ldots,m)$, then $L_0 = \mathrm{Pr}(\{p_i\},\frac{dm-r}{m-1}\frac{\lambda}{\min\{p_i\}}) > \frac{dm-r}{m-1}\frac{\lambda}{\min\{p_i\}}\min\{p_i\} = \frac{dm-r}{m-1}\lambda = L$.

(4) Actually, we have proved that the upper bounds L_F and L_0 have no strict bigger or smaller relationship in [26].

So, to demonstrate that L_F, L_0, and L do not have a strict bigger or smaller relationship, it is sufficient to construct examples as follows.

Example 1. Let $p_1 = \frac{2}{3}$, $p_2 = \frac{1}{3}$, and $\rho_1 = \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| + \frac{1}{3}|2\rangle\langle 2|$, $\rho_2 = \frac{1}{6}|0\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1| + \frac{2}{3}|3\rangle\langle 3|$. Then, we can calculate explicitly the values of the three upper bounds as follows: $L_F = \frac{5}{9}$, $L_0 = \frac{7}{9}$, and $L = \frac{4}{9}$. So, we have $L < L_F < L_0$.

Example 2. Let $p_1 = \frac{1}{2}$, $p_2 = p_3 = \frac{1}{4}$, and $\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$, $\rho_2 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|2\rangle\langle 2|$, and $\rho_3 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|3\rangle\langle 3|$. Then, we get $L_F = 1 - \frac{\sqrt{15}}{8}$, $L_0 = \frac{5}{8}$, and $L = \frac{3}{4}$. So, we have $L_F < L_0 < L$. ■

Remark 1. In Example 1, the mixed states satisfy the conditions in Theorem 1; when we take $\Pi_0 = |0\rangle\langle 0| + |1\rangle\langle 1|$, $\Pi_1 = |2\rangle\langle 2|$, $\Pi_2 = |3\rangle\langle 3|$, the success probability is 4/9. In other words, the average success probability can achieve the upper bound L.

In some quantum communication cases, we only know the rank and the spectral normal, and we cannot calculate the fidelity between the mixed states to be discriminated. The bounds L_0 and L may be very useful. Now, we consider the issue of unambiguous identification of unknown pure states.

V. AN UPPER BOUND ON UNAMBIGUOUS DISCRIMINATION BETWEEN UNKNOWN MULTICOPY QUDIT PURE STATES

In this section, we assume that the total quantum system consists of a probe system, labeled by the index 0, and m reference systems, labeled by the indices $1,\ldots,m$. The m reference systems are prepared in a set of m linearly independent states $|\psi_1\rangle,\ldots,|\psi_m\rangle$, which are in a d-dimensional Hilbert space ($m \leqslant d$), respectively, but we do not know the states in the set. The probe system is prepared in the states $|\psi_i\rangle$ with probability p_i $(i = 1,\ldots,m)$.

Let us now assume that the state of the probe qudit coincides with the state of the kth reference qudit, so that the state of the total system can be described as

$$|\Phi_k\rangle = |\psi_k\rangle|\psi_1\rangle|\psi_2\rangle\ldots|\psi_k\rangle\ldots|\psi_m\rangle.$$

For the issue of unambiguous identification of unknown pure states, we have the following result.

Theorem 2. To identify the linearly independent unknown states $|\psi_i\rangle$ with probability p_i $(i = 1,\ldots,m)$ in a d-dimensional Hilbert space ($m \leqslant d$), the success probability Pr_s^{Id} satisfies the following inequality:

$$\mathrm{Pr}_s^{Id} \leqslant \mathrm{Pr}\left(\{p_i\},\frac{m(d-1)}{(m-1)(d+1)}\right). \quad (20)$$

In more general case, let n_i $(i = 0,\ldots,m)$ denote the number copies of the ith system and let $\mu_k = c^{[n_0+n_k]}\prod_{i=1,i\neq k}^{m} c^{[n_i]}$, $c^{[n]} = \frac{(n+d-1)!}{n!(d-1)!}$, and $\theta = \frac{md^{\sum_{i=0}^{m} n_i} - \sum_{k=1}^{m}\mu_k}{m-1}$; then we have

$$\mathrm{Pr}_s^{Id} \leqslant \min\{\mathrm{Pr}(\{p_k\},\,\theta\max_k\{\mu_k\}),\theta\max_k\{p_k\mu_k\}\}.$$

Proof. The total quantum system we are considering consists of one probe qudit, labeled by the index 0, and m reference qudits, labeled by the indices $1,\ldots,m$. Let the states $|i\rangle_k$ denote orthonormal basis vectors for the kth qudit. The identity operator in the d-dimensional Hilbert space of the kth qudit is given by

$$I_k = \sum_{i=0}^{d-1}|i\rangle_k\langle i|_k \ (k = 0,1,\ldots,m). \quad (21)$$

