Zitterbewegung of Klein-Gordon particles and its simulation by classical systems

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The Klein-Gordon equation is used to calculate the *Zitterbewegung* (ZB, trembling motion) of spin-zero particles in the absence of fields and in the presence of an external magnetic field. Both Hamiltonian and wave formalisms are employed to describe ZB and their results are compared. It is demonstrated that if one uses wave packets to represent particles, then the ZB motion has a decaying behavior. It is also shown that the trembling motion is caused by an interference of two subpackets composed of positive- and negative-energy states, which propagate with different velocities. In the presence of a magnetic field, the quantization of the energy spectrum results in many interband frequencies contributing to ZB oscillations and the motion follows a collapse-revival pattern. In the limit of nonrelativistic velocities, the interband ZB components vanish and the motion is reduced to cyclotron oscillations. The exact dynamics of a charged Klein-Gordon (KG) particle in the presence of a magnetic field is described on an operator level. The trembling motion of a KG particle in the absence of fields is simulated using a classical model proposed by Morse and Feshbach—it is shown that a variance of a Gaussian wave packet exhibits ZB oscillations.

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I. INTRODUCTION

The phenomenon of *Zitterbewegung* (ZB, trembling motion) goes back to Schrödinger, who proposed it in 1930 for free relativistic electrons in a vacuum [1]. Schrödinger observed that due to noncommutativity of the velocity operators with the Dirac Hamiltonian, relativistic electrons experience a trembling motion in the absence of external fields. It was later recognized that ZB is due to an interference of electron states with positive and negative electron energies. A very high frequency of ZB in a vacuum, corresponding to $\hbar\omega_Z = 2m_ec^2$, and its very small amplitude on the order of the Compton wavelength $\lambda_c = \hbar/m_ec \simeq 3.86 \times 10^{-3}$ Å made it impossible to observe this effect in its original form with the currently available experimental methods. However, in a recent work, Gerritsma *et al.* [2] simulated the $1+1$ Dirac equation and the resulting *Zitterbewegung* with the use of trapped ions excited by laser beams. The important advantage of this method is that one can simulate also the basic parameters of the Dirac equation and tailor their desired values. The result of Gerritsma *et al.* allows one to expect that observable effects for relativistic particles in a vacuum can be convincingly reproduced with more "user-friendly" parameters. In general, there has recently been a revival of interest in the relativistic-type equations related to "the rise of graphene" [3], topological insulators, and similar systems in narrow-gap semiconductors [4].

The purpose of our paper is to describe the phenomenon of *Zitterbewegung* for charged Klein-Gordon (KG) spin-zero particles in the absence of fields and in the presence of a magnetic field [5–7]. The *Zitterbewegung* of KG particles in the absence of fields was described before; see [8–12]. However, in our treatment, we introduce a number of additional elements. First, we describe the particles by wave packets and show that this feature leads to a transient character of the resulting ZB motion. Second, we use both the Hamiltonian and wave forms of the Klein-Gordon equation (KGE) and show the equivalence of the two approaches as far as average physical quantities are concerned. Third, we point out that ZB is a result of interference between positive- and negative-energy subpackets propagating with different velocities. Fourth, we simulate classically the ZB motion using a simple mechanical system proposed by Morse and Feshbach [13]. Still, our main objective is to consider in detail the dynamics of a charged KG particle in the presence of an external uniform magnetic field and describe the phenomenon of ZB in this situation.

In the beginning, the meson theory was done by means of a field theory. Pauli and Weisskopf showed that there is no difficulty of interpretation if the KG equation is regarded as the equation of motion of a field and quantized in the usual fashion [14]. However, if one is interested in the problems of interaction between mesons and the electromagnetic field, then it is useful to describe mesons by a one-particle wave equation, similar to other particles: electrons, positrons, and photons. The first efforts in this direction were carried out by Petiau [15], Duffin [16], and Kemmer [17]. The Hamiltonian formulation of the equation of motion for the spinless mesons, as it is currently used today, was given by Sakata and Taketani (ST, Ref. [18]). In the ST theory for spinless particles, a set of 2×2 operators called the $\hat{\tau}_i$ matrices plays a role similar to that of the $\hat{\rho}_i$ matrices in the Dirac theory; see below. Symmetry properties of ST equations were discussed by Krajcik and Nieto [19]. It is known that the one-particle Klein-Gordon equation for spin-zero particles leads to some difficulties [12, 20]: the KG equation involves a second time derivative, the probability density is not positively definite, and there are problems with the position operator or vanishing square of the velocity operator. For this reason, in the present work, we calculate the ZB of *average current*, which has a well-defined meaning in the theory of the KG equation. For charged particles, the average current is proportional to the average particle velocity, so, in our work, we calculate one of these two quantities. In previous treatments of ZB for the Dirac equation, as well as simulations by trapped ions and solid state

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systems, the authors usually calculated the ZB of the position operator.

In our considerations, we encounter another interesting anomaly of the KG equation in the Sakata-Taketani version, namely, that particle velocities can exceed the speed of light for sufficiently large momenta. In other words, in contrast to the Dirac equation for electrons, the KGE does not possess an automatic "safety brake" for velocities to keep them below c. It is known, since the works of Velo and Zwanziger [21], that the two-component ST theory can lead to noncausal results (wave propagation with velocities higher than c) in the presence of even very weak electromagnetic potentials [22].

Our paper is organized as follows. In Sec. II, we calculate the ZB of a wave packet using the Hamiltonian formalism. In Sec. III, we obtain similar results with the use of KG waves and discuss explicitly the physical background for the transient behavior of ZB motion. Section IV contains a description of ZB for a charged KG particle in a magnetic field. In Sec. V, we simulate classically the ZB phenomenon using a system proposed by Morse and Feshbach. In Sec. VI, we discuss our results; the paper is concluded by a summary. Appendix A contains a derivation of particle dynamics in the presence of a magnetic field, Appendix B discusses the problem of high particle velocities, and in Appendices C and D, we give some mathematical details.

II. *ZITTERBEWEGUNG* IN VACUUM

We begin by considering a Klein-Gordon particle in the absence of external fields. The Klein-Gordon equation in the Hamiltonian form is [18,23]

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi. \tag{1}$$

Here the Hamiltonian is

$$\hat{H} = \frac{\tau_3 + i\tau_2}{2m}\hat{\boldsymbol{p}}^2 + \tau_3 mc^2, \tag{2}$$

where m is particle mass, $\hat{\boldsymbol{p}}$ is particle momentum, and τ_j ($j = 1,2,3$) are the Pauli matrices σ_j, respectively. The wave function Ψ is a two-component vector

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \tag{3}$$

In the Hamiltonian form, one can introduce the Heisenberg picture [23]. The zth component of the time-dependent velocity operator is

$$\hat{v}_z(t) = e^{i\hat{H}t/\hbar}\hat{v}_z(0)e^{-i\hat{H}t/\hbar}, \tag{4}$$

where $\hat{v}_z(0) = \partial\hat{H}/\partial\hat{p}_z$. In this representation, $\hat{v}_z(t)$ is a 2×2 matrix operator. Expanding $e^{i\hat{H}t/\hbar} = 1 + \hat{H}t + (1/2!)\hat{H}^2 + \cdots$, and noting that $\hat{H}^2 = E^2$, where the energy is $E = \pm cp_0$ with

$$p_0 = +\sqrt{m^2c^2 + p^2}, \tag{5}$$

we obtain

$$e^{i\hat{H}t/\hbar} = \cos(Et/\hbar) + \frac{i\hat{H}}{E}\sin(Et/\hbar). \tag{6}$$

The velocity operator in Eq. (4) is a product of three matrices. Its $(1,1)$ component is

$$(\hat{v}_z)_{11}(t) = \frac{\hat{p}_z}{m} + \frac{\hat{p}^2\hat{p}_z}{2m\hat{p}_0^2}[\cos(2Et/\hbar) - 1]. \tag{7}$$

The remaining elements of $\hat{v}_z(t)$ are calculated similarly. The $\hat{v}_x$ and $\hat{v}_y$ components of the velocity operator are obtained from $\hat{v}_z(t)$ by the replacement $\hat{p}_z \to \hat{p}_x,\hat{p}_y$, respectively. In the nonrelativistic limit $p \ll mc$, we obtain in Eq. (7) the classical motion $(\hat{v}_z)_{11}(t) \simeq \hat{p}_z/m$. In the absence of external fields, p_i are good quantum numbers. We introduce $\boldsymbol{p} = \hbar\boldsymbol{k}$ and $\boldsymbol{q} = \lambda_c\boldsymbol{k}$, where the effective Compton wavelength is $\lambda_c = \hbar/mc$. Also, we introduce a useful frequency $\omega_0 = (mc^2)/\hbar$. Both λ_c and ω_0 refer to particles of mass m. In the above notation, Eq. (7) becomes

$$(\hat{v}_z)_{11}(t) = cq_z + \frac{c}{2}\frac{q^2q_z}{(1+q^2)}[\cos(2\omega_0t\sqrt{1+q^2}) - 1]. \tag{8}$$

The first term in Eq. (8) corresponds to the classical motion of a particle, while the second term describes rapid oscillations of the velocity. The velocity oscillates from $v_{\max} = cq_z$ to $v_{\min} = cq_z/(1+q^2)$. Since the maximum velocity of the particle is c, there must be $|\boldsymbol{q}| \leqslant 1$. We notice that, in principle, Eq. (8) admits velocities above the speed of light. We discuss this issue in more detail in Appendix B. The frequency of oscillations varies from $\omega = 2\omega_0$ for low $\boldsymbol{q}$ to $\omega = 2\sqrt{2}\omega_0$ for $|\boldsymbol{q}| = 1$. The velocity oscillations taking place in the absence of external fields are called *Zitterbewegung*.

By integrating $(\hat{v}_z)_{11}(t)$ in Eq. (8) over time, we have

$$\hat{z}_{11}(t) = z_{11}(0) + cq_zt - \frac{c}{2}\frac{q^2q_z}{1+q^2}t + \frac{\lambda_c}{4}\frac{q^2q_z}{(1+q^2)^{3/2}}\sin(2\omega_0t\sqrt{1+q^2}). \tag{9}$$

The amplitude of ZB oscillations of the position operator is on the order of λ_c. The operator $\hat{z}_{11}(t)$ is obtained in a formal way; physical limitations to the position operator will be discussed below.

In order to obtain physical observables, one needs to average operator quantities over the wave packet. The average velocity $\langle\hat{v}_z(t)\rangle$ of the wave packet $|W\rangle$ is

$$\langle\hat{v}_z(t)\rangle = \langle W|\tau_3\hat{v}_z(t)|W\rangle = \sum_{\boldsymbol{p}\boldsymbol{p}'}\langle W|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\tau_3\hat{v}_z(t)|\boldsymbol{p}'\rangle\langle\boldsymbol{p}'|W\rangle. \tag{10}$$

For the KGE in the Hamiltonian form, the matrix elements of operators include the additional τ_3 factor [23]. We take the wave packet in the form of a two-component vector $\langle\boldsymbol{r}|W\rangle = (1,0)^T\langle\boldsymbol{r}|w\rangle$ with one nonvanishing component. Here $\langle\boldsymbol{r}|w\rangle \equiv w(\boldsymbol{r})$ is a Gaussian function with the nonzero momentum $\hbar\boldsymbol{k}_0$,

$$w(\boldsymbol{r}) = \frac{1}{(d\sqrt{\pi})^{3/2}}\exp[-r^2/(2d^2) + i\boldsymbol{k}_0\cdot\boldsymbol{r}]. \tag{11}$$

There is $w(\boldsymbol{k}) = \int e^{-i\boldsymbol{k}\cdot\boldsymbol{r}/\hbar}w(\boldsymbol{r})d^3\boldsymbol{r}$ and we have

$$\langle\boldsymbol{k}|w\rangle = (2d\sqrt{\pi})^{3/2}\exp[-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2/2]. \tag{12}$$

The wave packet $|W\rangle$ selects (1,1) component of the velocity matrix $\hat{v}_z(t)$. From Eqs. (10) and (12), we obtain

$$\langle\hat{v}_z(t)\rangle = c\frac{d_c^3}{\pi^{3/2}}\int \exp\left[-d_c^2(\boldsymbol{q}-\boldsymbol{q}_0)^2\right] \times\left\{q_z+\frac{1}{2}\frac{q^2q_z}{1+q^2}[\cos(\omega_0 t\sqrt{1+q^2})-1]\right\}d^3\boldsymbol{q}, \quad (13)$$

where $d_c = d/\lambda_c$. This integral is nonzero only if $\boldsymbol{q}_0$ has a nonzero zth component, so we take $\boldsymbol{q}_0 = (0,0,q_{0z})^T$. Selecting the z axis to be parallel to $\boldsymbol{q}_0$ and using the spherical coordinates, we calculate the integrals over the two angular variables. The remaining integral over q is computed numerically.

In Fig. 1, we plot the average packet velocity $\langle\hat{v}_z(t)\rangle$ calculated from Eq. (13) for three different packet widths d. The time on the horizontal axis is expressed in $t_c = \hbar/mc^2$ units, where $t_c = (m_e/m)\times 1.29\times 10^{-21}$ s, and m_e is the electron mass. In all cases, the motion has a transient character. Physically, the decay of ZB oscillations is due to different propagation velocities of subpackets corresponding to the positive- and negative-energy states. We analyze this effect below. It is seen that the final packet velocity differs from the initial value $\hbar k_{0z}/m$. In the limit of $d\to\infty$, the velocity oscillations do not decay in time.

