Exact solvability of the quantum Rabi model using Bogoliubov operators

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Within extended coherent states, a recent exact solution to the quantum Rabi model (QRM) [D. Braak, Phys. Rev. Lett. **107**, 100401 (2011)] can be recovered in an alternative simpler and more physical way, without use of any extra conditions. In the same framework, the two-photon QRM is solved exactly by treating extended squeezed states on an equal footing. Concise transcendental functions responsible for the exact solutions are derived. The isolated Juddian solutions are also analytically obtained in terms of degeneracy. Both the extended coherent states and squeezed states employed here are essentially Fock states in the space of the corresponding Bogoliubov operators, which result in free-particle number operators. The present approach can be summarized concisely in a unified way and easily extended to various spin-boson systems with multiple levels, even multiple modes.

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I. INTRODUCTION

Matter-light interaction is fundamental and ubiquitous in areas of modern physics ranging from quantum optics and quantum information science to condensed-matter physics. The simplest paradigm is a two-level atom (qubit) coupled to the electromagnetic mode of a cavity (oscillator), which is called the Rabi model [1]. If the coupling strength g/ω (ω is the cavity frequency) between the atom and the cavity mode exceeds the loss rate, the atom and the cavity can repeatedly exchange excitations before coherence is lost. Rabi oscillations can be observed in this atom-cavity system, which is usually known as cavity quantum electrodynamics (QED) [2]. Typically, the coupling strength in cavity QED reaches $g/\omega \sim 10^{-6}$. It can be described by the Jaynes-Cummings (JC) model [3] where the rotating-wave approximation (RWA) is made and analytically closed-form exact solutions are available.

Recently, for superconducting qubits, a one-dimensional (1D) transmission line resonator [4] or an *LC* circuit [5–8] has been shown to be able to play the role of the cavity; this is known today as circuit QED. More recently, an *LC* resonator inductively coupled to a superconducting qubit [9–11] has been realized experimentally, where the qubit-resonator coupling can be strengthened by 10%. In this ultrastrong-coupling regime of circuit QED, evidence for the breakdown of the RWA has been provided by the transmission spectra [9]. The remarkable Bloch-Siegert shift associated with the counter-rotating terms also demonstrates the failure of the RWA [10]. So the quantum Rabi model (QRM) has been reconsidered by many authors recently.

On the other hand, the two-photon QRM has also attracted a lot of attention. It is a phenomenological model describing a three-level system interacting with two photons. When the intermediate transition frequencies are strongly detuned from the cavity frequency, after adiabatically eliminating the intermediate levels, one arrives at the effective Hamiltonian. It can describe the two-photon processes occurring in rubidium atoms [12] and quantum dots [13]. The two-photon QRM has also been studied for a long time both with the RWA [14] and beyond the RWA [15–17].

More recently, Braak presented an exact solution [18] to the one-photon QRM, in a representation of bosonic creation and annihilation operators in the Bargmann space [19] of analytical functions in a complex variable. A transcendental function, whose zeros could give exact eigenvalues, was derived. By a proposed criterion for quantum integrability, Braak further shows that the QRM is integrable due to the parity symmetry. However, the derivations are outlined in a mathematical way. It was suggested [20] that an intense dialogue between mathematics and physics is needed. In other words, it is useful to shed some physical insights on Braak's mainly mathematical solutions.

In this paper, without the use of extra conditions, like analyticity of the eigenfunction in the Bargmann representation, we alternatively rederive the same transcendental functions as in Ref. [18] quantum-mechanically within extended coherent states [21,22]. Both zero-bias and biased QRM can be treated simultaneously. Our method is more intuitional and more easily understandable. Similarly, we also study the two-photon QRM [16,17] within extended squeezed states. Compact transcendental functions which are responsible for the exact solution are derived. The Juddian solutions [23] are then easily discussed.

The paper is organized as follows. In Sec. II, we describe the present approach in detail for the one-photon QRM. Braak's exact solution is recovered straightforwardly. Discussions and comparisons and a brief tutorial for the approach are also given. In Sec. III, the two-photon QRM is solved exactly on an equal footing, concise transcendental functions are derived, and Juddian solutions are discussed. A brief summary is given finally.