Since the m states $|\psi_k\rangle$ $(k = 1,\ldots,m)$ are unknown states in a d-dimensional Hilbert space H, they can be changed from preparation to preparation. We introduce the density operator $\Phi_k = |\Phi_k\rangle\langle\Phi_k|$ and take its average over the m unknown reference states.

$$\sigma_k = \int \Phi_k\, d\mu(\psi_1)d\mu(\psi_2)\cdots d\mu(\psi_m). \quad (22)$$

According to Lemma 2, we have

$$\sigma_k = \frac{2}{(d+1)d^m}P_{\mathrm{sym}}^{[2]}\bigotimes_{i=1,i\neq k}^{m} I_i,\ (k = 1,\ldots,m), \quad (23)$$

with probability p_k, where $P_{\mathrm{sym}}^{[2]}$ is defined as

$$P_{\mathrm{sym}}^{[2]} = \sum_{i=0}^{d-1}|i\rangle_0|i\rangle_k\langle i|_0\rangle i|_k + \sum_{j=1}^{d-1}\sum_{i=0}^{j-1}\frac{|i\rangle_0|j\rangle_k + |j\rangle_0|i\rangle_k}{\sqrt{2}}\frac{\langle i|_0\langle j|_k + \langle j|_0\langle i|_k}{\sqrt{2}}. \quad (24)$$

We can easily calculate that the rank of $P_{\mathrm{sym}}^{[2]}$ is $\frac{d(d+1)}{2}$ and $||\sigma_k|| = \frac{2}{(d+1)d^m}$. The rank of σ_k is $\frac{(d+1)d^m}{2}$. From Proposition 1, we have $L_0 < L$. So we can get

$$\mathrm{Pr}_s^{Id} \leqslant \mathrm{Pr}\left(\{p_i\},\frac{d'm-r}{m-1}\max_i\{||\rho_i||\}\right) \quad (25)$$

$$= \mathrm{Pr}\left(\{p_i\},\frac{d^{m+1}m - m\frac{(d+1)d^m}{2}}{m-1}\frac{2}{(d+1)d^m}\right)$$

$$= \mathrm{Pr}\left(\{p_i\},\frac{m(d-1)}{(m-1)(d+1)}\right). \quad (26)$$

In a more general case, the multiple copies of probe and reference systems are used in the input states. Let n_0 denote the number of copies of probe states and n_i $(i = 1,\ldots,m)$ denote the number of copies of the ith reference states. Assuming that the state of the probe qudit coincides with the state of the kth reference qudit, the state of the total system can be described as

$$|\Psi_k\rangle = |\psi_k\rangle^{\otimes n_0}|\psi_1\rangle^{\otimes n_1}|\psi_2\rangle^{\otimes n_2}\cdots|\psi_k\rangle^{\otimes n_k}\cdots|\psi_m\rangle^{\otimes n_m}.$$

We introduce the density operator $\Psi_k = |\Psi_k\rangle\langle\Psi_k|$ and take its average over the m unknown reference states,

$$\rho_k = \int \Psi_k\, d\mu(\psi_1)d\mu(\psi_2)\cdots d\mu(\psi_m). \quad (27)$$

From Lemma 2, we can obtain

$$\rho_k = \frac{1}{\mu_k}P_{\mathrm{sym}}^{[n_0+n_k]}\otimes P_{\mathrm{sym}}^{[n_1]}\otimes\cdots\otimes P_{\mathrm{sym}}^{[n_m]}, \quad (28)$$

where $\mu_k = c^{[n_0+n_k]}\prod_{i=1,i\neq k}^{m} c^{[n_i]}$, and $c^{[n]} = \frac{(n+d-1)!}{n!(d-1)!}$. Let

$$\theta = \frac{md^{\sum_{i=0}^{m} n_i} - \sum_{k=1}^{m}\mu_k}{m-1}. \quad (29)$$

According to Proposition 1, we have

$$\mathrm{Pr}_s^{Id} \leqslant \min\{\mathrm{Pr}(\{p_k\},\,\theta\max_k\{\mu_k\}),\theta\max_k\{p_k\mu_k\}\}.$$

We complete the proof. ■

VI. CONCLUDING REMARKS

In this paper, we first presented an upper bound L on the success probability of unambiguously discriminating between mixed states. Furthermore, we analyzed how this upper bound can be achieved by presenting a sufficient condition and presented a proposition to clarify the relationships between different bounds on the success probability of unambiguous discrimination. Then, we investigated the issue of unambiguous identification of unknown pure states. We derived an upper bound on the success probability of unambiguous identification of m unknown multicopy qudit pure states with a given *a priori* probability.

A further problem is whether there is a new upper bound for unambiguous discrimination of mixed states to improve these existing bounds. Optimum unambiguous identification of m unknown multicopy qudit pure states also is an interesting problem.

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