Now we calculate the average velocity by splitting the initial wave packet into two subpackets corresponding to the positive- and negative-energy states. First we introduce the unity operator [10]

$$\hat{1} = \sum_{\boldsymbol{k}s}|\boldsymbol{k}s\rangle\langle\boldsymbol{k}s|\tau_3 s, \quad (14)$$

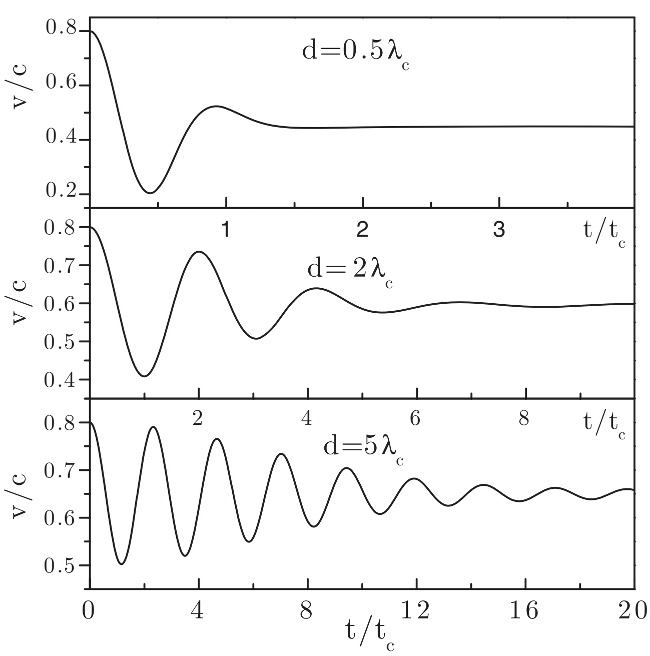

FIG. 1. Calculated velocity of wave packet in the absence of external fields for three packet widths d. The phenomenon of transient *Zitterbewegung* is seen. The initial packet wave vector is $\boldsymbol{k}_0 = (0,0,k_{0z})$ with $k_{0z} = 0.8\lambda_c^{-1}$. Time is expressed in $t_c = \hbar/mc^2$ units. The initial packet velocity is $v_{0z} = \hbar k_{0z}/m = 0.8c$; its final velocity depends on packet parameters.

where $s=\pm1$, and

$$\langle\boldsymbol{r}|\boldsymbol{k}s\rangle = \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{2\sqrt{mcp_0}}\begin{pmatrix}mc+sp_0\\ mc-sp_0\end{pmatrix} \quad (15)$$

are the two eigenstates of $\hat{H}$ corresponding to the positive and negative energies $E_s = scp_0$. These states are normalized according to $\langle\boldsymbol{k}s|\tau_3|\boldsymbol{k}'s'\rangle/(2\pi)^{3/2} = s\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{ss'}$. Then,

$$|W\rangle = \sum_{\boldsymbol{k}s}s|\boldsymbol{k}s\rangle\langle\boldsymbol{k}s|\tau_3|W\rangle = \sum_{\boldsymbol{k}s}s|\boldsymbol{k}s\rangle W_{\boldsymbol{k}s}, \quad (16)$$

where $W_{\boldsymbol{k}s} = \langle\boldsymbol{k}s|\tau_3|W\rangle$. The subpacket of positive-energy states is $|W+\rangle = \sum_{\boldsymbol{k}}|\boldsymbol{k}+\rangle W_{\boldsymbol{k}+}$, while the subpacket of negative-energy states is $|W-\rangle = \sum_{\boldsymbol{k}}|\boldsymbol{k}-\rangle W_{\boldsymbol{k}-}$. Using Eqs. (15) and (16), we find

$$W_{\boldsymbol{k}s} = (2d\sqrt{\pi})^{3/2}\frac{(mc+sp_0)}{2\sqrt{mcp_0}}e^{-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2/2}. \quad (17)$$

The average packet velocity is

$$\langle\hat{v}_z(t)\rangle = \sum_{\boldsymbol{k}\boldsymbol{k}'ss'}ss'W^*_{\boldsymbol{k}s}W_{\boldsymbol{k}'s'}\langle\boldsymbol{k}s|\tau_3\hat{v}_z(t)|\boldsymbol{k}'s'\rangle = \sum_{\boldsymbol{k}\boldsymbol{k}'ss'}ss'W^*_{\boldsymbol{k}s}W_{\boldsymbol{k}'s'}e^{i(\omega_s-\omega_{s'})t}\langle\boldsymbol{k}s|\tau_3\frac{\partial\hat{H}}{\partial\hat{p}_z}|\boldsymbol{k}'s'\rangle. \quad (18)$$

We define $\omega_s = s\omega_0\sqrt{1+(k\lambda_c)^2}$ and use the equality

$$\langle\boldsymbol{k}s|\tau_3 e^{i\hat{H}t/\hbar} = \langle\boldsymbol{k}s|e^{i\hat{H}^\dagger t/\hbar}\tau_3 = e^{i\omega_s t}\langle\boldsymbol{k}s|\tau_3, \quad (19)$$

which follows from the properties $\hat{H} = \tau_3\hat{H}^\dagger\tau_3$ and $\langle\boldsymbol{k}s|\hat{H}^\dagger = (\hat{H}|\boldsymbol{k}s\rangle)^\dagger = E_s\langle\boldsymbol{k}s|$. Another proof of the identity (19) is given in Appendix C. There is also

$$\langle\boldsymbol{k}s|\tau_3\frac{\partial\hat{H}}{\partial p_z}|\boldsymbol{k}'s'\rangle = (2\pi)^3\frac{cp_z}{p_0}\delta_{\boldsymbol{k}\boldsymbol{k}'}, \quad (20)$$

which does not depend on s and s'. By combining Eqs. (18)–(20), we obtain

$$\langle\hat{v}_z(t)\rangle = \frac{2d^3\pi^{3/2}}{(2\pi)^3 m}\int\frac{p_z}{p_0^2}e^{-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2}d^3\boldsymbol{k} \times\sum_{s,s'}ss'(mc+sp_0)(mc+s'p_0)e^{i(\omega_s-\omega_{s'})t}. \quad (21)$$

The average velocity in Eq. (21) is a sum of four terms. The term with $s=s'=+1$ describes the motion of the positive-energy subpacket, while the term with $s=s'=-1$ corresponds to the negative-energy subpacket,

$$\langle\hat{v}_z\rangle^{\pm} = \frac{d^3}{4m\pi^{3/2}}\int\left(1\pm\frac{mc}{p_0}\right)^2 p_z e^{-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2}d^3\boldsymbol{k}. \quad (22)$$

Thus the two subpackets move with different velocities. Their relative velocity is

$$\langle\hat{v}_z\rangle^{\rm rel} = \frac{cd^3}{\pi^{3/2}}\int\frac{p_z}{p_0}e^{-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2}d^3\boldsymbol{k}. \quad (23)$$

The two terms in Eq. (21) with $s\neq s'$, corresponding to an interference of the two packets, give rise to an oscillatory term

$$\langle\hat{v}_z(t)\rangle^{\rm osc} = \frac{d^3}{4m\pi^{3/2}}\int\left(1-\frac{m^2c^2}{p_0^2}\right)p_z \times\cos(2\omega_k t)e^{-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2}d^3\boldsymbol{k}, \quad (24)$$

where $\omega_k = \sqrt{1+(k\lambda_c)^2}$. According to the Riemann-Lebesgue theorem, this term has a transient character [24]. Performing the integrations in Eqs. (22) and (24), we obtain again Eq. (13). Thus we showed that the ZB oscillations arise from the interference of positive- and negative-energy states. After a certain time, the two subpackets are sufficiently far away from each other and the overlap between them vanishes, which results in the disappearance of ZB oscillations. This explains the behavior of velocity shown in Fig. 1.

To evaluate the decay time of ZB oscillations, we estimate the time after which the two subpackets will be separated from each other by the distance $2d$. Assuming that $k_0\lambda_c \simeq 1$, the relative velocity between the two subpackets is $\langle \hat{v}_z\rangle^{\rm rel} \simeq c(k_0\lambda_c)$. The time interval after which the distance between the subpackets exceeds $2d$ is

$$t_d \simeq \frac{2d}{ck_0\lambda_c}. \tag{25}$$

It is seen in Fig. 1 that the ZB oscillations nearly disappear after t_d. For example, there is $t_d = 5t_c$ for $d = 2\lambda_c$. Since the ZB frequency is $2\omega_0 = 2mc^2/\hbar$, a number of nonvanishing oscillations is approximately

$$N_{\rm osc} \simeq \frac{2\omega_0 t_d}{2\pi} = \frac{2}{\pi}\left(\frac{d}{\lambda_c}\right)\left(\frac{1}{k_0\lambda_c}\right). \tag{26}$$

The above estimation correctly evaluates the number of ZB oscillations seen in Fig. 1. The optimal conditions for an appearance of ZB are wide packets and small values of $|\boldsymbol{k}_0|$. On the other hand, for too small values of $|\boldsymbol{k}_0|$, one of the two subpackets disappears [see Eq. (22)], which reduces the amplitude of the ZB oscillations.

III. WAVE FORM OF THE KGE

Now we intend to demonstrate a relation between the ZB oscillations of the average packet velocity calculated above with the use of the Hamiltonian form of the KGE and an average current obtained from the wave form of the KGE. In the absence of external fields, the Klein-Gordon equation has the wave equation form

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(x) - \nabla^2\phi(x) + \frac{m^2c^2}{\hbar^2}\phi(x) = 0, \tag{27}$$

where $x = (ct,\boldsymbol{r})$ is the position four vector [12]. The solution of this equation is

$$\phi(x) = \frac{1}{(2\pi)^3}\int\sqrt{\frac{mc}{p_0}}[a(\boldsymbol{k})e^{-ik\cdot x} + b^*(\boldsymbol{k})e^{ik\cdot x}]d^3\boldsymbol{k}, \tag{28}$$

where $k = (\omega_k/c,\boldsymbol{k})$, $\omega_k = \omega_0\sqrt{1+(k\lambda_c)^2}$, and $a(\boldsymbol{k})$, $b^*(\boldsymbol{k})$ are complex coefficients. Function ϕ is normalized to

$$\frac{i\hbar}{2mc^2}\int\left[\phi^*\frac{\partial\psi}{\partial t} - \left(\frac{\partial\phi^*}{\partial t}\right)\psi\right]d^3\boldsymbol{r} = Q, \tag{29}$$

where $Q = \pm 1$ for charged particles and $Q = 0$ for neutral particles. In the following, we select $Q = +1$, which leads to

$$\int d^3\boldsymbol{k}[a^*(\boldsymbol{k})a(\boldsymbol{k}) - b^*(\boldsymbol{k})b(\boldsymbol{k})] = 1. \tag{30}$$

To determine the coefficients $a(\boldsymbol{k})$ and $b^*(\boldsymbol{k})$, we need two boundary conditions for ϕ and $\partial\phi/\partial t$ at $x = (0,\boldsymbol{r})$. Having specified $a(\boldsymbol{k})$ and $b^*(\boldsymbol{k})$, one can calculate the current density $\boldsymbol{j}(x)$ as

$$\boldsymbol{j}(x) = \frac{\hbar}{2im}[\phi^*(\nabla\phi) - (\nabla\phi^*)\phi], \tag{31}$$

and the average current $\langle \boldsymbol{j}(t)\rangle = \int \boldsymbol{j}(x)d^3\boldsymbol{r}$.

Our aim is to find a correspondence between the average packet velocity calculated in Eq. (13) and the average current $\langle \boldsymbol{j}(t)\rangle$ given in Eq. (31). To this end, we select the coefficients $a(\boldsymbol{k})$ and $b^*(\boldsymbol{k})$ in such a way that the function ϕ in the wave form of the KGE corresponds to the wave packet $(w(\boldsymbol{r}),0)^T$ in the Hamiltonian form of the KGE. Relations between ϕ, $\partial\phi/\partial t$, and the two-component wave function $\Psi = (\varphi,\chi)^T$ in the Hamiltonian form of the KGE are [12]

$$\phi = \varphi + \chi, \tag{32}$$

$$i\partial\phi/\partial t = mc^2(\varphi - \chi)/\hbar. \tag{33}$$

Since $(\varphi,\chi)^T = (w(\boldsymbol{r}),0)^T$, we find the coefficients $a(\boldsymbol{k})$ and $b^*(\boldsymbol{k})$ from Eqs. (32) and (33) by setting $\varphi(t=0,\boldsymbol{r}) = w(\boldsymbol{r})$ and $\chi = 0$. From Eq. (32), we have

$$\int\sqrt{\frac{mc}{p_0}}[a(\boldsymbol{k})e^{+i\boldsymbol{k}\cdot\boldsymbol{r}} + b^*(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}]d^3\boldsymbol{k} = (2d\sqrt{\pi})^{3/2}\int e^{-(\boldsymbol{k}-\boldsymbol{k}_0)^2d^2/2+i\boldsymbol{k}\cdot\boldsymbol{r}}d^3\boldsymbol{k}, \tag{34}$$

while from Eq. (33), we have

$$\int\sqrt{\frac{mc}{p_0}}[a(\boldsymbol{k})e^{+i\boldsymbol{k}\cdot\boldsymbol{r}} - b^*(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}]p_0d^3\boldsymbol{k} = (2d\sqrt{\pi})^{3/2}mc\int e^{-(\boldsymbol{k}-\boldsymbol{k}_0)^2d^2/2+i\boldsymbol{k}\cdot\boldsymbol{r}}d^3\boldsymbol{k}. \tag{35}$$

In the terms including $b^*(\boldsymbol{k})$, we replace $\boldsymbol{k}\to-\boldsymbol{k}$, solve Eqs. (34) and (35) for $a(\boldsymbol{k})$ and $b^*(-\boldsymbol{k})$, and obtain

$$\phi(\boldsymbol{r},t) = \frac{(2d\sqrt{\pi})^{3/2}}{2(2\pi)^3}\int d^3\boldsymbol{k}e^{-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2/2+i\boldsymbol{k}\cdot\boldsymbol{r}} \times\left[\left(1+\frac{mc}{p_0}\right)e^{-i\omega_k t} + \left(1-\frac{mc}{p_0}\right)e^{+i\omega_k t}\right]. \tag{36}$$

The above function ϕ includes both positive- and negative-energy amplitudes. For $p\to 0$, there is $1 + mc/p_0 \simeq 2$ and $1-(mc/p_0) \simeq p^2/2(mc)^2$. Thus the second term in Eq. (36) is much smaller than the first. In this limit, the packet consists of the positive-energy states alone.