II. THE QRM WITHIN BOGOLIUBOV OPERATORS

A. Rederivation of Braak's solution

The Hamiltonian of a generalized QRM can be describe as follows:

$$H = -\tfrac{1}{2}(\varepsilon\sigma_z + \Delta\sigma_x) + a^\dagger a + g(a^\dagger + a)\sigma_z, \tag{1}$$

where ε and Δ are the qubit static bias and tunneling matrix element, $a^\dagger$ and a are the photon creation and annihilation operators of the single-mode cavity with frequency ω, g is the qubit-cavity coupling constant, and σ_k $(k = x,y,z)$ are the Pauli matrices. To facilitate the study, we write the Hamiltonian in the matrix form in units of $\hbar = \omega = 1$,

$$H = \begin{pmatrix} a^\dagger a + g(a^\dagger + a) - \frac{\varepsilon}{2} & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & a^\dagger a - g(a^\dagger + a) + \frac{\varepsilon}{2} \end{pmatrix}. \tag{2}$$

To remove the linear terms in the $a^\dagger$ (a) operators, we perform the following two Bogoliubov transformations:

$$A = a + g, \quad B = a - g. \tag{3}$$

In Bogoliubov operators A (B), the matrix element H_{11} (H_{22}) can be reduced to the free-particle number operators $A^\dagger A$ $(B^\dagger B)$ plus a constant, which is very helpful for further study.

Unlike the previous ansatz, where the Hamiltonian is expressed in the two operators A and B at the same time [21], we here use the single operators one after the other. First, in terms of operator A, the Hamiltonian can be written as

$$H = \begin{pmatrix} A^\dagger A - \alpha & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & A^\dagger A - 2g(A^\dagger + A) + \beta \end{pmatrix}, \tag{4}$$

where

$$\alpha = g^2 + \frac{\varepsilon}{2}, \quad \beta = 3g^2 + \frac{\varepsilon}{2}.$$

The wave function is then proposed as

$$|\,\rangle = \begin{pmatrix} \sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A \\ \sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A \end{pmatrix}, \tag{5}$$

where e_n and f_n are the expansion coefficients. $|n\rangle_A$ is called an extended coherent state with the following properties:

$$|n\rangle_A = \frac{(A^\dagger)^n}{\sqrt{n!}} |0\rangle_A = \frac{(a^\dagger + g)^n}{\sqrt{n!}} |0\rangle_A, \tag{6}$$

$$|0\rangle_A = e^{-(1/2)g^2 - g a^\dagger} |0\rangle_a, \tag{7}$$

where the vacuum state $|0\rangle_A$ in Bogoliubov operators A is well defined as the eigenstate of the one-photon annihilation operator a, and is known as a pure coherent state [24].

The Schrödinger equation gives

$$\begin{aligned} &\sum_{n=0}^{\infty} (n - \alpha)\sqrt{n!} e_n |n\rangle_A - \frac{\Delta}{2} \sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A \\ &= E \sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A \\ &\quad -\frac{\Delta}{2} \sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A + \sum_{n=0}^{\infty} (n + \beta)\sqrt{n!} f_n |n\rangle_A \\ &\quad -2g \sum_{n=0}^{\infty} (\sqrt{n} f_n \sqrt{n!} |n-1\rangle_A + \sqrt{n+1}\sqrt{n!} f_n |n+1\rangle_A) \\ &= E \sum_{n=0}^{\infty} \sqrt{n!} f_n \,|n\rangle_A \,. \end{aligned}$$

Left-multiplying by ${}_A\langle m|$ gives

$$(m - \alpha - E) e_m = \frac{\Delta}{2} f_m, \tag{8}$$

$$(m + \beta - E) f_m - 2g(m+1) f_{m+1} - 2g f_{m-1} = \frac{\Delta}{2} e_m. \tag{9}$$

So the two coefficients e_m and f_m with the same index m are related by

$$e_m = \frac{\Delta}{2\,(m - \alpha - E)} f_m, \tag{10}$$

and the coefficient f_m can be defined recursively,

$$m f_m = \Omega(m-1) f_{m-1} - f_{m-2}, \tag{11}$$

$$\Omega(m) = \frac{1}{2g} \left((m + \beta - E) - \frac{\Delta^2}{4\,(m - \alpha - E)} \right), \tag{12}$$

with $f_0 = 1$ and $f_1 = \Omega(0)$

Similarly, using the second operator B, we can obtain the Hamiltonian as

$$H = \begin{pmatrix} B^\dagger B + 2g(B^\dagger + B) + \beta' & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & B^\dagger B - \alpha' \end{pmatrix}, \tag{13}$$

where

$$\alpha' = g^2 - \frac{\varepsilon}{2}, \quad \beta' = 3g^2 - \frac{\varepsilon}{2}.$$

