

## Dynamic generation of spin-squeezed states in bosonic Josephson junctions

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We analyze the formation of squeezed states in a condensate of ultracold bosonic atoms confined by a double-well potential. The emphasis is set on the dynamical formation of such states from initially coherent many-body quantum states. Two cases are described: the squeezing formation in the evolution of the system around the stable point, and in the short-time evolution in the vicinity of an unstable point. The latter is shown to produce highly squeezed states on very short times. On the basis of a semiclassical approximation to the Bose-Hubbard Hamiltonian, we are able to predict the amount of squeezing, its scaling with  $N$ , and the speed of coherent spin formation with simple analytical formulas which successfully describe the numerical Bose-Hubbard results. This method of producing highly squeezed spin states in systems of ultracold atoms is compared to other standard methods in the literature.

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### I. INTRODUCTION

Condensates of ultracold atoms provide an exceptional tool to understand and control a number of phenomena in the fields of condensed matter, many-body quantum mechanics, and quantum information and computation [1,2]. Condensates are bosonic many-body quantum systems, the Hamiltonian of which can be tuned via Feshbach resonance techniques or by varying the trapping conditions.

In particular, we shall be interested here in condensates of ultracold bosonic atoms trapped in an external double-well potential, thus giving rise to the so-called external Josephson dynamics [3–7]. The case of atoms with two internal states trapped in a common harmonic potential is similar. In this case, the Josephson dynamics takes place between the two internal states [8]. A first relevant observation for these systems was that of the predicted self-trapped regime [9,10], which appears already in the semiclassical description of the two-site Bose-Hubbard Hamiltonian. Later, the emphasis has been set on producing strongly correlated quantum states with appealing quantum properties such as entangled states [6,11,12], or squeezed states with possible application in quantum metrology [7,13–15]. Recently, the limits imposed by finite temperature on the maximal attainable spin squeezing have been discussed in Ref. [16].

Most of the studies have concentrated on quantum many-body properties present in the ground state. Notably studying the possibility of having catlike many-body ground states [17–22] or largely squeezed states [6]. In this paper, we focus on the dynamical generation of squeezed states, that is, we consider a condensate initially prepared in a coherent state which is left to evolve in a suitable Hamiltonian so as to give rise to entangled many-body states during the time evolution. Our aim is thus to build those particular states from initial states that can be constructed with present experimental techniques. We will use the Bose-Hubbard Hamiltonian to study numerically the time evolution by solving the corresponding time-dependent Schrödinger equation (TDSE). Alternatively, we apply a semiclassical approximation (based on a perturbative expansion in  $1/N$ , with  $N$  the number of atoms) to obtain

simple and yet accurate expressions describing the dynamics of the relevant expectation values. Similar methods have been used in recent years to study the thermodynamic limit of the Lipkin-Meshkov-Glick model [23], which can be mapped into the usual two-site Bose-Hubbard, finding exact expressions for the ground state in the thermodynamic limit [24] and characterizing entanglement properties of the ground state in the same limit [25].

The article is organized as follows. First, we introduce the Bose-Hubbard (BH) Hamiltonian in Sec. II, and give a short reminder of the semiclassical approximation in Sec. III. In Secs. IV and V, we propose an experimentally feasible setup for producing dynamically a new kind of squeezed states and study their properties. A comparison with the adiabatic and diabatic one-axis squeezing is presented in Sec. VI. In Sec. VII, we outline our conclusions.

### II. TWO-SITE BOSE-HUBBARD HAMILTONIAN AND SQUEEZING

Let us consider a many-body system of bosons described by a two-site Bose-Hubbard Hamiltonian of the form  $\hbar H_{\text{BH}}$  with

$$H_{\text{BH}} = -J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U}{2}[\hat{n}_1(\hat{n}_1 - 1) + \hat{n}_2(\hat{n}_2 - 1)], \quad (1)$$

where  $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$  and  $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$ .  $J$  is the hopping strength, taken positive, and  $U$  is the nonlinear coupling strength.  $U > 0$  and  $U < 0$  correspond to repulsive and attractive interactions, respectively. To remain close to ongoing experimental realizations, we will concentrate on the case of repulsive interactions among the atoms. The time-dependent Schrödinger equation is written as

$$i\partial_t |\Psi\rangle = H_{\text{BH}} |\Psi\rangle. \quad (2)$$

An appropriate many-body basis for this bosonic system is the Fock basis [26]  $\{|N_1, N_2\rangle\}$ , with  $N_1 + N_2 = N$ . Since the total number of atoms  $N$  is taken to be constant, it will be more convenient to introduce a different notation:

$N_1 = k$ ,  $N_2 = N - k$ . A general many-body state  $|\Psi\rangle$  can then be written in this basis as

$$|\Psi\rangle = \sum_{k=0}^N c_k |k, N-k\rangle. \quad (3)$$

The low-energy stationary states of the system are characterized by values of  $c_k$  that vary smoothly with  $k$  and that take vanishingly small values when  $k \rightarrow 0$  or  $k \rightarrow N$ , which corresponds to negligible probabilities for finding almost all the atoms on one of the two sites.