In Fig. 2, we plot the time evolution of the wave packet ϕ in one dimension. The packet propagates according to a one-dimensional version of Eq. (36). The initial packet is assumed in a Gaussian form

$$\phi(x,0) = \frac{d}{\sqrt{\pi}}e^{-x^2/(2d^2)+ik_0x}. \tag{37}$$

Its absolute value is indicated in Fig. 2 by the thick line. Each thin line describes $|\phi(x,t)|$ in successive time intervals, $2t_c = 2\hbar/(mc^2)$. It is seen that the packet splits into two subpackets moving with different velocities. The subpacket at the right corresponds to positive energies, while the subpacket at the left corresponds to negative energies. The difference in the amplitudes of subpackets results from different contributions of the positive- and negative-energy states in the initial packet

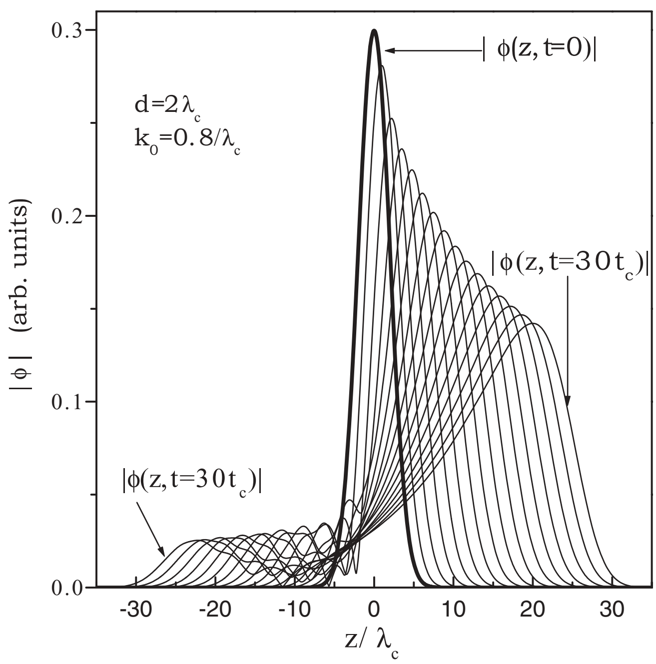

FIG. 2. Time evolution of the wave packet (absolute value) according to the one-dimensional version of Eq. (36). The initial wave packet (thick line) splits into two subpackets moving with different velocities. The thin lines show shapes of subpackets in successive time intervals, $2t_c = 2\hbar/(mc^2)$.

at $t = 0$; see Eqs. (34) and (35). The *Zitterbewegung* occurs only when the subpackets overlap. Each of the subpackets slowly spreads in time, but the spreading time is much larger than the overlapping time, so the ZB vanishes much faster than the spreading of subpackets.

Now we continue the calculation of average current given in Eq. (31) using the function ϕ of Eq. (36). This function has the form of an integral over $\boldsymbol{k}$. To calculate the spatial derivative $\nabla\phi$, we change the order of integration and differentiation, which can be done for any function decaying exponentially for $k \to \infty$. Using the identity $1 + (mc/p_0)^2 = 2 - (p/p_0)^2$, we obtain for the first term of the average current,

$$\begin{aligned}\frac{\hbar}{2im}\int \phi^*\frac{\partial\phi}{\partial z}d^3\boldsymbol{r} &= \frac{d^3\hbar}{8im\pi^{3/2}}\int d^3\boldsymbol{k}e^{-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2}(ik_z)\\ &\quad\times\left|\left(1+\frac{mc}{p_0}\right)e^{-i\omega_k t}+\left(1-\frac{mc}{p_0}\right)e^{+i\omega_k t}\right|^2\\ &= \frac{d^3}{2\pi^{3/2}}\int d^3\boldsymbol{k}e^{-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2}\left\{\frac{p_z}{m}+\frac{p_z p^2}{2mp_0}[\cos(2\omega_k t)-1]\right\}.\end{aligned} \tag{38}$$

The calculation of the second term in the current, $\hbar/(2im)\int(\partial\phi^*/\partial z)\phi d^3\boldsymbol{r}$, gives the same result but with the opposite sign, so that both terms in Eq. (31) add together. By comparing Eq. (38) with Eqs. (31) and (13), we conclude that the current density $\langle j_z(t)\rangle$ averaged over the packet $\phi(x)$ in Eq. (28) is equal to the average velocity $\langle v_z(t)\rangle$ of the packet in the Hamiltonian form of the KGE multiplied by the particle charge. In this way, we establish an equivalence of *Zitterbewegung* in the Hamiltonian and wave equation formalisms.

The above equivalence is valid for the average values only. In the Hamiltonian form of the KGE, one can define the time-dependent velocity operator $\hat{\boldsymbol{v}}(t) = e^{i\hat{H}t/\hbar}\hat{\boldsymbol{v}}(0)e^{-i\hat{H}t/\hbar}$, which can be expressed in a closed form without specifying the wave packet; see Eq. (7). But an analogous current operator in the wave form of the KGE can be defined as a current density $\boldsymbol{j}(x)$, which strongly depends on the form of function ϕ.

Even more significant differences between the Hamiltonian and wave descriptions of the ZB appear in the analysis of the position operator $\hat{\boldsymbol{r}}(t)$. In the Hamiltonian form of the KGE, the position operator written in the Heisenberg picture is $\hat{\boldsymbol{r}}(t) = e^{i\hat{H}t/\hbar}\hat{\boldsymbol{r}}(0)e^{-i\hat{H}t/\hbar}$ and, for the field-free KGE, it can be calculated in a compact form; see Eq. (9) and Ref. [10]. On the other hand, there is no well-defined position operator $\hat{\boldsymbol{r}}$ for the wave form of the KGE since this operator is not Hermitian; see Ref. [20]. However, one can calculate an *average position operator* for the wave form of the KGE by integrating the average current over time,

$$\langle \boldsymbol{r}(t)\rangle = \langle \boldsymbol{r}(0)\rangle + \frac{1}{Q}\int\langle \boldsymbol{j}(t)\rangle dt, \tag{39}$$

where the charge $Q \neq 0$. This example indicates that the equivalence between the *Zitterbewegung* for the Hamiltonian and wave equation formalisms holds for the average values only.

IV. *ZITTERBEWEGUNG* IN A MAGNETIC FIELD

In the presence of a magnetic field, the KG Hamiltonian for a charged particle reads [12]

$$\hat{H} = \frac{\tau_3 + i\tau_2}{2m}(\hat{\boldsymbol{p}} - q\boldsymbol{A})^2 + \tau_3 mc^2, \tag{40}$$

where q is the particle charge and $\boldsymbol{A}$ is the vector potential of a magnetic field. We assume the magnetic field $\boldsymbol{B}$ to be parallel to the z axis and describe it by the asymmetric gauge $\boldsymbol{A} = B(-y,0,0)$. The eigenstates of the Hamiltonian are of the form

$$\Psi(\boldsymbol{r}) = e^{ik_x x + ik_z z}\Phi(y), \tag{41}$$

and the resulting eigenenergy equation is $\hat{H}\Psi = E\Psi$, with

$$\hat{H} = (\tau_3 + i\tau_2)\frac{1}{2m}\left[(\hbar k_x + qBy)^2 + \hbar^2k_y^2 + \hbar^2k_z^2\right] + \tau_3 mc^2. \tag{42}$$

We introduce the magnetic radius $L = \sqrt{\hbar/|q|B}$ and define $\xi = k_x L + \eta_q y/L$, where $\eta_q = \pm 1$ is the sign of q. Then, there is $\eta_q y = \xi L - k_x L^2$ and $\partial/\partial y = (1/L)\partial/\partial\xi$. The eigenenergies are $E_n = sE_{n,k_z}$, where [25]

$$E_{n,k_z} = \sqrt{m^2c^4 + 2mc^2\hbar\omega_c(n + 1/2) + (c\hbar k_z)^2}. \tag{43}$$

The corresponding eigenstates $|\mathrm{n}\rangle$ are characterized by four quantum numbers: $|\mathrm{n}\rangle = |n,k_x,k_z,s\rangle$, where n labels the Landau levels, k_x and k_z are wave-vector components, and $s = \pm 1$ label positive- and negative-energy branches. The wave functions are [26]

$$\Psi_{\mathrm{n}}(\boldsymbol{r}) \equiv \langle \boldsymbol{r}|\mathrm{n}\rangle = \frac{e^{ik_x x + ik_z z}}{4\pi}\phi_n(\xi)\begin{pmatrix}\mu^+_{n,k_x,s}\\ \mu^-_{n,k_x,s}\end{pmatrix}, \tag{44}$$

where $\phi_n(\xi)$ are the harmonic-oscillator functions

$$\phi_n(\xi) = \frac{1}{\sqrt{LC_n}}\mathrm{H}_n(\xi)e^{-1/2\xi^2}, \quad (45)$$

in which $\mathrm{H}_n(\xi)$ are the Hermite polynomials, and $C_n = \sqrt{2^n n!\sqrt{\pi}}$. We defined $\mu^{\pm}_{n,k_x,s} = \nu_{n,k_x} \pm s/\nu_{n,k_x}$, where $\nu_{n,k_x} = \sqrt{mc^2/E_{n,k_z}}$.

We want to calculate an average packet velocity in a magnetic field. We can, as before, introduce the Heisenberg picture for the time-dependent velocity operator. Then, the jth component of the average velocity is [see Eq. (18)]

$$\langle\hat{v}_j(t)\rangle = \langle W|\tau_3 e^{i\hat{H}t/\hbar}\hat{v}_j e^{-i\hat{H}t/\hbar}|W\rangle, \quad (46)$$

where $\hat{v}_j = \partial\hat{H}/\partial\hat{p}_j$. For the Hamiltonian (40) in the asymmetric gauge, we find

$$\hat{v}_x = (\tau_3 + i\tau_2)\left(\frac{\hat{p}_x - qBy}{m}\right), \quad (47)$$

$$\hat{v}_y = (\tau_3 + i\tau_2)\frac{\hat{p}_y}{m}, \quad (48)$$

$$\hat{v}_z = (\tau_3 + i\tau_2)\frac{\hat{p}_z}{m}. \quad (49)$$

The unity operator is now

$$\hat{1} = \sum_{\mathrm{n}} |\mathrm{n}\rangle\langle\mathrm{n}|s_n\tau_3, \quad (50)$$

where the states $\langle\boldsymbol{r}|\mathrm{n}\rangle$ are given in Eq. (44), and $s_n = \pm 1$ are the quantum numbers associated with the states $|\mathrm{n}\rangle$. The proof of the above identity is given in Appendix C. Using the unity operator, we expand the packet $|W\rangle$ in terms of the eigenstates of $\hat{H}$ [see Eq. (16)],

$$|W\rangle = \sum_{\mathrm{n}} s_{\mathrm{n}}|\mathrm{n}\rangle\langle\mathrm{n}|\tau_3|W\rangle \equiv \sum_{\mathrm{n}} s_{\mathrm{n}}|\mathrm{n}\rangle W_{\mathrm{n}}, \quad (51)$$

where $W_{\mathrm{n}} = \langle\mathrm{n}|\tau_3|W\rangle$. By inserting $|W\rangle$ into Eq. (46), one obtains [see Eq. (18)]

$$\langle\hat{v}_j(t)\rangle = \sum_{\mathrm{nm}} s_{\mathrm{n}}s_{\mathrm{m}}W^*_{\mathrm{n}}W_{\mathrm{m}}\langle\mathrm{n}|\tau_3 e^{i\hat{H}t/\hbar}\hat{v}_j e^{-i\hat{H}t/\hbar}|\mathrm{m}\rangle. \quad (52)$$

There is $e^{-i\hat{H}t/\hbar}|\mathrm{n}\rangle = e^{-i\omega_{\mathrm{n}}t}|\mathrm{n}\rangle$, where $\omega_{\mathrm{n}} = s_{\mathrm{n}}E_{n,k_x}/\hbar$. Proceeding the same way as in Sec. II, we have

$$\langle\mathrm{n}|\tau_3 e^{i\hat{H}t/\hbar} = \langle\mathrm{n}|e^{i\hat{H}^\dagger t/\hbar}\tau_3 = e^{i\omega_{\mathrm{n}}t}\langle\mathrm{n}|\tau_3, \quad (53)$$

which finally gives

$$\langle\hat{v}_j(t)\rangle = \sum_{\mathrm{nm}} s_{\mathrm{n}}s_{\mathrm{m}}W^*_{\mathrm{n}}W_{\mathrm{m}}e^{i(\omega_{\mathrm{n}}-\omega_{\mathrm{m}})t}\langle\mathrm{n}|\tau_3\hat{v}_j|\mathrm{m}\rangle. \quad (54)$$