The wave function can also be written in terms of B as

$$|\,\rangle = \begin{pmatrix} \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f'_n |n\rangle_B \\ \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} e'_n |n\rangle_B \end{pmatrix}. \tag{14}$$

Proceeding as before, the two coefficients f'_n and e'_n satisfy

$$e'_m = \frac{\frac{\Delta}{2}}{m - \alpha' - E} f'_m, \tag{15}$$

and the recursive relation is given by

$$m f'_m = \Omega'(m-1) f'_{m-1} - f'_{m-2}, \tag{16}$$

$$\Omega'(m) = \frac{1}{2g} \left[(m + \beta' - E) - \frac{\Delta^2}{4(m - \alpha' - E)} \right], \tag{17}$$

with $f'_0 = 1$ and $f'_1 = \Omega'(0)$.

If both wave functions (5) and (14) are true eigenfunctions for a nondegenerate eigenstate with eigenvalue E, they should be in principle only different by a complex constant r,

$$\sum_{n=0}^{\infty} \sqrt{n!} e_n \,|n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f'_n \,|n\rangle_B \,, \tag{18}$$

$$\sum_{n=0}^{\infty} \sqrt{n!} f_n \,|n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} e'_n \,|n\rangle_B \,. \tag{19}$$

Left-multiplying the original vacuum state $\langle 0|$ by both side of the above equations yields

$$\sum_{n=0}^{\infty} \sqrt{n!} e_n \langle 0||n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f'_n \langle 0||n\rangle_B, \tag{20}$$

$$\sum_{n=0}^{\infty} \sqrt{n!} f_n \langle 0||n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} e'_n \langle 0||n\rangle_B, \tag{21}$$

where

$$\sqrt{n!}\langle 0|n\rangle_A = (-1)^n\sqrt{n!}\langle 0|n\rangle_B = e^{-g^2/2}g^n. \tag{22}$$

Eliminating the ratio constant r gives

$$\sum_{n=0}^{\infty} e_n g^n \sum_{n=0}^{\infty} e'_n g^n = \sum_{n=0}^{\infty} f_n g^n \sum_{n=0}^{\infty} f'_n g^n.$$

With the help of Eqs. (10) and (15), we arrive at

$$\sum_{n=0}^{\infty} \frac{\Delta/2}{n-\alpha-E} f_n g^n \sum_{n=0}^{\infty} \frac{\Delta/2}{n-\alpha'-E} f'_n g^n = \sum_{n=0}^{\infty} f_n g^n \sum_{n=0}^{\infty} f'_n g^n. \tag{23}$$

If we set $f_n{=}K_n^-$, $f'_n{=}K_n^+$, and $E = x - g^2$, we can recover Braak's exact solution [18]

$$G_\epsilon(x) = \left(\frac{\Delta}{2}\right)^2 \overline{R}^+(x)\overline{R}^-(x) - R^+(x)R^-(x) = 0, \tag{24}$$

where

$$R^\pm(x) = \sum_{n=0}^{\infty} K_n^\pm(x)\, g^n,$$

$$\overline{R}^\pm(x) = \sum_{n=0}^{\infty} \frac{K_n^\pm(x)}{x-n\mp\frac{\varepsilon}{2}}\, g^n.$$

If $\varepsilon = 0$, the above equation can obviously be reduced to the following zero-bias case [18]:

$$G_0^\pm(x) = \sum_{n=0}^{\infty} f_n(x)\left(1 \mp \frac{\Delta/2}{x-n}\right) g^n = 0. \tag{25}$$

Therefore Braak's G functions are completely rederived in a very intuitive and concise way.

The G functions can be written [25] in terms of so-called Heun functions [26]. The zeros of these Heun functions cannot be given analytically; a numerical technique to locate the zeros is still needed, so a cutoff for the summation is unavoidable in the practical evaluation.

B. Discussion

In the above derivation, the crucial step is the proportionality of the two wave functions (5) and (14) with the same eigenvalue. Both Hilbert spaces in the two Bogoliubov operators are complete, if truncation is not done, and the proportionality is justified naturally for nondegenerate states.

Next, we link the degenerate states to the Juddian solutions [23]. Koc *et al.* [27] have obtained isolated exact solutions in the QRM, which are just the Juddian solutions with doubly degenerate eigenvalues. The degenerate eigenstates are excluded in principle in solutions based on proportionality. It naturally follows that the Juddian solutions are exceptional ones as discussed by Braak [18].