It is customary to define three operators  $\hat{\mathbf{J}} \equiv (\hat{J}_x, \hat{J}_y, \hat{J}_z)$  [26,27]:

$$\begin{aligned} \hat{J}_x &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \\ \hat{J}_y &= \frac{1}{2i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1), \\ \hat{J}_z &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2). \end{aligned} \quad (4)$$

In terms of these, the Hamiltonian reads as

$$H_{\text{BH}} = -2J\hat{J}_x + U\hat{J}_z^2 + U\left(\frac{\hat{N}^2}{4} - \frac{\hat{N}}{2}\right). \quad (5)$$

An important consequence of the form of this Hamiltonian is the existence of squeezed-spin eigenstates in the Fock representation [11]. This pseudospin is the one defined in Eq. (4). These states are of special importance as they incorporate correlations which are beyond mean field.

Here instead we will study the dynamical generation of squeezing: we assume that at  $t = 0$  the system is initially prepared in a coherent state characterized by  $(\theta, \phi)$  [27]:

$$|\Psi_{\theta, \phi}\rangle = \sum_k \binom{N}{k}^{1/2} (\cos \theta/2)^k (e^{i\phi} \sin \theta/2)^{N-k} |k, N-k\rangle,$$

which corresponds to a state in which all atoms populate the same single-particle state  $\cos(\theta/2)|1\rangle + e^{i\phi} \sin(\theta/2)|2\rangle$  where  $|1\rangle = \hat{a}_1^\dagger |\text{vac}\rangle$  and  $|2\rangle = \hat{a}_2^\dagger |\text{vac}\rangle$ . Such states have been recently engineered, producing and characterizing them in a wide range of values of  $(\theta, \phi)$  [8].

Moreover, coherent states have simple expectation values of  $\hat{J}_x$ ,  $\hat{J}_y$ , and  $\hat{J}_z$  [27],

$$\begin{aligned} \langle \Psi_{\theta, \phi} | \hat{J}_x | \Psi_{\theta, \phi} \rangle &= \frac{N}{2} \sin \theta \cos \phi, \\ \langle \Psi_{\theta, \phi} | \hat{J}_y | \Psi_{\theta, \phi} \rangle &= \frac{N}{2} \sin \theta \sin \phi, \\ \langle \Psi_{\theta, \phi} | \hat{J}_z | \Psi_{\theta, \phi} \rangle &= \frac{N}{2} \cos \theta, \end{aligned} \quad (6)$$

which allow us to represent them on the surface of a sphere of radius  $N/2$ . They can be used to define a Husimi distribution of any given many-body state  $|\Phi\rangle$  [19]:

$$\rho_{\text{H}}(\theta, \phi) = |\langle \Psi_{\theta, \phi} | \Phi \rangle|^2. \quad (7)$$

As an example, it is useful to note that the Husimi distribution of a coherent state characterized by  $(\theta', \phi')$  is given by

$$\begin{aligned} \rho_{\text{H}}(\theta, \phi) &= 2^{-N} [1 + \cos(\theta) \cos(\theta') \\ &\quad + \cos(\phi' - \phi) \sin(\theta) \sin(\theta')]^N, \end{aligned} \quad (8)$$

which has a maximum of 1 for  $(\theta, \phi) = (\theta', \phi')$ .

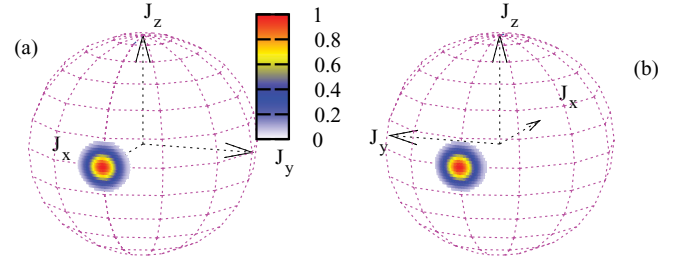


FIG. 1. (Color online) Husimi distribution  $\rho_{\text{H}}(\theta, \phi)$  of the state  $\Psi_{\pi/2,0}$  (a), and  $\Psi_{\pi/2,\pi}$  (b).  $N = 200$ .