The matrix elements of velocity operators calculated between the states $|\mathrm{n}\rangle$, $|\mathrm{m}\rangle$ are

$$\langle\mathrm{n}|\tau_3\hat{v}_y|\mathrm{m}\rangle = c\frac{\lambda_c}{i\sqrt{2}L}\nu_{n,k_z}\nu_{m,k_z}\delta_{k_x,k'_x}\delta_{k_z,k'_z} \times(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}), \quad (55)$$

$$\langle\mathrm{n}|\tau_3\hat{v}_x|\mathrm{m}\rangle = c\frac{\lambda_c}{\sqrt{2}L}\nu_{n,k_z}\nu_{m,k_z}\delta_{k_x,k'_x}\delta_{k_z,k'_z} \times(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}), \quad (56)$$

$$\langle\mathrm{n}|\tau_3\hat{v}_z|\mathrm{m}\rangle = \frac{p_z}{m}\nu_{n,k_z}\nu_{m,k_z}\delta_{k_x,k'_x}\delta_{k_z,k'_z}\delta_{m,n}. \quad (57)$$

The matrix elements of $\hat{v}_y$ and $\hat{v}_x$ are nonzero for the states with $m = n \pm 1$ and arbitrary indexes s_n and s_m. The matrix elements of $\hat{v}_z$ are nonzero for $m = n$ and arbitrary indexes s_n and s_m. To simplify further analysis, we assume the initial wave packet $W(\boldsymbol{r})$ to be in a separable form $w(\boldsymbol{r}) = w_{xy}(x,y)w_z(z)$ [cf. Eq. (11)], which gives

$$W(\boldsymbol{r}) = W_{xy}(x,y)W_z(z). \quad (58)$$

Then, there is

$$W_{\mathrm{n}} = \langle\mathrm{n}|\tau_3|W\rangle = \mu^+_{n,k_z}g_z(k_z)F_n(k_z), \quad (59)$$

where

$$F_n(k_x) = \frac{1}{\sqrt{2LC_n}}\int_{-\infty}^{\infty} g_{xy}(k_x,y)e^{-\frac{1}{2}\xi^2}\mathrm{H}_n(\xi)dy, \quad (60)$$

in which

$$g_{xy}(k_x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w_{xy}(x,y)e^{ik_xx}dx \quad (61)$$

and

$$g_z(k_z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w_z(z)e^{ik_zz}dz. \quad (62)$$

For $\langle\hat{v}_y(t)\rangle$, we obtain

$$\begin{aligned}\langle\hat{v}_y(t)\rangle = c\frac{\lambda_c}{2\sqrt{2}iL}\sum_{n,m=0}^{\infty}\int_{-\infty}^{\infty} dk_z|g_z(k_z)|^2(\sqrt{n+1}\delta_{m,n+1}\\ - \sqrt{n}\delta_{m,n-1})U_{n,m}\{(1+\nu_m^2\nu_n^2)\cos(\omega_m t - \omega_n t)\\ + (\nu_m^2\nu_n^2 - 1)\cos(\omega_m t + \omega_n t)\\ + i(\nu_m^2 + \nu_n^2)\sin(\omega_m t - \omega_n t)\\ + i(\nu_m^2 - \nu_n^2)\sin(\omega_m t + \omega_n t)\}.\end{aligned}$$

In the above expressions, we use the notation $\nu_n \equiv \nu_{n,k_z}$ and $\omega_n = E_{n,k_z}/\hbar$, and

$$U_{n,m} = \int_{-\infty}^{\infty} F^*_n(k_x)F_m(k_x)dk_x. \quad (63)$$

For the Gaussian packet of Eq. (11), one can obtain analytical expressions for $U_{n,m}$; see Appendix D. After performing the summation over m and changing $n \to n+1$ in $\delta_{m,n-1}$ terms, we finally obtain

$$\begin{aligned}\langle\hat{v}_y(t)\rangle = -c\frac{\lambda_c}{2\sqrt{2}L}\sum_{n=0}^{\infty}\sqrt{n+1}(U_{n+1,n} + U_{n,n+1})\\ \times\int_{-\infty}^{\infty}|g_z(k_z)|^2\{(\nu_{n+1}^2 + \nu_n^2)\sin(\omega_{n+1}t - \omega_n t)\\ + (\nu_{n+1}^2 - \nu_n^2)\sin(\omega_{n+1}t + \omega_n t)\}dk_z,\end{aligned} \quad (64)$$

$$\begin{aligned}\langle\hat{v}_x(t)\rangle = -c\frac{\lambda_c}{2\sqrt{2}L}\sum_{n=0}^{\infty}\sqrt{n+1}(U_{n+1,n} + U_{n,n+1})\\ \times\int_{-\infty}^{\infty}|g_z(k_z)|^2\{(1 + \nu_{n+1}^2\nu_n^2)\cos(\omega_{n+1}t - \omega_n t)\\ + (1 - \nu_{n+1}^2\nu_n^2)\cos(\omega_{n+1}t + \omega_n t)\}dk_z,\end{aligned} \quad (65)$$

$$\langle \hat{v}_z(t)\rangle = \frac{c\lambda_c}{2}\sum_{n=0}^{\infty} U_{n,n}\int_{-\infty}^{\infty} k_z|g_z(k_z)|^2 \times\left\{\left(1+\nu_n^4\right)+\left(1-\nu_n^4\right)\cos(2\omega_n t)\right\}dk_z. \quad (66)$$

Equations (64)–(66) are our final results for the average velocity of the wave packet in a magnetic field. Both the arguments of sine and cosine functions as well as coefficients ν_n and ν_{n+1} depend on k_z, so all of the integrals vanish in the limit $t \to \infty$, as a consequence of the Riemann-Lebesgue theorem, and the resulting oscillations have a transient character. The velocity of the packet oscillates with many frequencies, $\omega_{n+1} \pm \omega_n$ (or $2\omega_n$ for $\hat{v}_z$), but in practice the spectrum is limited to a few frequencies related to the largest coefficients $U_{n+1,n}$ and $U_{n,n}$. The frequencies $\omega_{n+1} - \omega_n$ correspond to the intraband transitions and they can be interpreted as the cyclotron resonances. These frequencies do not appear in $\hat{v}_z$ velocity. On the other hand, the frequencies $\omega_{n+1} + \omega_n$ and $2\omega_n$ (for $\hat{v}_z$) correspond to the interband transitions and they can be interpreted as the *Zittebewegung* components of the motion, in analogy to the situation at zero field. The motion in the x-y directions requires that $k_{0x} \neq 0$ because for $k_{0x} = 0$ all of the coefficients $U_{n+1,n}$ and $U_{n,n+1}$ vanish [27]. For the motion in the z direction, one needs only that $k_{0z} \neq 0$ because the coefficients $U_{n,n}$ are nonzero for any k_{0x} value [27].

Considering the nonrelativistic limit in Eqs. (64)–(66), there is $\hbar\omega_c \ll mc^2$ and $\hbar k_z \ll mc$, so that $\omega_{n+1} - \omega_n \simeq \hbar\omega_c$ and $\omega_{n+1} + \omega_n \simeq 2mc^2/\hbar$. In this limit, there is $\nu_{n+1} \simeq \nu_n \simeq 1$, and the ZB part of the velocity is nearly zero. In this case, we may decouple in Eqs. (64)–(66) the summation over n and integration over k_z. This gives [27]

$$\sum_{n=0}^{\infty}\sqrt{n+1}\,U_{n+1,n} = -\frac{k_{0x}L}{\sqrt{2}}, \quad (67)$$

$$\sum_{n=0}^{\infty} U_{n,n} = 1. \quad (68)$$

By integrating over k_z, one gets

$$\langle \hat{v}_y(t)\rangle \simeq \frac{\hbar k_{0x}}{m}\sin(\omega_c t), \quad (69)$$

$$\langle \hat{v}_x(t)\rangle \simeq \frac{\hbar k_{0x}}{m}\cos(\omega_c t), \quad (70)$$

$$\langle \hat{v}_z(t)\rangle \simeq \frac{\hbar k_{0z}}{m}. \quad (71)$$

Thus, in the nonrelativistic limit, the particle moves on a circular orbit with the cyclotron frequency in the x-y plane and a constant velocity in the z direction. Let us introduce a measure of intensity of a magnetic field by its relation to an effective Schwinger field, $\hbar e B_s/m = mc^2$, or, equivalently, by $L_s = \hbar/mc$. There is $B_s = 4.41 \times 10^9 (m/m_e)^2$ T, where m_e is the electron mass. Below we perform calculations for pions π^+ having the mass $m \simeq 273.1\, m_e$, so the effective Schwinger field is $B_s = 3.29 \times 10^{14}$ T.

In Fig. 3, we plot the average packet velocity for three values of the magnetic field. The ellipsoidal packet is selected with a nonzero initial momentum k_{0x}. We assume that the five parameters d_x, d_y, d_z, L, and k_{0x}^{-1} have similar orders of the magnitude, which are the optimal conditions for the appearance of the *Zitterbewegung* phenomenon. In Fig. 3, we

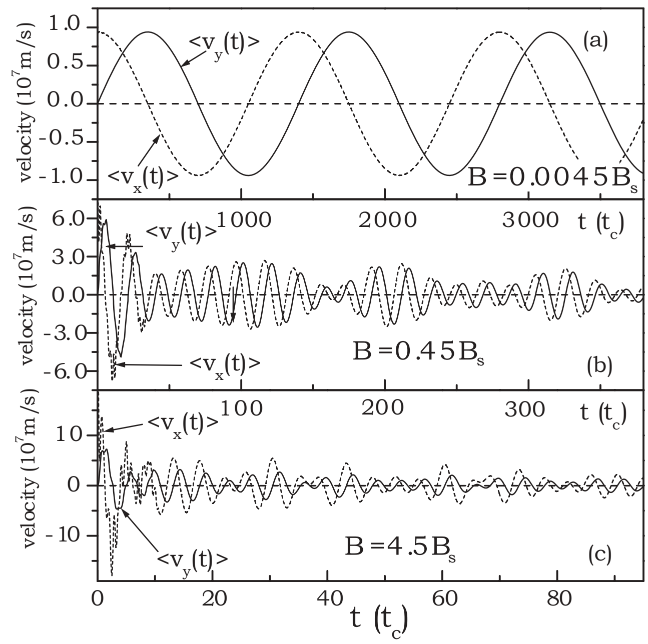

FIG. 3. Time-dependent velocity components for an ellipsoidal wave packet at various magnetic fields. For (a) low fields, cyclotron motion is obtained; for (b), (c) higher fields, packet velocity includes both cyclotron and *Zitterbewegung* frequencies. In all cases, the motion decays in time.

selected parameters $d_x = 0.91(B_s/B)\lambda_c$, $d_y = 0.82(B_s/B)\lambda_c$, $d_z = 0.68(B_s/B)\lambda_c$, $k_{0x} = 0.7(B/B_s)\lambda_c^{-1}$, and $k_{0z} = 0$, where B_s is the effective Schwinger field. For $B = 4.5B_s$, we set $k_{0x} = \lambda_c^{-1}$. For low fields ($B = 0.0045B_s$), the packet moves on a circular orbit; see Eqs. (69) and (70). For such fields, the ZB components of the motion are negligible. For higher fields, the packet motion includes both the intraband and interband (ZB) components so that several frequencies give significant contributions to the motion. In all cases, the motion has a transient character, but for low fields its decay time is very long. In Fig. 4, we plot components of the average velocity of a spherical packet in a longer time scale. The collapse-and-revival patterns occur for both velocity components. After a sufficiently long time, the oscillations disappear. In Fig. 5, we show the average velocity $\langle \hat{v}_z(t)\rangle$ of an ellipsoidal packet having the same parameters as those used in Fig. 3. For large magnetic fields, the motion in the z direction is similar to that in the field-free case exhibiting ZB oscillations; see Fig. 1. For smaller fields, the ZB oscillations disappear and only the classical motion remains; see Eq. (71). Finally, it should be mentioned that in the two-dimensional case, the ZB oscillations do not disappear in time [27].