It is very interesting to note that, in the whole derivation above, we do not need any extra conditions, such as the analyticity of the eigenfunction in Bargmann representation as in Braak's work [18]. We believe that the extra condition in the Bargmann representation is covered in the vacuum state in the space of the Bogoliubov operators. These vacuum states are well defined and known as coherent states, so the present derivation is more physical and simpler.

In addition, the validity of the present approach is independent of the parity symmetry. Parity symmetry would be contained self-consistently in the final G functions if the system really has, e.g., $\varepsilon = 0$.

C. Comparisons and remarks

Based on two Bogoliubov operators A and B, three of the present authors and one collaborator have used the following wave function to the Hamiltonian(1) to analyze the spectrum in qubit-oscillator systems [cf. Eq. (6) in Ref. [22]]:

$$|\,\rangle = \begin{pmatrix} \sum_{n=0}^{N} c_n|n\rangle_A \\ \sum_{n=0}^{N} d_n|n\rangle_B \end{pmatrix}, \tag{26}$$

where N is the truncated number. Numerical exact diagonalization (ED) in the space of the two Bogoliubov operators gave the spectrum exactly. The coefficients c_n and $d_n$can be obtained also.

It is interesting to link coefficients in wave function (26) and those in wave functions (5) and (14) as

$$c_n = \sqrt{n!}e_n, \quad d_n = r\,(-1)^n\,\sqrt{n!}e'_n,$$

although the former are obtained from ED and the latter by the zeros of the G functions. It can also be confirmed numerically. For practical purposes, there are perhaps no essential differences, except that the avenues used to obtain basically the same coefficients are different. The latter is described in a mathematical way and is of more conceptual interest.

In the mathematical sense, we cannot rule out the possibility that ED gives good results for small N but gets worse for higher N without a practical evaluation, although we know empirically that it is generally not the case for large N. For low-order perturbation theory, it happens that third-order perturbation theory will give worse results than second-order perturbation theory in some parameter regime, for instance, but this may not be that case in very high-order perturbation theory. In the calculation, we find that the difference between the exact results, which are easily obtained to any desired accuracy, and those for the cutoff N decrease monotonically with increasing N, and convergence can be arrived at easily. One can determine that the Heun series converges before numerical calculations.

Braak's G functions exhibit a very compact form in power series, which motivates us to reshape our previous work. By use of tunable extended bosonic coherent states, the QRM can be mapped to a polynomial equation with a single variable [28]. We can also write this polynomial in power series in the following more concise form for large truncated number M:

$$F(\alpha) = \sum_{j=0}^{M} \frac{(2\alpha)^j}{j!} c_{M-j} = 0, \tag{27}$$

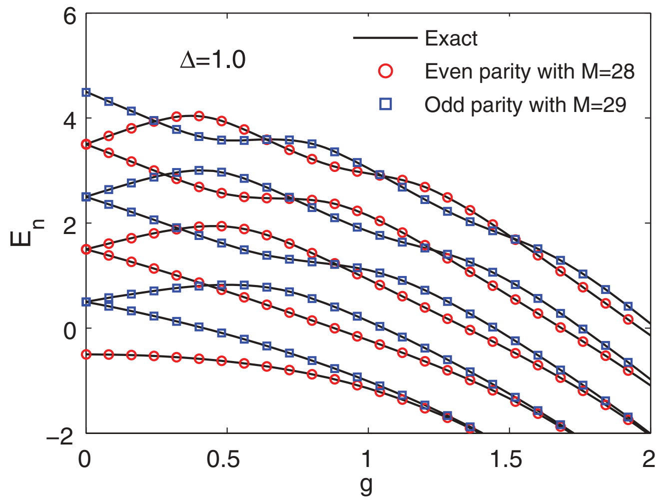

FIG. 1. (Color online) Spectrum of the one-photon quantum Rabi model as zeros of functions $F(\alpha)$ in Eq. (27) at resonance. The numerical solutions are also shown with solid lines.

where α is the key tunable variable we seek, and the coefficients are also related recursively with the following scheme:

$$(m+1)gc_{m+1} = -\left(m \pm \frac{\Delta}{2}\right)c_m - (\alpha+g)\,c_{m-1} \pm (-1)^m \frac{\Delta}{2}\sum_{j=0}^{m}\frac{(2\alpha)^j}{j!}c_{m-j}, \tag{28}$$

initiated from $c_0 = 1.0$ and $c_1 = 0$, because the coefficients with the two highest indices, M and $M-1$, are negligibly small due to the required convergence and can thus be omitted. The zeros of the above function $F(\alpha)$ can also give the exact eigenvalue through

$$E^{\pm} = \alpha g \mp \frac{\Delta}{2}, \tag{29}$$

where $\pm$ denotes the parity. The results are shown in Fig. 1. In Eq. (20) at the end of Ref. [28], it was demonstrated that the wave function is equivalent to expansion in the Fock space of displaced operators with tunable displacements, which are different from the present Bogoliubov operators with fixed displacements.

