Nonlocal multipartite correlations from local marginal probabilities

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(Received 21 June 2012; published 20 September 2012)

Understanding what can be inferred about a multiparticle quantum system given only the knowledge of its subparts is a highly nontrivial task. Clearly, if a global system does not contain an information resource of some kind, neither do its subparts. For the case of entanglement as an information resource, it is known that the converse of this last statement is not true: Some nonentangled reduced states are only compatible with global states which are entangled. We extend this result to correlations and provide local marginal correlations that are only compatible with global genuinely tripartite nonlocal correlations. Quantum nonlocality can thus be deduced from the mere observation of local marginal correlations.

DOI: 10.1103/PhysRevA.86.032117 PACS number(s): 03.65.Ud

I. INTRODUCTION

In contrast to classical systems, multipartite quantum systems can be entangled and exhibit nonlocal correlations. Beyond their fundamental interest, both properties are resources for quantum information theory [1,2]. It is thus a relevant question to understand the types of quantum states and correlations that are possible in composite quantum systems.

In a multipartite system, every subset of parties constitutes a proper system in itself. The fact that these subsystems describe parts of the same total system requires them to satisfy some *compatibility conditions*. For instance, a bipartite quantum state ϱ_{AB} is compatible with a tripartite state ϱ_{ABC} if and only if $\varrho_{AB} = \mathrm{tr}_C(\varrho_{ABC})$. While it is straightforward to check whether some reduced states are compatible with a given global state, the question becomes much subtler when the global state is unknown and one is interested in knowing whether there exists a quantum state compatible with the given marginals. Finding the conditions for compatibility among reduced quantum states is known as the *quantum marginal problem* [3–6]. It is the quantum counterpart of the classical marginal problem, which is concerned with the compatibility of marginal probability distributions.

The quantum marginal problem is trivial in the bipartite case: Two reduced states, ϱ_A and ϱ_B, are always compatible with the product bipartite quantum state $\varrho_{AB} = \varrho_A \otimes \varrho_B$. However, the situation becomes more interesting when more than two parties are involved. For instance, it is well known that if two parties share a maximally entangled state, then any tripartite quantum state compatible with it must be such that the third party is uncorrelated to the first two. This phenomenon is known as the monogamy of entanglement [7,8] and implies that a maximally entangled state $|\phi^+\rangle_{AB}$ is incompatible with any correlated state ρ_{AC} or ρ_{BC}. A similar property, known as the monogamy of nonlocality, is displayed by nonlocal correlations [2]. Parts of a system can thus constrain the set of possible full systems in ways that show up in other parts of the same system.

In this work we are interested in the question of what can be inferred about the correlations of a global state given only the knowledge of some of its subparts. It is clear that if subparts of a system display entanglement or nonlocality so does the global system. However, is the converse also true? For the case of entanglement it is known that the answer to this question is negative: There are separable states of two qubits that are only compatible with entangled multipartite states [9,10]. To show this, the authors of Refs. [9,10] used spin-squeezing inequalities to detect entanglement and found entangled multiqubit states whose reduced two-qubit states are separable. As the entanglement criteria they used only rely on two-body correlations, this demonstrates the existence of nonentangled reduced states that are only compatible with entangled global states.

Here we pose a similar question in the context of no-signaling correlations, where one deals with the raw correlations of classical inputs and outputs described by a joint conditional probability distribution. Therefore, one does not assume the whole Hilbert space formalism of quantum mechanics, but just the validity of the no-signaling principle. Our goal then is to see whether there are local marginal correlations that are only compatible with multiparite nonlocal correlations. We show that this is indeed the case and that, similarly to what happens with entanglement, nonlocality of multipartite correlations can be certified from marginal correlations that admit a local description. We further provide a quantum state and corresponding measurements that exhibit this type of correlations. In this case we also demonstrate that the nonlocality present in the full correlations can be genuinely multipartite [11,12]. Concerning the question of certifying entanglement from separable marginals, we further provide new examples of separable reduced states that are only compatible with an entangled global state. Our findings show how the compatibility conditions lead to nontrivial results even when acting on *a priori* useless marginals: It is possible to witness the presence of useful correlations in the global system from useless reduced states.