V. SIMULATION OF ZB

The phenomenon of *Zitterbewegung* for relativistic particles in a vacuum has an unfavorable high frequency corresponding to the energy gap between the positive- and negative-energy branches, $\hbar\omega_0 \simeq 2mc^2$, and a very small amplitude on the order of the effective Compton wavelength, $\Delta \boldsymbol{r} \simeq \hbar/(mc)$; see Eq. (9). Thus, similarly to the case of relativistic electrons, one cannot hope at present to observe

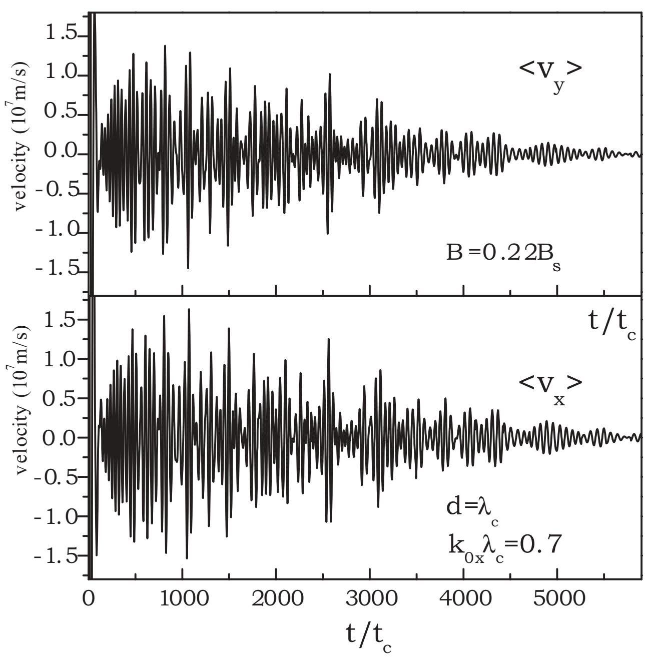

FIG. 4. Average velocity of the spherical wave packet in a longer time scale. The collapse-revival patterns are seen in the ZB oscillations. The motion has a transient character.

directly the ZB in a vacuum. However, it was recently demonstrated by Gerritsma *et al.* that one can simulate the ZB of electrons in a vacuum using trapped ions interacting with laser beams [2]. In this experiment, the authors simulated the linear momentum $\hat{p}_i$ appearing in the Dirac equation with the use of Jaynes-Cumminngs interaction between the electrons on trapped ion levels and the electromagnetic radiation. The

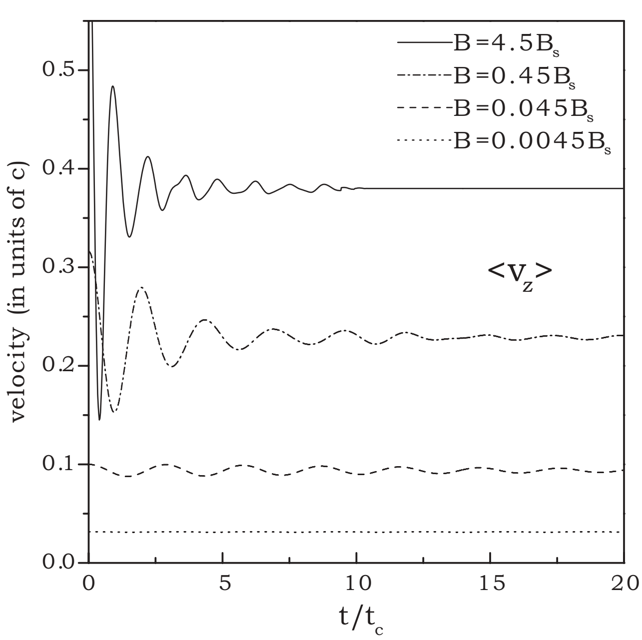

FIG. 5. Average packet velocity $\langle \hat{v}_z(t) \rangle$ in the direction parallel to the magnetic field vs time for four values of B. The transition to the nonrelativistic limit is visible. Parameters are the same as those used for Fig. 3.

decisive advantage of such a simulation is that one can tailor the frequency and amplitude of ZB, making them considerably more favorable than the values for a vacuum. Clearly, it would be of interest to simulate the ZB of a Klein-Gordon particle using similar methods. The problem is that in the KGE, one deals with *squares* of momentum components $\hat{p}^2$, which are more difficult to simulate with the Jaynes-Cumminngs interaction. For this reason, we choose a different route.

The Klein-Gordon equation appears in several *classical* systems, usually as a modification of the wave equation $\Box\phi = 0$. Under some conditions, the KGE is used to describe sound waves in ducts [28,29], electromagnetic waves in the ionosphere [30,31], transverse modes of wave guides [32], and oceanic waves [33]. Below we examine in more detail a model proposed by Morse and Feshbach in which one can simulate the KGE with the use of a piano string and a thin rubber sheet [13]. By employing this example, we demonstrate similarities and differences between ZB in the relativistic KGE and its classical analogues.

Let us consider flexible one-dimensional string in the x direction; see Fig. 6. We assume that the string is uniform with a linear density ρ. A uniform tension T is applied to each element dx of the string. We neglect all other forces acting on the string (e.g., gravity) and the stiffness of the string. Let $y(x,t)$ be a displacement of the element dx of the string from its equilibrium position at an instant t. We assume that $y(x,t)$ is small compared to the length of the string and to the distances to each end of the string. The restoring force acting on each element dx of the string is $F_T = T dx(\partial^2 y/\partial x^2)$ and the displacement $y(x,t)$ of the released string changes according to the wave equation [13]

$$\frac{1}{u^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \tag{72}$$

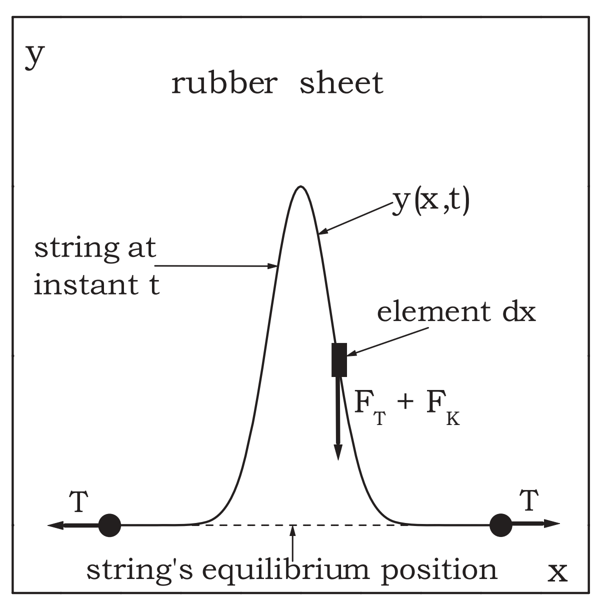

FIG. 6. Classical simulation of the KGE according to Morse and Feshbach [13]. A flexible string is anchored at two points and tension T is applied to each end. The string is also attached to a thin rubber sheet. At instant t, the shape of the string is given by $y(x,t)$. There are two forces acting on each element dx of the string: restoring force F_T due to applied tension and elastic force F_K of stretched rubber.

where $u^2 = T/\rho$. Now we attach the string to an elastic substrate, e.g., to a thin sheet of rubber which can shrink or expand in the y direction. Then, in addition to the restoring force due to the tension, there will be another restoring force due to the elastic rubber acting on each element of the string. If the element dx is displaced to $y(x,t)$ and the rubber sheet obeys the Hook law, then the restoring force acting on the element dx of the string is $F_K(x,t) = -Ky(x,t)dx$, where K is the elastic constant of the rubber sheet. The second Newton law for the element dx of the string having mass $dm = dx\rho$ is $dx\rho(\partial^2 y/\partial t^2) = F_T + F_K$, so the equation of motion of the released string is

$$\frac{1}{u^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - \nu^2 y, \tag{73}$$

where $\nu^2 = K/T$. Equation (73) has the form of the wave KGE with the light speed replaced by u and the mass term $m^2c^2/\hbar^2$ replaced by ν^2. By comparing Eq. (73) with Eq. (27), we find the following correspondence between parameters of the two systems:

$$\frac{T}{\rho} \leftrightarrow c^2, \tag{74}$$

$$\frac{K}{T} \leftrightarrow \frac{m^2c^2}{\hbar^2} = \lambda_c^{-2}. \tag{75}$$

Thus, one can simulate values of c and λ_c by changing material parameters ρ, K, and T.

However, there exist also limitations of such a simulation and they affect a possibility of observation of ZB motion in classical analogues of the KGE. The first difference between the relativistic KGE and its classical counterpart is that the wave function ϕ in the relativistic KGE is not an observable. On the other hand, all classical analogues of ϕ (such as a displacement of the string, the pressure of sound or the oceanic waves, the intensity of the electromagnetic field, etc.) are observable quantities. The second difference is that the relativistic function ϕ is a function of a complex variable, while its classical counterpart is a function of a real variable. A direct consequence of these limitations for observation of ZB in classical systems is that for any real function $\xi(\boldsymbol{r},t)$ being the solution of the KGE, the current density associated with this function is always zero: $\boldsymbol{j} \propto [\xi^*\nabla\xi - (\nabla\xi^*)\xi] = 0$. Therefore, we are not able to simulate directly the current or velocity oscillations calculated in the previous sections.

To overcome this problem, let us consider the motion of a *neutral* particle described by a real field ξ. For simplicity, we assume a one-dimensional KGE that can be simulated by a flexible string attached to an elastic substrate described above. In our calculations, we use the relativistic form of the KGE, but the final results will be presented for parameters corresponding to the flexible string model. We assume the initial wave packet to be a real Gaussian function without an initial momentum,

$$w_0(x) = \frac{1}{(d\sqrt{\pi})^{1/2}}\exp[-x^2/(2d^2)]. \tag{76}$$

Its Fourier transform is

$$w_0(k) = (2d\sqrt{\pi})^{1/2}\exp[-d^2k^2/2]. \tag{77}$$

A real solution $\xi(x,t)$ of the KGE is

$$\xi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w_0(k)\cos(kx-\omega_k t)dk, \tag{78}$$

where $\omega_k = \omega_0\sqrt{1+(k\lambda_c)^2}$. The average current for the real wave packet of Eq. (76) is zero and no ZB occurs. Thus, we turn to other physical operators which do not commute with the KG Hamiltonian (1). Namely, we calculate a *variance* of the position operator for the above real function $\xi(x,t)$,

$$V = \langle\xi|\hat{x}^2|\xi\rangle - \langle\phi|\hat{x}|\phi\rangle^2 = \langle\xi|\hat{x}^2|\xi\rangle, \tag{79}$$

since $\langle\phi|\hat{x}|\phi\rangle = 0$. Assuming $\xi(x,t)$ in the form (78), we have

$$\begin{aligned} V =& \iiint_{\infty}^{\infty} w_0(k)w_0(k')\cos(kx-\omega_k t) \\ &\times \cos(k'x-\omega_{k'}t)x^2dxdkdk' \\ =& \iiint_{\infty}^{\infty} [B_kB_{k'}e^{ix(k+k)} + B_kB_{k'}^*e^{ix(k-k')} \\ &\times B_k^*B_{k'}e^{-ix(k-k')} + B_k^*B_{k'}^*e^{-ix(k+k')}]x^2dxdkdk', \end{aligned} \tag{80}$$

where $B_k = w_0(k)e^{-i\omega_k t}/(4\pi)$. Consider the first of the four terms given above. Because $w_0(k)$ and B_k decay exponentially for $k \to \pm\infty$, one can change the order of integration over x, k, and k', and replace $x^2 \to (\partial/\partial ik)(\partial/\partial ik')$. Then we integrate by parts over k and k', and obtain

$$\begin{aligned} &\iiint_{\infty}^{\infty} B_kB_{k'}e^{ix(k+k)}x^2dxdkdk' \\ &= -\iiint_{\infty}^{\infty} \frac{\partial B_k}{\partial k}\frac{\partial B_{k'}}{\partial k'}e^{ix(k+k)}dxdkdk' \\ &= 2\pi\int_{\infty}^{\infty} \left.\frac{\partial B_k}{\partial k}\frac{\partial B_{k'}}{\partial k'}\right|_{k'=k} dk. \end{aligned} \tag{81}$$

The other three terms in Eq. (80) are calculated similarly. After some manipulations, we find

$$V = V_1^c + V_1^{\rm osc} + V_2^c + V_2^{\rm osc} + V_3, \tag{82}$$

where

$$V_1^c = \frac{d^3}{2\sqrt{\pi}}\int_{\infty}^{\infty} e^{-d^2k^2}(kd)^2dk, \tag{83}$$

$$V_1^{\rm osc} = \frac{d^3}{2\sqrt{\pi}}\int_{\infty}^{\infty} e^{-d^2k^2}(kd)^2\cos(2\omega_k t)dk, \tag{84}$$

$$V_2^c = \frac{d(ct)^2}{2\sqrt{\pi}}\int_{\infty}^{\infty} \frac{e^{-d^2k^2}(k\lambda_c)^2}{1+(k\lambda_c)^2}dk, \tag{85}$$

$$V_2^{\rm osc} = -\frac{d(ct)^2}{2\sqrt{\pi}}\int_{\infty}^{\infty} \frac{e^{-d^2k^2}(k\lambda_c)^2}{1+(k\lambda_c)^2}\cos(2\omega_k t)dk, \tag{86}$$

$$V_3 = \frac{d^2(ct)}{\sqrt{\pi}}\int_{\infty}^{\infty} \frac{e^{-d^2k^2}(k\lambda_c)}{\sqrt{1+(k\lambda_c)^2}}\sin(2\omega_k t)dk. \tag{87}$$

The term V_3 is odd in k and it vanishes upon the integration. For $t = 0$, the variance in Eq. (82) is equal to the variance $V_0 = d^2/2$ of the initial packet $w_0(x)$. The variance given in Eq. (82) consists of oscillating and nonoscillating terms. For large times, the nonoscillating terms grow in time as $d^2/2 + Ct^2$, where C is a constant depending on d. The quadratic dependence of the variance on time is similar to

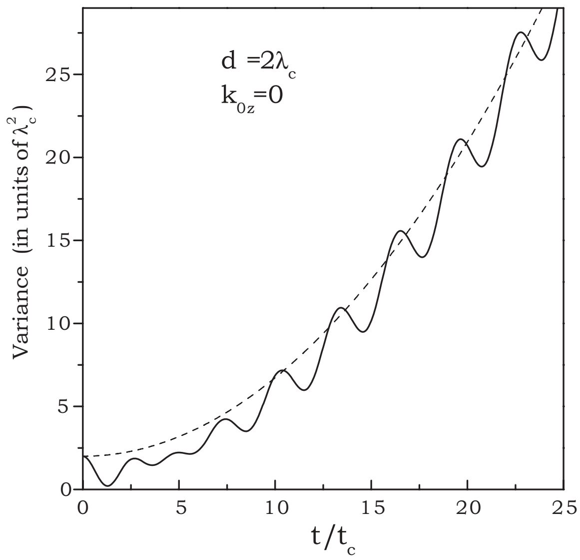

FIG. 7. Calculated classical variance of the position of the wave packet propagating according to the KGE: nonoscillating part of variance (dashed line) and total variance (solid line). For string oscillations analyzed in the text, there is $\lambda_c^s = 4.47$ mm and $t_c^s = 2.37 \times 10^{-5}$ s.