It is very interesting to note that the zeros of both functions defined through different power series in Eqs. (25) and (27) can give the exact eigenvalues. In our practical evaluation, it is not more difficult to locate the zeros for the function in Eq. (27) than those in Eq. (25), because the poles at $x = n$ emerging in the latter are not present in the former. The main difference between Eqs. (25) and (27) is that the former is well defined without restriction and the latter is well defined with built-in truncation.

It is implied in the viewpoint of [20] that the QRM might have been solved exactly with an analytical closed-form solution in Ref. [18]. Nevertheless, whether Braak's exact solution could be called closed form is subtle and therefore still controversial in our opinion. The so-called Heun functions can be basically called closed form because they are well defined, although much more complicated than, e.g., the hypergeometric functions. But the eigenvalues are given by the zeros of the Heun functions, which cannot be obtained without truncation in the power series. As shown in wave functions (5) and (14), the expansion cannot be closed naturally as in the JC model. It is generally accepted that the QRM has no trivial closed-form solution like that in the JC model, due to the counter-rotating terms. The QRM can have closed-form solutions only with a vanishing qubit tunneling $\Delta = 0$ [28,29]. Perhaps the question of "closed-form" solutions is academic and not of real value. Braak's solution is mainly interesting for the integrability of the QRM.

D. Tutorial for Bogoliubov operator approach

The present approach using the Bogoliubov operators can be generally described as follows; the description helpful for further applications. The main task is to find the corresponding Bogoliubov operators. Then, one can expand the wave functions in terms of each Bogoliubov operator separately. Eliminating the constant ratio of these wave functions will give transcendental functions, which are defined through power series in model parameter-dependent quantities with coefficients related recursively. Finally, the zeros of these transcendental functions give the eigenvalues exactly, where numerical solutions to the one-variable (or finite variables in other multilevel systems for example) nonlinear equation must be required. Although the power series are defined through an infinite summation formally, in a practical calculation, they should be truncated to a finite summation. Fortunately, the obtained transcendental function $G(x)$ can be written in terms of so-called Heun functions, from which we can determine the convergence before numerical solution. The unavoidable "cutoff" in the summation of the G functions in practical calculations means that some states in the Hilbert spaceare not considered, according to the wave functions (5) and (14); even though their contribution is negligibly small, they still belong to the Hilbert space of Bogoliubov operators.

Following the approach outlined aobve, we will study the two-photon QRM in the next section.

III. THE TWO-PHOTON QRM

The Hamiltonian of the two-photon QRM takes the following matrix form:

$$H = \begin{pmatrix} a^\dagger a + g[(a^\dagger)^2 + a^2] & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & a^\dagger a - g[(a^\dagger)^2 + a^2] \end{pmatrix}. \tag{30}$$

First, we perform a Bogoliubov transformation,

$$b = ua + va^\dagger, \quad b^\dagger = ua^\dagger + va, \tag{31}$$

to generate a new bosonic operator. Comparing to the Hamiltonian, if we set

$$u = \sqrt{\frac{1+\beta}{2\beta}}, \quad v = \sqrt{\frac{1-\beta}{2\beta}}, \tag{32}$$

with $\beta = \sqrt{1-4g^2}$, we have a simple quadratic form of one diagonal Hamiltonian matrix element:

$$H_{11} = a^\dagger a + g[(a^\dagger)^2 + a^2] = \frac{b^\dagger b - v^2}{u^2 + v^2}.$$

Similarly, we can introduce another operator

$$c = ua - va^\dagger, \quad c^\dagger = ua^\dagger - va, \tag{33}$$

which yields a simple quadratic form of the other diagonal Hamiltonian matrix element

$$H_{22} = a^\dagger a - g[(a^\dagger)^2 + a^2] = \frac{c^\dagger c - v^2}{u^2 + v^2}.$$

In terms of the Bogoliubov operator b, the Hamiltonian can be written as

$$H = \begin{pmatrix} \frac{b^\dagger b - v^2}{u^2+v^2} - v^2 & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & H_{22}' \end{pmatrix}, \tag{34}$$

with

$$H_{22}' = (u^2 + v^2 + 4guv)b^\dagger b \\ -[uv + g(u^2 + v^2)][(b^\dagger)^2 + b^2] + 2guv + v^2.$$