II. NONLOCALITY FROM LOCAL MARGINALS

Quantum nonlocality represents a quantum property inequivalent to entanglement. In the paradigm of

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device-independent quantum information processing, nonlocality has been identified as an alternative resource for quantum information protocols, necessary, for instance, for secure key distribution [13] or randomness generation [14]. The corresponding scenario consists of different distant observers that can input a classical setting x_i into this part of the system and obtain an output a_i. The correlations of the inputs and outputs are encapsulated in the joint conditional probability distribution $P(a_1,\dots,a_N|x_1,\dots,x_N)$ that denotes the probability of obtaining the outputs $a_1,\dots,a_N$ when inputs $x_1,\dots,x_N$ are used.

In what follows we consider a tripartite scenario where each party can choose from two different inputs, denoted by 0 and 1, and obtain two different outputs, denoted by -1 and $+1$, that is, $x,y,z\in\{0,1\}$ and $a,b,c\in\{-1,1\}$. It is useful to consider the following parametrization of the probabilities

$$P(abc|xyz)=\tfrac{1}{8}[1+a\langle A_x\rangle+b\langle B_y\rangle+c\langle C_z\rangle+ab\langle A_xB_y\rangle +ac\langle A_xC_z\rangle+bc\langle B_yC_z\rangle+abc\langle A_xB_yC_z\rangle], \tag{1}$$

where $\langle A_x\rangle=P(a=1|x)-P(a=-1|x)$ is the expectation value for the outcome of the first party A given input x, $\langle A_xB_y\rangle=P(ab=1|xy)-P(ab=-1|xy)$ is the expectation value for the product of the outcomes of A and B given the inputs x and y, and so on.

Given the fact that entanglement can be deduced from the observation of separable reduced states only [9,10], it seems natural to ask whether one can infer that some tripartite correlations are nonlocal, only from the observation of local bipartite marginals. To answer this question in the affirmative one needs to find three local bipartite nonsignaling distributions P_{AB},P_{AC},P_{BC} such that any tripartite nonsignaling distribution P_{ABC} compatible with them is nonlocal. Being compatible in this context means that one must have

$$\sum_c P_{ABC}(abc|xyz)=P_{AB}(ab|xy), \tag{2}$$

$$\sum_b P_{ABC}(abc|xyz)=P_{AC}(ac|xz), \tag{3}$$

$$\sum_a P_{ABC}(abc|xyz)=P_{BC}(bc|yz), \tag{4}$$

where the left-hand sides are defined independently of the third input as P_{ABC} is assumed to be nonsignaling. In what follows we provide several examples of distributions satisfying these requirements.

In the first example, we fix the one-party expectation values as

$$\langle A_x\rangle=\langle B_y\rangle=\langle C_z\rangle=\tfrac{1}{3},\quad x,y,z\in\{0,1\}, \tag{5}$$

and the two-party expectation values as

$$\langle A_xB_y\rangle=\langle A_xC_y\rangle=\langle B_xC_y\rangle=\begin{cases}1 & \text{if } x=y=0,\\ -\frac{1}{3} & \text{otherwise.}\end{cases} \tag{6}$$

These values define the three bipartite marginals univocally. One can check that these bipartite correlations are local, as they satisfy all possible permutations of the Clauser-Horne-Shimony-Holt (CHSH) inequality [15], which is the only relevant Bell inequality for two parties having binary inputs and outputs [16].

However, only one tripartite nonsignaling distribution has (5) and (6) as its marginals. To see this, consider any tripartite nonsignaling distribution P_{ABC} that is compatible with the given marginals. The positivity constraints $P_{ABC}(abc|xyz)\geqslant 0$ together with the fixed values for the one- and two-party expectation values lead to lower bounds on $\langle A_xB_yC_z\rangle$ and $-\langle A_xB_yC_z\rangle$ that ultimately only allow for the assignment

$$\langle A_xB_yC_z\rangle=\begin{cases}\frac{1}{3} & \text{if } x+y+z\in\{0,1\},\\ -1 & \text{otherwise.}\end{cases} \tag{7}$$

Equations (5) through (7) define an extremal point of the tripartite nonsignaling polytope, the box number 29 in the classification of Ref. [17]. This point is genuinely nonlocal as it violates a Svetlichny-Bell inequality [17,18]. Thus we found some bipartite correlations that are local, but only compatible with (unique) genuinely tripartite nonlocal correlations.