that of a Gaussian wave packet in the nonrelativistic quantum mechanics. The oscillations in Eqs. (84) and (86) have the same interband frequency $2\omega_k$ as the velocity oscillation given in Eq. (8). Therefore, the oscillations of variance of the position operator can be interpreted as a signature of *Zitterbewegung* in classical systems. The term $V_1^{\rm osc}$ has a decaying character and it vanishes after a few oscillations. The $V_2^{\rm osc}$ term gives persistent oscillations because of the presence of the t^2 factor in front of the integral. To estimate the time dependence of these oscillations, we consider the limit of large packet widths $d \gg \lambda_c$. In this case, the Gaussian function in Eq. (86) restricts the integration to small values of k. Then we may disregard the $(k\lambda_c)^2$ term in the denominator of the integrand and expand ω_k under the cosine function. This gives approximately

$$V_2^{\rm osc} \simeq -\frac{d(ct)^2}{2\sqrt{\pi}} \int_{\infty}^{\infty} e^{-d^2k^2}(k\lambda_c)^2 \cos\left[\omega_0 t\left(2 + k^2\lambda_c^2\right)\right]dk$$
$$= -\frac{d(ct)^2}{4} \sum_{\eta=\pm 1} \frac{e^{2i\eta\omega_0 t}}{(d^2 + i\eta\omega_0 t)^{3/2}}. \quad (88)$$

For large time, we may approximate, in Eq. (88),

$$V_2^{\rm osc} \simeq -C_d t^{1/2} \cos(2\omega_0 t), \quad (89)$$

where C_d is a constant depending on d. Thus, the oscillations of variance are persistent, their amplitude increases with time as $t^{1/2}$, and their frequency is $2\omega_0$. Since nonoscillating terms V_2^c increase as t^2, the total variance of the packet has a quadratic time dependence with superimposed oscillations. This behavior is illustrated in Fig. 7. In our classical considerations, we do not face the problem of negative variances that can occur for some quantum systems; see Refs. [34,35]. For $t < 5t_c$, the oscillations have an irregular character because of the contribution of the $V_1^{\rm osc}$ term.

Estimating the characteristic frequency $2\omega_0$ for the flexible string attached to the elastic substrate, we have

$$\omega_0^2 = \frac{m^2c^4}{\hbar^2} = c^2 \times \frac{1}{\lambda_c^2} \longleftrightarrow \left(\frac{T}{\rho}\right)\left(\frac{K}{T}\right) = \frac{K}{\rho}, \quad (90)$$

so that the analog of the relativistic frequency ω_0 does not depend on the applied tension. Taking a piano copper string of the bulk density $\rho_{3D} = 8940$ kg/m^3 and having a cross section of radius $r = 1$ mm, one gets a linear density $\rho = \pi r^2 \rho_{3D} = 2.81 \times 10^{-2}$ kg/m. We identify the rubber elastic constant K with the Young modulus $K = 0.05 \times 10^9$ N/m^2. Then the analog of the ZB frequency given in Eq. (90) is $2\omega_0^s = 8.44 \times 10^4$ s^{-1}, i.e., the corresponding frequency is $f_0 = 13.43$ kHz, which can be heard by the human ear. The characteristic time of the ZB oscillations is $t_c^s = 1/\omega_0^s = 2.37 \times 10^{-5}$ s. Assuming the tension of the string $T = 1000$ N, we find from Eq. (75) that the simulated Compton wavelength is $\lambda_c^s = 4.47$ mm. The initial wave packet should have widths d on the order of a few λ_c^s, i.e., of a few centimeters, and it will move with the velocity $u = 188.7$ m/s; see Eq. (74). Thus, it is really possible to simulate and observe the *Zitterbewegung*phenomenon in this system. Finally, we observe that in classical simulations, all of the involved quantities are well-defined observables. Since the classical KGE does not reproduce but only *simulates* the quantum KGE, we are allowed to consider quantities which are not well defined in the quantum world.

VI. DISCUSSION

Our main results for the ZB of KG particles in the absence of fields are shown in Fig. 1 and in the presence of a magnetic field in Figs. 3–5. It is not our purpose here to consider the difficulties of the one-particle Klein-Gordon equation, but we keep them in mind. In particular, we do not consider particle trajectories as they are believed to not be well defined; see [20]. On the other hand, we describe average particle velocities and currents both in the Hamiltonian and wave formalisms. The results can be compared to those for relativistic electrons in a vacuum described by the Dirac equation as well as for electrons in solids.

Similarly to the Dirac electrons, the ZB phenomenon of KG particles is due to the interference of positive- and negative-energy states. In the nonrelativistic limit, one of the two components progressively vanishes and the ZB contribution to the motion disappears. This can be clearly seen in Figs. 3 and 5, as well as in Fig. 3 of Ref. [27] for the Dirac electrons. If particles are described by wave packets, then the ZB motion decays in time; see our Fig. 1 for KE particles and Fig. 2 of Ref. [36] for the Dirac electrons. This is a general consequence of the Riemann-Lebesgue theorem, as indicated by Lock [24], calculated by the present authors [40], and experimentally confirmed by Gerritsma *et al.* [2]. In all cases, the basic frequency of the ZB oscillations is given by the energy difference between the positive- and negative-energy branches, $\hbar\omega_Z \simeq 2mc^2$, with the corresponding particle mass. The main difference with the Dirac electrons is the spin. For KG particles, the interband ZB frequencies in a magnetic field do not include the spin energies, and one does not deal with the Fermi sea for the negative-energy branches, etc. The KG Hamiltonian is quadratic in momenta, which does not allow a direct simulation with the use of the Jaynes-Cummings interaction. In our calculations, we assume the KG particle to be represented by Gaussian wave packets; cf. Eq. (11). It is not at present possible to prepare in a vacuum a mesonlike particle in this form. However, one can produce Gaussian wave packets in simulations by trapped ions, as well as in the case of solid-state electrons [37].

It should be mentioned that similarly to the case of the Dirac equation, one can define for a KG particle a *mean* position operator [10,12,23]

$$\hat{\boldsymbol{X}} = \hat{\boldsymbol{x}} - i\hbar\frac{\tau_1 \boldsymbol{p}}{2p_0^2}, \tag{91}$$

where $p_0 = \sqrt{m^2c^2 + p^2}$. The time derivative of $\hat{\boldsymbol{X}}$ satisfies the relation

$$\frac{d\hat{\boldsymbol{X}}}{dt} = \frac{c\boldsymbol{p}\tau_3}{p_0}, \tag{92}$$

so that the mean velocity (92) does not exhibit the *Zitterbewegung* phenomenon, but has a strictly classical behavior. The eigenfunctions of $\hat{\boldsymbol{X}}$ are not localized in configuration space, but rather extend over a radius on the order of $\lambda_c = \hbar/mc$ [23]. The operator $\hat{\boldsymbol{X}}$ can be obtained by means of a transformation similar to that of Foldy-Wouthuysen [10,23], or by using a set of physically reasonable requirements [38].

One could ask whether the noncausality aspects of the Klein-Gordon equation in the ST form mentioned in Sec. I do not interfere with the viability of the *Zitterbewegung* of spin-zero particles. Guertin and coworkers considered both the ZB [9] and noncausal features of the KG equation [22] and did not find inconsistency between the two. We believe that the additional factors we introduced into the problem, namely, the wave packet and a constant magnetic field, should not interfere with the above consistency. In Appendix B, we explicitly indicate how to eliminate nonphysical components of the wave packet.

As to the *Zitterbewegung* of electrons in narrow-gap semiconductors and, in particular, in zero-gap monolayer graphene, one should emphasize that although it is also described using a two-band model of band structure [4], its physical nature is completely different from the ZB of particles in a vacuum. The ZB in semiconductors or in graphene results from the electron motion in a periodic potential [39]. In a zero-gap situation in graphene, the ZB frequency is given by the difference of energies between positive- and negative-energy bands corresponding to the average value of quasimomentum $\hbar\boldsymbol{k}_0$ for the wave packet [40]. A one-dimensional system which strongly resembles the KG particle in a vacuum is presented by electrons in carbon nanotubes: one can neglect the electron spin and have an energy gap controlled by the tube's diameter [41]. The resulting ZB frequency and amplitude have values easily accessible experimentally. On the other hand, it is at present unclear how to follow the dynamics of a single electron in a solid. As to KG particles in a vacuum, one is bound to recourse to simulations since the ZB frequency and amplitude as well as field intensities necessary to see ZB effects in the presence of a magnetic field exceed the present experimental possibilities.

We present a classical simulation of ZB by using a mechanical system and calculate the oscillating variance of the position of the wave packet. The variance of the position operator for the Dirac Hamiltonian was calculated by Barut and Malin [42], who found it to be sensitive to the ZB of electrons in a vacuum. The present authors analyzed in Ref. [40] the variance of the position operator in bilayer graphene and found its oscillating character with the frequency equal to that of ZB. Our present simulation of variance (the second moment of the position operator) uses a real wave function and the variance is always positive. It was shown by Lev *et al.* [34] that when the wave function is complex, the variance can be positive or negative. In this case, one may be forced to calculate the first or third moment of the position operator.

One should finally remark that attempts are constantly made in the literature to overcome the above-mentioned difficulties in the interpretation of the position operator in the KG equation. In particular, Mostafazadeh [43] proposed a redefinition of the scalar product of solutions to the KGE, which allows one to obtain a positively defined probability distribution of the position. Semenov *et al.* [35] proposed to limit the allowed solutions of the KG equation to those having positive-definite probability distributions. They showed that the physical solutions of the KGE fulfill this criterion. If the above attempts are accepted, then one could analyze the ZB of the position operator for KG particles; see Eq. (9).

VII. SUMMARY

We considered the trembling motion (*Zitterbewegung*) of relativistic spin-zero particles in the absence of fields and in the presence of a magnetic field using the Klein-Gordon equation. We aimed to describe physical observables (currents and velocities), calculating quantities averaged with the use of Gaussian wave packets. Surprisingly, the calculated particle velocities can exceed the velocity of light for sufficiently large momenta, indicating that the KGE does not possess the automatic restriction of relativity. We showed that the trembling motion has a decaying character resulting from an interference of positive- and negative-energy subpackets moving with different velocities. In the presence of a magnetic field, there exist many interband frequencies that contribute to the *Zitterbewegung*. On the other hand, in the limit of nonrelativistic energies, the interband ZB components vanish, while the intraband components reduce to the cyclotron motion with a single frequency. The trembling motion was simulated using the classical system obeying the Klein-Gordon equation—a stretched string attached to a rubber sheet. The calculated variance of the position of the string shaped initially as a Gaussian packet exhibits oscillations corresponding to the *Zitterbewegung* with the correct frequency.

APPENDIX A

In this appendix, we calculate an exact time dependence of current operators for a KG particle in a magnetic field. We define the creation and annihilation operators

$$\begin{aligned} \hat{a} &= (\xi + \partial/\partial\xi)/\sqrt{2}, \\ \hat{a}^{\dagger} &= (\xi - \partial/\partial\xi)/\sqrt{2}, \end{aligned} \tag{A1}$$

and rewrite Eq. (40) in the form

$$\hat{H} = \hat{\mathcal{T}}\left[\hbar\omega_c\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)+\frac{\hbar^2k_z^2}{2m}\right]+\tau_3 mc^2, \tag{A2}$$

where $\omega_c = qB/m$ is the cyclotron frequency and $\hat{\mathcal{T}} = (\tau_3 + i\tau_2)$. The current density is

$$\boldsymbol{j} = \frac{\hbar}{2im}[\psi^\dagger\tau_3\hat{\mathcal{T}}\nabla\psi - (\nabla\psi^\dagger)\tau_3\hat{\mathcal{T}}\psi] - \frac{e}{mc}\boldsymbol{A}\psi^\dagger\tau_3\hat{\mathcal{T}}\psi, \tag{A3}$$

and the average current is $\langle \boldsymbol{j}\rangle = \int \boldsymbol{j}d^3\boldsymbol{r}$. We introduce a *current operator* $\hat{\boldsymbol{J}}$ in such a way that for $\boldsymbol{j}$ given in Eq. (A3), there is

$$\langle\psi|\tau_3\hat{\boldsymbol{J}}|\psi\rangle = \int \boldsymbol{j}d^3\boldsymbol{r}. \tag{A4}$$

Note the presence of τ_3 in the matrix element. In the asymmetric gauge, one has

$$\langle j_x\rangle = -\frac{i\hbar}{2m}\int\left(\psi^\dagger\tau_3\hat{\mathcal{T}}\frac{\partial\psi}{\partial x} - \frac{\partial\psi^\dagger}{\partial x}\tau_3\hat{\mathcal{T}}\psi\right)d^3\boldsymbol{r} - \frac{qB}{mc}\int(\psi^\dagger\tau_3\hat{\mathcal{T}}y\psi)d^3\boldsymbol{r}, \tag{A5}$$

$$\langle j_y\rangle = -\frac{i\hbar}{2m}\int\left(\psi^\dagger\tau_3\hat{\mathcal{T}}\frac{\partial\psi}{\partial y} - \frac{\partial\psi^\dagger}{\partial y}\tau_3\hat{\mathcal{T}}\psi\right)d^3\boldsymbol{r}. \tag{A6}$$

Below we assume the function ψ to be Gaussian-like. In that case, we may simplify the above expressions for the average current by integrating by parts the terms including derivatives of $\psi^\dagger$,

$$\langle j_x\rangle = -\frac{i\hbar}{m}\int\left(\psi^\dagger\tau_3\hat{\mathcal{T}}\frac{\partial\psi}{\partial x}\right)d^3\boldsymbol{r} - \frac{qB}{mc}\int(\psi^\dagger\tau_3\hat{\mathcal{T}}y\psi)d^3\boldsymbol{r}, \tag{A7}$$

$$\langle j_y\rangle = -\frac{i\hbar}{m}\int\left(\psi^\dagger\tau_3\hat{\mathcal{T}}\frac{\partial\psi}{\partial y}\right)d^3\boldsymbol{r}, \tag{A8}$$

so that the components of the current operator are

$$\hat{J}_x = -\frac{i\hbar}{m}\hat{\mathcal{T}}\frac{\partial}{\partial x} - \frac{qB}{mc}\hat{\mathcal{T}}\hat{y}, \tag{A9}$$

$$\hat{J}_y = -\frac{i\hbar}{m}\hat{\mathcal{T}}\frac{\partial}{\partial y}. \tag{A10}$$

In the Heisenberg picture, the time-dependent current operator is

$$\hat{\boldsymbol{J}}(t) = e^{i\hat{H}t/\hbar}\hat{\boldsymbol{J}}(0)e^{-i\hat{H}t/\hbar}, \tag{A11}$$

where $\hat{H}$ is given in Eq. (A2). Our task is to calculate the time evolution of the current operators $\hat{J}_x(t)$ and $\hat{J}_y(t)$. By averaging these operators over the state ψ, as shown in Eq. (A4), one obtains the time-dependent charge current corresponding to ψ.