The wave function is suggested as

$$|\,\rangle = \begin{pmatrix} \sum_{n=0} \sqrt{n!}e_n\,|n\rangle_b \\ \sum_{n=0} \sqrt{n!}f_n\,|n\rangle_b \end{pmatrix}, \tag{35}$$

where

$$|n\rangle_b = \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle_b = \frac{(ua^\dagger + va)^n}{\sqrt{n!}}|0\rangle_b, \tag{36}$$

$$b\,|0\rangle_b = 0. \tag{37}$$

$|0\rangle_b$ is the vacuum state of a linear combination of $a^\dagger$ and a, which is well known as the single-mode squeezed state [30]; $|n\rangle_b$ is thus called an extended squeezed state. Following the procedures in [31], we derive for later use the explicit expression of $|0\rangle_b$ in terms of the operator a involving either even- or odd-number states as follows:

$$|0\rangle_b^{(e)} \propto \sum_k \frac{\sqrt{2k!}}{2^k k!}\left(-\frac{v}{u}\right)^k |2k\rangle_a, \tag{38}$$

$$|0\rangle_b^{(o)} \propto \sum_k \frac{2^k k!}{\sqrt{(2k+1)!}}\left(-\frac{v}{u}\right)^k |2k+1\rangle_a. \tag{39}$$

The Schrödinger equation gives

$$\sum_{n=0} \frac{b^\dagger b - v^2}{u^2 + v^2}\sqrt{n!}e_n\,|n\rangle_b - \frac{\Delta}{2}\sum_{n=0}\sqrt{n!}f_n\,|n\rangle_b \\ = E\sum_{n=0}\sqrt{n!}e_n|n\rangle_b,$$

$$(u^2 + v^2 + 4guv)b^\dagger b\sum_{n=0}\sqrt{n!}f_n|n\rangle_b \\ -[uv + g(u^2+v^2)][(b^\dagger)^2 + b^2]\sum_{n=0}\sqrt{n!}f_n\,|n\rangle_b \\ + (2guv + v^2)\sum_{n=0}\sqrt{n!}f_n\,|n\rangle_b - \frac{\Delta}{2}\sum_{n=0}\sqrt{n!}e_n\,|n\rangle_b \\ = E\sum_{n=0}\sqrt{n!}f_n\,|n\rangle_b\,.$$

Left-multiplying by ${}_b\langle m|$ gives

$$\left(\frac{m - v^2}{u^2 + v^2} - E\right)e_m - \frac{\Delta}{2}f_m = 0,$$

$$-[uv + g(u^2+v^2)][f_{m-2} + (m+2)(m+1)f_{m+2}] \\ + [(u^2+v^2)m + v^2 + 2guv(2m+1) - E]f_m \\ - \frac{\Delta}{2}e_m = 0.$$

Thus we have built a one-to-one relation for coefficients e_m and f_m:

$$e_m = \frac{\Delta}{2[\frac{m-v^2}{u^2+v^2} - E]}f_m, \tag{40}$$

which will considerably simplify the problem. The reclusive relation is then obtained as

$$(m+2)(m+1)\,f_{m+2} = -f_{m-2} + \frac{\Omega(m)}{uv + g(u^2+v^2)}f_m, \tag{41}$$

where

$$\Omega(m) = (u^2+v^2)m + v^2 + 2guv\,(2m+1) \\ -E - \frac{\Delta^2}{4\left(\frac{m-v^2}{u^2+v^2} - E\right)}.$$

The Hamiltonian can also be expressed in terms of the other Bogoliubov operator c:

$$H = \begin{pmatrix} H_{11}' & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & \frac{c^\dagger c - v^2}{u^2+v^2} \end{pmatrix}, \tag{42}$$

with

$$H_{11}' = (v^2 + u^2 + 4guv)c^\dagger c \\ +[uv + g(v^2+u^2)][(c^\dagger)^2 + c^2] + 2guv + v^2.$$

The wave function then can be expanded in the Fock space of the c operator in the form

$$|\,\rangle = \begin{pmatrix} \sum_{n=0} i^l\sqrt{n!}f_n'\,|n\rangle_c \\ \sum_{n=0} i^l\sqrt{n!}e_n'\,|n\rangle_c \end{pmatrix}, \tag{43}$$

where $l = n$ for $n = 2k$ and $l = n+1$ for $n = 2k+1$. Therefore only two values of $i^l = \pm 1$ are possible.