While this first example answers our original question, it is not entirely satisfactory, as no measurements on a quantum system can achieve all bipartite correlations (5) and (6) at the same time. Indeed, the only possible extension of these correlations, namely box 29 in Ref. [17], violates the "Guess-Your-Neighbor-Input" inequality [19], which is satisfied by quantum correlations. Let us thus provide a general characterization of marginals that are only compatible with nonlocal probability distributions. To this end, consider the set Π of bipartite marginals with binary inputs and outputs, which result from a tripartite local and nonsignaling probability distribution

$$\Pi=\{(P_{AB},P_{AC},P_{BC})|\exists P_{ABC}\ \text{local s.t. (2), (3), (4) hold}\}. \tag{8}$$

Clearly, the set Π is convex and has a finite number of extreme points. It is then a polytope and can be described by a finite number of inequalities that only involve the marginal correlations P_{AB},P_{AC},P_{BC}. If the bipartite marginals of some tripartite nonsignaling correlations violate any of these inequalities, then they cannot be compatible with a local tripartite distribution. Thus, any extension of these marginals to a tripartite nonsignaling distribution must be nonlocal. On the other hand, if some bipartite correlations satisfy all the inequalities that define Π, then they are necessarily compatible with some tripartite local correlations.

Similarly, one can check whether some marginals are compatible with genuinely tripartite nonlocal correlations by considering the polytope

$$\Pi'=\{(P_{AB},P_{AC},P_{BC})|\exists P_{ABC}\ \text{bilocal s.t. (2), (3), (4) hold}\}. \tag{9}$$

Here we consider the definition of bilocality given in Refs. [11,12], which solves some inconsistencies of the original definition of bilocality by Svetlichny [18]. Since the constraints of the polytope Π' are strictly weaker than those of Π, one has $\Pi\subset\Pi'$. Any inequality satisfied by Π' is thus also a valid inequality for Π.

An example of inequality satisfied by Π' (and Π) is

$$\begin{aligned}&-\langle A_0(1+B_0+B_1+C_0)\rangle,\\ &-\langle A_1(1+B_0+C_0+C_1)\rangle,\\ &-\langle B_0+C_0+B_0C_0+B_1C_1\rangle\leqslant 4.\end{aligned} \tag{10}$$

The violation of this inequality implies that the correlations compatible with the given marginals must be genuinely tripartite nonlocal. The inequality (10) can be violated by measuring the noisy W state $\varrho_W(p)$ for $p > 0.9548$, where

$$\varrho_W(p) = p|W\rangle\langle W| + \frac{1-p}{8}\mathbb{I}, \tag{11}$$

with $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ and $0 \leqslant p \leqslant 1$. The corresponding measurement settings are

$$\begin{aligned} A_0 &= \cos\alpha\sigma_z + \sin\alpha\sigma_x, & A_1 &= \cos\alpha\sigma_z - \sin\alpha\sigma_x,\\ B_0 &= -\sigma_z, & B_1 &= \cos\beta\sigma_z + \sin\beta\sigma_x,\\ C_0 &= -\sigma_z, & C_1 &= \cos\beta\sigma_z - \sin\beta\sigma_x, \end{aligned} \tag{12}$$

and $\alpha = 3.6241$ and $\beta = 2.0221$. The reduced states of two parties of $\varrho_W(p)$ are all equal and have the form

$$\varrho_{\text{red}}(p) = \frac{2p}{3}|\psi^+\rangle\langle\psi^+| + \frac{p}{3}|00\rangle\langle 00| + \frac{1-p}{4}\mathbb{I}, \tag{13}$$

where $|\psi^+\rangle = 1/\sqrt{2}(|01\rangle + |10\rangle)$. Since these reduced states satisfy the Horodecki criterion for the violation of the CHSH inequality [20] for every $0 \leqslant p \leqslant 1$, any pair of two-outcome measurements on $\varrho_W(p)$ is necessarily local. Thus we have obtained an example of local quantum marginal correlations which are only compatible with genuine tripartite nonlocal correlations.

III. ENTANGLEMENT FROM SEPARABLE MARGINALS

Regarding the problem of entanglement detection from separable marginals, note that the global state of a system is known to be generally determinable from its marginals, if one has the promise that the global state is pure [3]. Indeed, consider the bipartite marginals $\varrho_{AB} = \varrho_{AC} = \varrho_{BC} = \varrho = (|00\rangle\langle 00| + |11\rangle\langle 11|)/2$. If the global state of the systems is pure, it follows from its Schmidt decomposition that it must be the Greenberger-Horne-Zeilinger (GHZ) state $|\text{GHZ}\rangle = 1/\sqrt{2}(|000\rangle + e^{i\phi}|111\rangle)$. While these bipartite marginals are separable, the GHZ state is entangled and, thus, the observation of separable marginals can only be compatible with an entangled pure state.