It is convenient to rewrite current operators in Eqs. (A9) and (A10) in the form

$$\hat{J}_x = -\frac{i\hbar}{m}\hat{\mathcal{P}} - \frac{qB}{\sqrt{2}mc}(\hat{\mathcal{J}} + \hat{\mathcal{J}}^\dagger), \tag{A12}$$

$$\hat{J}_y = -\frac{i\hbar}{\sqrt{2}m}(\hat{\mathcal{J}} - \hat{\mathcal{J}}^\dagger), \tag{A13}$$

where we introduce three auxiliary operators:

$$\hat{\mathcal{P}} = (\tau_3 + i\tau_2)\frac{\partial}{\partial x} \equiv \hat{\mathcal{T}}\frac{\partial}{\partial x}, \tag{A14}$$

$$\hat{\mathcal{J}} = (\tau_3 + i\tau_2)\hat{a} \equiv \hat{\mathcal{T}}\hat{a}, \tag{A15}$$

$$\hat{\mathcal{J}}^\dagger = (\tau_3 + i\tau_2)\hat{a}^\dagger \equiv \hat{\mathcal{T}}\hat{a}^\dagger. \tag{A16}$$

We calculate the time dependence of $\hat{\mathcal{J}}$, $\hat{\mathcal{J}}^\dagger$, and $\hat{\mathcal{P}}$ in a way similar to that described in Ref. [27]. Consider first the operator $\hat{\mathcal{P}}$. From the equation of motion $\hat{\mathcal{P}}_t = (i/\hbar)[\hat{H},\hat{\mathcal{P}}]$, one has

$$\hat{\mathcal{P}}_t = \frac{imc^2}{\hbar}[\tau_3,\hat{\mathcal{P}}] = 2i\omega_0\tau_1\frac{\partial}{\partial x}, \tag{A17}$$

where we used $\hat{\mathcal{T}}^2 = 0$. Since $\{\hat{H},\hat{\mathcal{P}}_t\} = 0$, there is $[\hat{H},\hat{\mathcal{P}}_t] = 2\hat{H}\hat{\mathcal{P}}_t - \{\hat{H},\hat{\mathcal{P}}_t\} = 2\hat{H}\hat{\mathcal{P}}_t$, and one obtains

$$\hat{\mathcal{P}}_{tt} = \frac{2i}{\hbar}\hat{H}\hat{\mathcal{P}}_t. \tag{A18}$$

We solve this equation for $\hat{\mathcal{P}}_t$ and then integrate the solution over time,

$$\hat{\mathcal{P}}(t) = \frac{\hbar}{2i\hat{H}}e^{2i\hat{H}t/\hbar}\hat{\mathcal{P}}_t(0) + \hat{\mathcal{C}}, \tag{A19}$$

where $\hat{\mathcal{C}}$ is a constant of integration. Applying the initial conditions $\hat{\mathcal{P}}(0) = \hat{\mathcal{T}}(\partial/\partial x)$, $\hat{\mathcal{P}}_t(0) = 2i\omega_0\tau_1(\partial/\partial x)$, and using the identity $\hat{H}^{-1} = \hat{H}/E^2$, we have

$$\hat{\mathcal{P}}(t) = \hat{\mathcal{T}}\frac{\partial}{\partial x} + \frac{\hbar\omega_0\hat{H}}{E^2}(e^{2i\hat{H}t/\hbar} - 1)\tau_1\frac{\partial}{\partial x}. \tag{A20}$$

It is seen that $\hat{\mathcal{P}}(t)$ in Eq. (A20) satisfies the initial conditions for $\hat{\mathcal{P}}(0)$ and $\hat{\mathcal{P}}_t(0)$. The form of $\hat{\mathcal{P}}(t)$ given above resembles results obtained for the position operator in the field-free case by Fuda and Furlani [10].

Now we turn to the operators $\hat{\mathcal{J}}$ and $\hat{\mathcal{J}}^\dagger$. From Eqs. (A15) and (A16), one has

$$\hat{\mathcal{J}}_t = 2i\omega_0\tau_1\hat{a}, \tag{A21}$$

$$\hat{\mathcal{J}}_t^\dagger = 2i\omega_0\tau_1\hat{a}^\dagger, \tag{A22}$$

where $\omega_0 = mc^2/\hbar$. Using $[\hat{a},\hat{a}^\dagger] = 1$, one obtains

$$\{\hat{H},\hat{\mathcal{J}}_t\} = -2i\omega_0\hbar\omega_c\hat{\mathcal{J}}, \tag{A23}$$

$$\{\hat{H},\hat{\mathcal{J}}_t^\dagger\} = +2i\omega_0\hbar\omega_c\hat{\mathcal{J}}^\dagger. \tag{A24}$$

Upon applying the identities

$$[\hat{H},\hat{\mathcal{J}}_t] = +2\hat{H}\hat{\mathcal{J}}_t - \{\hat{H},\hat{\mathcal{J}}_t\}, \tag{A25}$$

$$[\hat{H},\hat{\mathcal{J}}_t^\dagger] = -2\hat{\mathcal{J}}_t^\dagger\hat{H} + \{\hat{H},\hat{\mathcal{J}}_t^\dagger\}, \tag{A26}$$

we get

$$\hat{\mathcal{J}}_{tt} = +(2i/\hbar)\hat{H}\hat{\mathcal{J}}_t - 2\omega_0\omega_c\hat{\mathcal{J}}, \tag{A27}$$

$$\hat{\mathcal{J}}_{tt}^\dagger = -(2i/\hbar)\hat{\mathcal{J}}_t^\dagger\hat{H} - 2\omega_0\omega_c\hat{\mathcal{J}}^\dagger. \tag{A28}$$

In Eqs. (A27) and (A28), we eliminate terms with the first derivatives using the substitutions $\hat{\mathcal{J}} = \exp(+i\hat{H}t/\hbar)\hat{\mathcal{B}}$ and

$\hat{\mathcal{J}}^\dagger = \hat{\mathcal{B}}^\dagger \exp(-i\hat{H}t/\hbar)$, respectively. This gives

$$\hat{\mathcal{B}}_{tt} = -(\hat{\Omega}^2 + 2\omega_c\omega_0)\hat{\mathcal{B}}, \tag{A29}$$

$$\hat{\mathcal{B}}^\dagger_{tt} = -\hat{\mathcal{B}}^\dagger(\hat{\Omega}^2 + 2\omega_c\omega_0), \tag{A30}$$

where $\hat{\Omega} = \hat{H}/\hbar$. In the above equations, the operator

$$\hat{\mathcal{M}}^2 = \hat{\Omega}^2 + 2\omega_c\omega_0 \tag{A31}$$

stands on the left-hand side of $\hat{\mathcal{B}}$, but on the right-hand side of $\hat{\mathcal{B}}^\dagger$. The solutions to Eqs. (A29) and (A30) are

$$\hat{\mathcal{B}} = e^{-i\hat{\mathcal{M}}t}\hat{\mathcal{C}}_1 + e^{i\hat{\mathcal{M}}t}\hat{\mathcal{C}}_2, \tag{A32}$$

$$\hat{\mathcal{B}}^\dagger = \hat{\mathcal{C}}_1^\dagger e^{-i\hat{\mathcal{M}}t} + \hat{\mathcal{C}}_2^\dagger e^{i\hat{\mathcal{M}}t}, \tag{A33}$$

where $\hat{\mathcal{M}} = +\sqrt{\hat{\mathcal{M}}^2}$ is the positive root of $\hat{\mathcal{M}}^2$. Both $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ and their complex conjugates are time-independent operators.

Using the initial conditions $\hat{\mathcal{B}}(0) = \hat{\mathcal{J}}(0) = \hat{\mathcal{T}}\hat{a}$ and $\hat{\mathcal{B}}_t(0) = \hat{\mathcal{J}}_t(0) = +2i\omega_0\tau_1\hat{a}$ [see Eq. (A21)] and similar expressions for $\hat{\mathcal{B}}^\dagger(0)$ and $\hat{\mathcal{B}}^\dagger_t(0)$, we find that $\hat{\mathcal{J}}(t) = \hat{\mathcal{J}}_1(t) + \hat{\mathcal{J}}_2(t)$, where

$$\hat{\mathcal{J}}_1(t) = \tfrac{1}{2}e^{i\hat{\Omega}t}e^{-i\hat{\mathcal{M}}t}[\hat{\mathcal{J}}(0) + \hat{\mathcal{M}}^{-1}\hat{\mathcal{J}}(0)\hat{\Omega}], \tag{A34}$$

$$\hat{\mathcal{J}}_2(t) = \tfrac{1}{2}e^{i\hat{\Omega}t}e^{+i\hat{\mathcal{M}}t}[\hat{\mathcal{J}}(0) - \hat{\mathcal{M}}^{-1}\hat{\mathcal{J}}(0)\hat{\Omega}]. \tag{A35}$$

Similarly, one can express $\hat{\mathcal{J}}^\dagger(t) = \hat{\mathcal{J}}_1^\dagger(t) + \hat{\mathcal{J}}_2^\dagger(t)$, where

$$\hat{\mathcal{J}}_1^\dagger(t) = \tfrac{1}{2}[\hat{\mathcal{J}}^\dagger(0) + \hat{\Omega}\hat{\mathcal{J}}^\dagger(0)\hat{\mathcal{M}}^{-1}]e^{+i\hat{\mathcal{M}}t}e^{-i\hat{\Omega}t}, \tag{A36}$$

$$\hat{\mathcal{J}}_2^\dagger(t) = \tfrac{1}{2}[\hat{\mathcal{J}}^\dagger(0) - \hat{\Omega}\hat{\mathcal{J}}^\dagger(0)\hat{\mathcal{M}}^{-1}]e^{-i\hat{\mathcal{M}}t}e^{-i\hat{\Omega}t}. \tag{A37}$$

The results are given in terms of operators $\hat{\Omega}$ and $\hat{\mathcal{M}}$. To finalize the description, one needs to specify the physical sense of functions appearing in Eqs. (A34)–(A37).

For a reasonable function $f(\hat{D})$ of an operator $\hat{D}$ having eigenvalues $\lambda_{\rm d}$ and eigenstates $|{\rm d}\rangle$, there exists the following relationship: $f(\hat{D})|{\rm d}\rangle = f(\lambda_{\rm d})|{\rm d}\rangle$, provided that $f(\lambda_{\rm d})$ exists. To find the meanings of the operators $\hat{\mathcal{M}}^{-1}$ and $e^{\pm i\hat{\mathcal{M}}t}$, we express them as functions of the operator $\hat{\mathcal{M}}^2 = \hat{H}^2/\hbar^2 + 2\omega_0\omega_c$; see Eq. (A31). From the definition of $\hat{\mathcal{M}}^2$, it follows that its eigenstates are equal to the eigenstates $|{\rm n}\rangle$ of $\hat{H}$. The eigenvalues $\lambda_{\rm n}^2$ of the operator $\hat{\mathcal{M}}^2$ are $\lambda_{n,k_z}^2 = E_{n+1,k_z}^2$, and we obtain

$$\hat{\mathcal{M}}^{\pm 1}|{\rm n}\rangle = (\hat{\mathcal{M}}^2)^{\pm 1/2}|{\rm n}\rangle = \eta E_{n+1,k_z}^{\pm 1}|{\rm n}\rangle, \tag{A38}$$

$$e^{\pm i\hat{\mathcal{M}}t}|{\rm n}\rangle = e^{\pm i(\hat{\mathcal{M}}^2)^{1/2}t}|{\rm n}\rangle = e^{\pm i\eta E_{n+1,k_z}}|{\rm n}\rangle, \tag{A39}$$

where $\eta = +1$ or $\eta = -1$. As seen from Eqs. (A34)–(A37), the sums $\hat{\mathcal{J}}_1(t) + \hat{\mathcal{J}}_2(t)$ and $\hat{\mathcal{J}}_1^\dagger(t) + \hat{\mathcal{J}}_2^\dagger(t)$ do not depend on the sign of η, so we select $\eta = +1$.