Similarly, by the Schrödinger equation, we can obtain the following relations for the coefficients:

$$e_m' = \frac{\Delta}{2[\frac{m-v^2}{u^2+v^2} - E]}f_m', \tag{44}$$

and the reclusive relation

$$(m+2)(m+1)\,f_{m+2}' = -f_{m-2}' + \frac{\Omega'(m)}{uv + g(u^2+v^2)}f_m', \tag{45}$$

with

$$\Omega'(m) = v^2 + (u^2+v^2)m + 2guv(2m+1) \\ -E - \frac{\Delta^2}{4\left(\frac{m-v^2}{u^2+v^2} - E\right)}.$$

Note that the two sets of coefficients in the two wave functions have the same form.

Similarly, the two wave functions with the same eigenvalue should be in principle proportional to each other for the nondegenerate state,

$$\begin{pmatrix} \sum_{n=0} \sqrt{n!}e_n \,|n\rangle_b \\ \sum_{n=0} \sqrt{n!}f_n \,|n\rangle_b \end{pmatrix} = r \begin{pmatrix} \sum_{n=0} i^l \sqrt{n!}f_n' \,|n\rangle_c \\ \sum_{n=0} i^l \sqrt{n!}e_n' |n\rangle_c \end{pmatrix}. \quad (46)$$

Left-multiplying $\langle 0|$ on both equations gives

$$\sum_{n=0} \sqrt{n!}e_n \langle 0|n\rangle_b = r \sum_{n=0} i^l \sqrt{n!}f_n' \langle 0|n\rangle_c,$$
$$\sum_{n=0} \sqrt{n!}f_n \langle 0|n\rangle_b = r \sum_{n=0} i^l \sqrt{n!}e_n' \langle 0|n\rangle_c.$$

By using Eqs. (38) and (39), we always have

$$i^l \sqrt{n!}\langle 0\,|n\rangle_c = \sqrt{n!}\langle 0\,|n\rangle_b = L_n^{e,o}, \quad (47)$$

where

$$L_{n=2k}^e = \frac{(2k)!(uv)^k}{2^k} \sum_{j=0}^{k} \frac{\left(-\frac{v^2}{u^2}\right)^j}{j!(k-j)!}, \quad (48)$$

$$L_{n=2k+1}^o = \frac{(2k+1)!v(uv)^k}{2^k} \sum_{j=0}^{k} \frac{2^{2j} j!\left(-\frac{v^2}{u^2}\right)^j}{(2j+1)!(k-j)!}, \quad (49)$$

for even and odd particle numbers in the Bogoliubov operators b and c, respectively. Then we have

$$\sum_n e_n L_n^{e,o} = r \sum_n f_n' L_n^{e,o}, \quad \sum_n f_n L_n^{e,o} = r \sum_n e_n' L_n^{e,o}.$$

Now the summation $\sum_n$ is taken for the series with either even or odd number n. Eliminating the constant r yields

$$\sum_n \frac{\Delta}{2\left(\frac{n-v^2}{u^2+v^2} - E\right)} f_n L_n^{e,o} \sum_n \frac{\Delta}{2\left(\frac{n-v^2}{u^2+v^2} - E\right)} f_n' L_n^{e,o}$$
$$= \sum_n f_n' L_n^{e,o} \sum_n f_n L_n^{e,o}, \quad (50)$$

with the use of Eqs. (40) and (44). Setting $f_n = f_n'$ and $-x = -v^2 - E(u^2+v^2)$, we finally have

$$G_{e,o}^{\pm} = \sum_n f_n \left[1 \pm \frac{\Delta(u^2+v^2)}{2(n-x)}\right] L_n^{e,o} = 0, \quad (51)$$

where the coefficient f_n is initiated from $f_0 = 1$ ($f_1 = 1$) for the case of the even (odd) n in the recurrence scheme Eq. (41), and $\pm$ denotes the parity. Thus, G functions for the two-photon QRM have been obtained. The zeros of the G functions give the exact eigenvalues, as shown in Fig. 2. It should be straightforward to extend to the biased two-photon QRM, but it is not shown here.

Travěnec [17] has extended Braak's approach to solve this two-photon model, but a concise G function as in Eq. (51) was not obtained. The coefficients are entangled in the two coupled equations, which may prevent such a simple description for the G-functions. In the present Eqs. (40) and (44), the two coefficients are related one-to-one with the same index n, which facilitates the derivations. This is also the advantage of Bogoliubov operators, which result in free-particle number operators.