Now, if the global state is not assumed to be pure, then the above analysis immediately fails. For instance, the reduced states of the GHZ state are also compatible with the three-party mixed state $\varrho_{ABC} = 1/2(|000\rangle\langle 000| + |111\rangle\langle 111|)$, which is separable. Thus the observation of these marginals without further knowledge on the full state does not guarantee entanglement in the whole system. Actually, this result applies to every graph state: For any such state there is always a separable state that has the same two-body reductions [21]. So no criterion relying on two-particle correlations can detect graph-state entanglement.

However, as mentioned before, it was shown that there are separable two-qubit states that are only compatible with an entangled global state [9,10]. Here, we present further examples of this feature involving the reduced states of three-qubit states. The starting point for our investigation is again a noisy W state. The reduced states (13) are separable for $0 \leqslant p \leqslant p_{\text{sep}} = 3/(1 + 2\sqrt{5})$. We are interested to see if there exists a value of p with $p \leqslant p_{\text{sep}}$ such that every three-qubit state compatible with these reductions must be entangled.

TABLE I. Values for separability of the reduced two-party states of the noisy W state p_{sep} and for the solution to the SDP problem (14) $p^\star$ for a different number of parties.

n	3	4	5	6	7
$p^\star$	0.4899	0.6180	0.7464	0.8279	0.8787
p_{sep}	0.5482	0.7071	0.8050	0.8640	0.9009

To do that, we need to look for the maximal value of p such that every three-qubit state having $\varrho_{\text{red}}(p)$ as its reductions is *not* entangled. For simplicity, let us relax this last constraint, allowing the three-qubit state to have a positive partial transposition (PPT) instead of being separable [22]. After this relaxation, the maximal value of p corresponds to the solution $p^\star$ of the following instance of a semidefinite program (for an introduction to SDP see, for instance, the textbook [23]):

$$\begin{aligned} p^\star = \underset{\varrho, p}{\text{maximize}} \quad & p \\ \text{subject to} \quad & \varrho \succeq 0, \\ & \text{tr}_X \varrho = \varrho_{\text{red}}(p) \text{ for } X = A,B,C, \\ & \varrho^{T_X} \succeq 0 \text{ for } X = A,B,C. \end{aligned} \tag{14}$$

Note that the normalization condition $\text{tr}(\varrho) = 1$ is ensured by the constraints on the bipartite marginals $\text{tr}_X \varrho$.

By constructing the dual to the previous problem, it is possible to prove that the solution of Eq. (14) is $p^\star = 3/(2 + \sqrt{17}) \simeq 0.4899$ (see the Appendix). Therefore, the reduced states (13) with $p^\star < p \leqslant p_{\text{sep}}$ certify the presence of entanglement in the global state despite being separable.

The above considerations can be generalized to the case of more than three parties. Starting from the noisy W state of n qubits we found a similar behavior: One can choose separable two-party states that are only compatible with an entangled global state of n qubits. The value of p_{sep} for which the two-party reduced states become separable reads $p_{\text{sep}} = n/(4 - n + 2\sqrt{n^2 - 4n + 8})$, while solving the corresponding SDPs yields a value for $p^\star$. Table I summarizes our results for $n \leqslant 7$.

IV. CONCLUSION

To conclude, we have demonstrated how the compatibility constraints among marginal distributions allow one to certify the presence of nonlocal correlations in a global state from marginals that allow a local description. In particular, we have provided examples of local bipartite marginals that are only compatible with nonlocal probability distributions, and even with genuinely tripartite nonlocal distributions. This result reveals that local models reproducing some (local) bipartite marginal correlations can be fundamentally incompatible with each other since the full correlations representing their joint behavior admit no such model.

Furthermore, for the case of entanglement we have presented a collection of three separable two-qubit states that are only compatible with an entangled tripartite state. From a general viewpoint, our work proves how compatibility constraints lead to nontrivial results even when acting on separable or local states.