Finally, we show that the matrix elements of the operator $\hat{\mathcal{J}}(t) = \hat{\mathcal{J}}_1(t) + \hat{\mathcal{J}}_2(t)$ are equal to the matrix elements of the current operator $\hat{\mathcal{J}}_H(t) = e^{i\hat{\Omega}t}\hat{\mathcal{J}}(0)e^{-i\hat{\Omega}t}$ in the Heisenberg picture. The operator $\hat{\mathcal{J}}$ is proportional to the annihilation operator $\hat{a}$ whose nonvanishing matrix elements are $\langle n'|\hat{a}|n\rangle = \sqrt{n+1}\delta_{n',n+1}$. We select two eigenstates of KG Hamiltonian $|{\rm n}\rangle = |n,s\rangle$ and $|{\rm n}'\rangle = |n+1,z\rangle$; see Eq. (44). Here we omitted quantum numbers k_x and k_z. For $\hat{\mathcal{J}}_H(t)$, one has

$$\langle {\rm n}|\tau_3\hat{\mathcal{J}}_H(t)|{\rm n}'\rangle = e^{is\omega_n t}e^{-iz\omega_{n+1}t}\hat{\mathcal{J}}(0)_{{\rm nn}'}, \tag{A40}$$

where we define $\hat{\mathcal{J}}(0)_{{\rm nn}'} = \langle {\rm n}|\tau_3\hat{\mathcal{J}}(0)|{\rm n}'\rangle$. To calculate the matrix elements of $\hat{\mathcal{J}}_1(t)$, we use Eqs. (A38) and (A39) and obtain

$$\langle {\rm n}|\hat{\mathcal{M}}^{-1}\hat{\mathcal{J}}(0)\hat{\Omega}|{\rm n}'\rangle = \frac{\hbar}{E_{n+1}}\hat{\mathcal{J}}(0)_{{\rm nn}'}\frac{zE_{n+1}}{\hbar} = z\hat{\mathcal{J}}(0)_{{\rm nn}'}, \tag{A41}$$

which finally gives

$$\langle {\rm n}|\hat{\mathcal{J}}_1(t)|{\rm n}'\rangle = \frac{1+z}{2}\hat{\mathcal{J}}(0)_{{\rm nn}'}e^{is\omega_n t}e^{-i\omega_{n+1}t}, \tag{A42}$$

$$\langle {\rm n}|\hat{\mathcal{J}}_2(t)|{\rm n}'\rangle = \frac{1-z}{2}\hat{\mathcal{J}}(0)_{{\rm nn}'}e^{is\omega_n t}e^{+i\omega_{n+1}t}. \tag{A43}$$

The matrix elements of $\hat{\mathcal{J}}_1(t)$ are nonzero for $z = +1$ only, while the matrix elements of $\hat{\mathcal{J}}_2(t)$ are nonzero for $z = -1$ only. Comparing Eqs. (A42) and (A43) with Eq. (A40), we see that for each of the four combinations of $s = \pm 1$ and $z = \pm 1$, the matrix elements of $\hat{\mathcal{J}}_H(t)$ are equal to the matrix elements of $\hat{\mathcal{J}}(t) = \hat{\mathcal{J}}_1(t) + \hat{\mathcal{J}}_2(t)$, which is what we wanted to show. The calculations for $\hat{\mathcal{J}}^\dagger(t)$ are similar to those for $\hat{\mathcal{J}}(t)$. The compact equations (A34)–(A37) are our final results for the time dependence of the $\hat{\mathcal{J}}(t)$ and $\hat{\mathcal{J}}^\dagger(t)$ operators. These equations are exact and they are quite fundamental for relativistic spin-zero particles in a magnetic field. If we calculate the average currents of Eqs. (A12) and (A13) with the use of expressions (A42) and (A43) and the wave packet (11), then one obtains results corresponding to the velocities given in Sec. IV.

APPENDIX B

In this appendix, we analyze in more detail the relation of the particle velocity to the speed of light. We consider the $(1,1)$ component of the velocity operator for a KG particle given in Eq. (7). For the wave packet $\langle \boldsymbol{r}|w\rangle = w(\boldsymbol{r})(1,0)^T$ with one nonzero component, the average velocity is given by the average of $(\hat{v}_z)_{11}(t)$ over the function $w(\boldsymbol{r})$. The unexpected feature of operator $(\hat{v}_z)_{11}(t)$ is that for large $\boldsymbol{p}$, this velocity can exceed the speed of light c.

There are two possible ways to overcome this problem. We can *additionally* assume that $|p| \leqslant mc$, which ensures that the velocity $(\hat{v}_z)_{11}(t)$ does not exceed c. This condition is equivalent to $|q| \leqslant 1$ in the text; see Eq. (7). Alternatively, one can take the initial wave packet $w(\boldsymbol{r})$, which does not contain components with $|\boldsymbol{p}| > mc$. Then, the Gaussian packet in Eq. (12) must be replaced by a non-Gaussian packet $w'(\boldsymbol{r})$ of the form

$$\langle \boldsymbol{k}|w'\rangle = (2d\sqrt{\pi})^{3/2}\exp[-d^2(\boldsymbol{k}-\boldsymbol{k}_0)^2/2]\Theta(\lambda_c - |\boldsymbol{k}|), \tag{B1}$$

where $\Theta(\xi)$ is the step function.

For the Dirac Hamiltonian $\hat{H}_D = c\sum_j \hat{\alpha}_j\hat{p}_j + mc^2\hat{\beta}$, the situation is different. By expanding $e^{i\hat{H}_D t/\hbar}$ in a power series, one obtains an expression analogous to $e^{i\hat{H}t/\hbar}$ given in Eq. (6). After some algebra, we find

$$(\hat{v}_z)_{11}^D(t) = \frac{mc^2 p_z}{m^2c^2 + p^2}[1 - \cos(2Et/\hbar)]. \tag{B2}$$

In contrast to the KG case, the velocity operator given in Eq. (B2) has correct relativistic behavior for all values of $\boldsymbol{p}$. In

Eq. (B2), the expression in square brackets oscillates between zero and two. The factor $v^D(\boldsymbol{p}) = mc^2p_z/(m^2c^2+p^2)$ tends to zero for both large and small values of $\boldsymbol{p}$. Its maximum is at $\boldsymbol{p}^{\max} = (0,0,mc)$, for which one obtains

$$(\hat{v}_z)^D_{11}(t) = \frac{c}{2}[1-\cos(2\sqrt{2}\omega_0)]. \tag{B3}$$

The above velocity never exceeds the speed of light. Therefore, when calculating the average velocity of the wave packet for the Dirac Hamiltonian, there is no need for an artificial truncation of the high-momentum components of the wave packet, as proposed in Eq. (B1) for a KG particle.

APPENDIX C

We prove here some identities appearing in the previous sections. We begin with the identity in Eq. (50). Closing Eq. (50) with the use of states $\langle \boldsymbol{r}|$ and $|\boldsymbol{r}'\rangle$, employing Eq. (44), and writing explicitly the summations and integrations over the quantum numbers, we obtain

$$\begin{aligned}\delta_{\boldsymbol{r},\boldsymbol{r}'} &= \sum_{\rm n}\langle \boldsymbol{r}|{\rm n}\rangle\langle {\rm n}|\boldsymbol{r}'\rangle s_n\tau_3 = \frac{1}{16\pi^2}\sum_{n=0}^{\infty}\phi_n(\xi)\phi_n(\xi')\\ &\quad\times\int_{-\infty}^{\infty}e^{ik_x(x-x')}dk_x\int_{-\infty}^{\infty}e^{ik_z(z-z')}dk_z\\ &\quad\times\sum_{s=\pm1}\begin{pmatrix}\mu^+\\ \mu^-\end{pmatrix}(\mu^+,\mu^-)s\tau_3,\end{aligned} \tag{C1}$$

where $\mu^{\pm}\equiv\mu^{\pm}_{n,k_z,s}$. In the above equation, the summation over n gives $\delta_{\xi,\xi'}$, and the product of the two integrals is $4\pi^2\delta_{x,x'}\delta_{z,z'}$, so the product of the three terms equals $4\pi^2\delta_{\boldsymbol{r},\boldsymbol{r}'}$. By taking the explicit form of $\mu^{\pm} = \nu \pm s/\nu$ where $\nu = \sqrt{mc^2/E_{n,k_z}}$, we obtain, for the last line of Eq. (C1),

$$\sum_{s=\pm1}\begin{pmatrix} s(\nu+s/\nu)^2 & s(\nu^2-1/\nu^2)\\ s(\nu^2-1/\nu^2) & s(\nu-s/\nu)^2\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = 4. \tag{C2}$$

Collecting all numerical factors, we see that the right-hand side of Eq. (C1) equals $\delta_{\boldsymbol{r},\boldsymbol{r}'}$.

Next we prove the identity used in the derivation of Eqs. (19) and (53). Let $\hat{O}$ be any operator for which $\hat{O} = \tau_3\hat{O}^\dagger\tau_3$, where the dagger signifies the Hermitian conjugate. We want to show that

$$\tau_3 e^{\hat{O}} = e^{\hat{O}^\dagger}\tau_3. \tag{C3}$$

To this end, we expand the exponents and get

$$\tau_3\left[1+\frac{\hat{O}}{1!}+\frac{\hat{O}^2}{2!}+\cdots\right] = \left[1+\frac{\hat{O}^\dagger}{1!}+\frac{\hat{O}^{\dagger2}}{2!}+\cdots\right]\tau_3. \tag{C4}$$

Since $\hat{O} = \tau_3\hat{O}^\dagger\tau_3$, there is $\hat{O}^\dagger = \tau_3\hat{O}\tau_3$. Then, for $n \geqslant 0$, there is $\hat{O}^{\dagger n} = \tau_3\hat{O}^n\tau_3$, and we obtain, for the right-hand side of Eq. (C4),

$$\begin{aligned}\left[1+\frac{\hat{O}^\dagger}{1!}+\frac{\hat{O}^{\dagger2}}{2!}+\cdots\right]\tau_3 &= \left[1+\tau_3\frac{\hat{O}}{1!}\tau_3+\tau_3\frac{\hat{O}^2}{2!}\tau_3+\cdots\right]\tau_3\\ &= \tau_3\left[1+\frac{\hat{O}}{1!}+\frac{\hat{O}^2}{2!}+\cdots\right] = \tau_3 e^{\hat{O}},\end{aligned}$$

which is the desired result.

APPENDIX D

In this appendix, we quote for completeness all formulas necessary for the calculation of coefficients $U_{m,n}$ in Eqs. (64)–(66). Here we assume the initial wave vector in the form $\boldsymbol{k}_0 = (k_{0x},0,k_{0z})$. Using the definitions of $g_{xy}(k_x,y)$, $F_n(k_x)$, and $U_{m,n}$, we obtain (see Ref. [27])

$$g_{xy}(k_x,y) = \sqrt{\frac{d_x}{\pi d_y}}e^{-\frac{1}{2}d_x^2(k_x-k_{0x})^2}e^{-\frac{y^2}{2d_y^2}}, \tag{D1}$$

and

$$F_n(k_x) = \frac{A_n\sqrt{Ld_x}}{\sqrt{2\pi d_y}C_n}e^{-\frac{1}{2}d_x^2(k_x-k_{0x})^2}e^{-\frac{1}{2}k_x^2D^2}\,{\rm H}_n(-k_xc), \tag{D2}$$

where $D = L^2/\sqrt{L^2+d_y^2}$, $c = L^3/\sqrt{L^4-d_y^4}$, and

$$A_n = \frac{\sqrt{2\pi}d_y}{\sqrt{L^2+d_y^2}}\left(\frac{L^2-d_y^2}{L^2+d_y^2}\right)^{n/2}, \tag{D3}$$

$$\begin{aligned}U_{m,n} &= \frac{A_m^*A_nLQd_x\sqrt{\pi}\,e^{-W^2}}{\pi C_mC_nd_y}\sum_{l=0}^{\min\{m,n\}}2^l l!\binom{m}{l}\binom{n}{l}\\ &\quad\times[(1-(cQ)^2]^{(m+n-2l)/2}{\rm H}_{m+n-2l}\left[\frac{-cQY}{\sqrt{1-(cQ)^2}}\right],\end{aligned} \tag{D4}$$

in which $Q = 1/\sqrt{d_x^2+D^2}$, $W = d_xDQk_{0x}$, and $Y = d_x^2k_{0x}Q$. For the special case of $d_y = L$, the formula for $U_{m,n}$ is much simpler:

$$\begin{aligned}U_{m,n} &= 2\frac{\sqrt{\pi}\,(-i)^{m+n}\,d_x}{C_mC_nL}\left(\frac{L}{2P}\right)^{m+n+1}\\ &\quad\times\exp\left(-\frac{d_x^2k_{0x}^2L^2}{2P^2}\right){\rm H}_{m+n}\left(\frac{-id_x^2k_{0x}}{P}\right),\end{aligned} \tag{D5}$$

where $P = \sqrt{d_x^2+\frac{1}{2}L^2}$. In the above expressions, the coefficients $U_{m,n}$ are real numbers and they are symmetric in m,n indices. For further discussion of $U_{m,n}$, see Refs. [27,44].

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