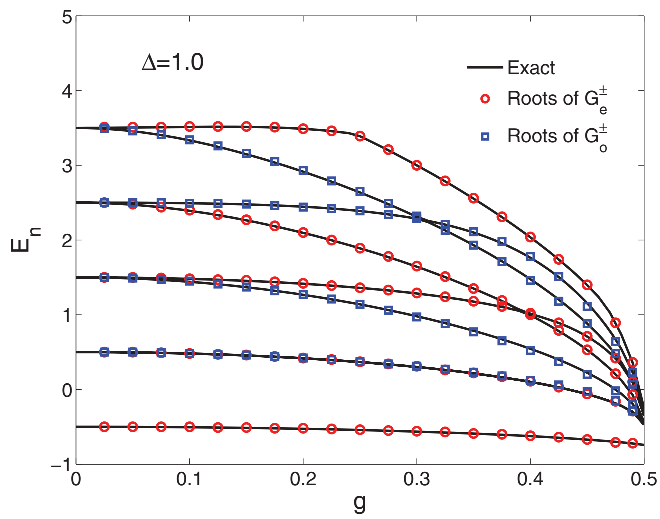

FIG. 2. (Color online) Spectrum of the two-photon quantum Rabi model as zeros of G functions in Eq. (51). The numerical solutions are also shown with solid lines.

The Juddian solution to the two-photon QRM has been studied by Emary and Bishop [32]. With these G functions at hand, we can also discuss the Juddian solution readily in an alternative way, similar to the one-photon model [18]. The G function is also not analytic in x but has simple poles at $x = 0,1,2,\ldots$. For special values of the model parameter g, there are eigenvalues which do not correspond to zeros of Eq. (51); these are the exceptional solutions. All exceptional eigenvalues are given by the positions of the poles:

$$E = \left(n + \tfrac{1}{2}\right)\beta - \tfrac{1}{2}, \quad (52)$$

which is exactly the isolated solution obtained in Ref. [32]. The necessary and sufficient condition for the occurrence of the eigenvalue is $f_n(x = n) = 0$, which provides a condition on the model parameters g and Δ. They occur when the pole of $G_{e,o}^{\pm}(x)$ at $x = n$ is lifted because its numerator in Eq. (51) vanishes. The condition can be obtained readily by Eq. (41) as follows for $n = 2$, 3, and 4, respectively:

$$2 - 6\beta^2 + \left(\frac{\Delta}{2}\right)^2 = 0, \quad 6 - 10\beta^2 + \left(\frac{\Delta}{2}\right)^2 = 0,$$

$$8(3 - 30\beta^2 + 35\beta^4) + 2(7 - 17\beta^2)\left(\frac{\Delta}{2}\right)^2 + \left(\frac{\Delta}{2}\right)^4 = 0,$$

which are exactly the same as those in Ref. [32]. These constraints on the model parameters for the Juddian solutions have not been derived in the direct extension of the Braak's approach to the two-photon model [17]. From Eq. (50), we know that the proportionality is justified only for even or odd photonic numbers, respectively. The Juddian solutions correspond to those states which are degenerate, and therefore are excluded within this proportionality, so the level crossing points of lines from G_e^+ and G_e^- and those from G_o^+ and G_o^- correspond to Juddian solutions.

IV. SUMMARY

In this paper, by using extended coherent states, Braak's exact solution in the QRM is recovered explicitly in an alternative more physical way. In a fully analogous way, the two-photon QRM is also exactly solved with the use of extended squeezed states. The corresponding G functions with a similar form to Braak's G function are derived. The isolated Juddian solutions can also be discussed analytically via the properties of degeneracy. Both models can be treated in a unified way by the expansion in the Fock space of the Bogoliubov operators. Further extensions to other more complicated systems, such as the multilevel, even multimode, spin-boson model, are straightforward, although perhaps a little tedious sometimes.

For a multilevel spin-boson model, such as the finite-sized Dicke model [33], quantum chaos has been discussed [34]. We have expanded the wave function in $N+1$ Bogoliubov operators for the Dicke model with finite-N two-level atoms, and obtained numerically exact solutions previously [21]. According to the above discussion and the link with Braak's solutions, exact solvability is ensured without doubt in this system. The quasi-integrability and quantum chaos in this system should be very interesting. On the other hand, the multimode QRM has also been realized experimentally in circuit QED systems [9]. Extensions to these systems are in progress.

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