ACKNOWLEDGMENTS

We thank Géza Tóth, Otfried Gühne, Yeong-Cherng Liang, and Marco Piani for useful discussions. This work was supported by the ERC starting grant PERCENT, the CHIST-ERA DIQIP project, the European EU FP7 Q-Essence and QCS projects, the Spanish FIS2010-14830 project, the Swiss NCCRs QP and QSIT, the European ERC-AG QORE, the Hungarian National Research Fund OTKA (PD101461), and a János Bolyai Grant of the Hungarian Academy of Sciences.

APPENDIX

This Appendix provides details on the solution of the SDP from main text. Defining $M = 2/3|\psi^+\rangle\langle\psi^+| + 1/3|00\rangle\langle 00| - 1/4\mathbb{I}$, the corresponding dual problem of Eq. (14) can be written as

$$\begin{aligned} d^\star = \underset{N_X,Q_X}{\text{minimize}} \quad & \frac{1}{4}\mathrm{tr}(N_A + N_B + N_C) \\ \text{subject to} \quad & Q_X \preceq 0 \text{ for } X = A,B,C, \\ & \mathrm{tr}[M(N_A + N_B + N_C)] = -1 \\ & \sum_X \mathbb{I}_X \otimes N_X + Q_X^{T_X} \succeq 0, \end{aligned} \tag{A1}$$

where N_X are 4×4 matrices and Q_X are 8×8 matrices; the expression $\mathbb{I}_X \otimes N_X$ denotes the operator that acts as the identity on particle X and as N_X on the rest.

From weak duality one always has $d^\star \geqslant p^\star$. Every feasible point for the primal problem gives a lower bound $p' \leqslant p^\star$ and every dual feasible point gives an upper bound $d' \geqslant d^\star$. The following choice of the variables ϱ, N_X, Q_X satisfy all the constraints of Eqs. (14) and (A1), while yielding the same bounds $d' = p' = 3/(2+\sqrt{17}) \simeq 0.4899$. Thus, we have $p^\star = d^\star = 3/(2+\sqrt{17})$,

$$\begin{aligned} \varrho = {} & \frac{p^\star}{2}(|W\rangle\langle W| + |\overline{W}\rangle\langle\overline{W}|) + \frac{3(1-p^\star)}{4}\sigma \\ & + \frac{p^\star}{6}|000\rangle\langle 000| + \frac{3-5p^\star}{12}|111\rangle\langle 111|, \end{aligned} \tag{A2}$$

with $\sigma = 1/3(|001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100|)$ and $|\overline{W}\rangle = 1/\sqrt{3}(|011\rangle + |101\rangle + |110\rangle)$,

$$\begin{aligned} N_X = {} & \left(1 + \frac{5}{3\sqrt{17}}\right)\frac{p^\star}{2}|00\rangle\langle 00| + (1-\sqrt{17})\frac{p^\star}{12}(|01\rangle\langle 01| \\ & + |10\rangle\langle 10|) - \left(1 + \frac{11}{\sqrt{17}}\right)\frac{p^\star}{6}(|01\rangle\langle 10| + |10\rangle\langle 01|) \\ & + 2\left(\frac{1}{3} + \frac{1}{\sqrt{17}}\right)p^\star|11\rangle\langle 11|, \end{aligned} \tag{A3}$$

for $X = A,B,C$, and

$$\begin{aligned} Q_A = {} & \left(1 + \frac{5}{3\sqrt{17}}\right)\frac{p^\star}{4}(-|000\rangle\langle 000| + |000\rangle\langle 110| \\ & + |000\rangle\langle 101| + \text{H.c.}) \\ & - \left(\frac{1}{3} - \frac{1}{\sqrt{17}}\right)p^\star(|001\rangle + |010\rangle)(\langle 001| + \langle 010|) \\ & + \frac{4}{3\sqrt{17}}p^\star(|001\rangle\langle 111| + |010\rangle\langle 111| + \text{H.c.}) \\ & - \left(\frac{3}{5} - \frac{1}{3\sqrt{17}}\right)\frac{p^\star}{2}(|101\rangle\langle 101| + |110\rangle\langle 110|) \\ & + \left(\frac{1}{5} - \frac{7}{3\sqrt{17}}\right)\frac{p^\star}{4}(|101\rangle\langle 110| + |110\rangle\langle 101|) \\ & - 2\left(\frac{1}{3} + \frac{1}{\sqrt{17}}\right)p^\star|111\rangle\langle 111|, \end{aligned} \tag{A4}$$

where H.c. stands for Hermitian conjugate. Q_B and Q_C are equal to Q_A after permutating the parties so that B or C take the role of A.

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