In our study, we will consider as initial states two different coherent states:

$$\begin{aligned} |\Psi_{\pi/2,0}\rangle &= \mathcal{N}_0 (\hat{a}_1^\dagger + \hat{a}_2^\dagger)^N |\text{vac}\rangle, \\ |\Psi_{\pi/2,\pi}\rangle &= \mathcal{N}_\pi (\hat{a}_1^\dagger - \hat{a}_2^\dagger)^N |\text{vac}\rangle, \end{aligned} \quad (9)$$

with  $\mathcal{N}_{0,\pi}$ , normalization constants. The coefficients  $|c_k|^2$  obey in both cases a binomial distribution

$$|c_k|^2 = \frac{1}{2^N} \binom{N}{k}, \quad (10)$$

and their Husimi distributions are

$$\rho_{\text{H}}(\theta, \phi) = \left( \frac{1 \pm \cos(\phi) \sin(\theta)}{2} \right)^N, \quad (11)$$

where the  $+$  and  $-$  signs correspond to the  $(\pi/2, 0)$  and  $(\pi/2, \pi)$ , respectively. For large  $N$ , the equiprobability lines correspond to circles around  $(\theta, \phi) = (\pi/2, 0)$  and  $(\pi/2, \pi)$ , respectively. The distributions are presented in Fig. 1.

Both initial states are especially interesting for two reasons: (1) they correspond to two relevant limiting cases which can be prepared in the laboratory, and (2) they give rise to different dynamical evolutions for  $\Lambda \neq 0$ . Starting from the  $(\pi/2, 0)$  state, the system evolves in the vicinity of a stable point in the semiclassical limit, producing in a natural way periodic dynamics. In contrast, a system initially prepared in the  $(\pi/2, \pi)$  state evolves in the vicinity of an unstable point [28] in the semiclassical picture. That difference causes the very different maximal coherent squeezing found in the two cases. This will be discussed in greater detail in the next sections.

### A. Squeezing parameters

As customary [6], the number-squeezing parameter is defined as

$$\xi_N^2(t) = \frac{\Delta \hat{J}_z^2}{(\Delta \hat{J}_z^2)_{\text{bin}}}, \quad (12)$$

where  $\Delta \hat{J}_z^2 \equiv \langle \hat{J}_z^2 \rangle - \langle \hat{J}_z \rangle^2$  and  $(\Delta \hat{J}_z^2)_{\text{bin}} = N/4$  in the binomial case (10). The many-body state is said to be squeezed

if  $\xi_N < 1$ . A second parameter, which takes into account the coherence of the state, is the so-called coherent spin-squeezing parameter defined as [14]<sup>1</sup>

$$\xi_S^2 = \frac{2J(\Delta \hat{J}_z^2)}{\langle \hat{J}_x \rangle^2} = \frac{\xi_N^2}{\alpha^2}, \quad (13)$$

where the phase coherence is given by

$$\alpha(t) = \langle \Psi(t) | \hat{\alpha} | \Psi(t) \rangle, \quad \hat{\alpha} = 2 \frac{\hat{J}_x}{N}. \quad (14)$$

The two initial states we are considering have  $\alpha(0) = 1$  and  $-1$  corresponding to  $(\pi/2, 0)$  and  $(\pi/2, \pi)$ , respectively.

If a state exhibits  $\xi_S < 1$ , it can be employed in a Ramsey-type atom interferometer with an increased phase precision compared to the coherent spin state. This gain in precision can be directly related to entanglement in the system [29].

### B. Angle of maximal squeezing

Number squeezing of a many-body state can occur along an axis different from the  $z$  axis considered above. In that case, one can generalize the squeezing parameter for an arbitrary direction  $\mathbf{u} \equiv (u_x, u_y, u_z)$  ( $\mathbf{u}^2 = 1$ ) as

$$\xi_{N;\mathbf{u}}^2 = \frac{\Delta(\mathbf{u} \cdot \hat{\mathbf{J}})^2}{N/4}, \quad (15)$$

where the denominator is again the fluctuation of the binomial distribution. The squeezing along any direction in the  $(y, z)$  plane only requires to calculate  $\langle (\mathbf{u} \cdot \hat{\mathbf{J}})^2 \rangle$  as  $\langle \mathbf{u} \cdot \hat{\mathbf{J}} \rangle = 0$ . The corresponding generalization of the coherent spin-squeezing parameter of Eq. (13) reads as

$$\xi_{S;\hat{\mathbf{u}}}^2 = \frac{\xi_{N;\mathbf{u}}^2}{\alpha^2}. \quad (16)$$

As the wave packet evolves in time, there is a certain direction  $z'$  in which the spin squeezing is maximal. In a frame rotated an angle  $\beta$  around the  $x$  axis we have

$$\begin{aligned} \hat{J}_{y'} &= \cos \beta \hat{J}_y + \sin \beta \hat{J}_z, \\ \hat{J}_{z'} &= -\sin \beta \hat{J}_y + \cos \beta \hat{J}_z. \end{aligned} \quad (17)$$

Since

$$\langle \hat{J}_{z'}^2 \rangle = \sin^2 \beta \langle \hat{J}_y^2 \rangle + \cos^2 \beta \langle \hat{J}_z^2 \rangle - \sin \beta \cos \beta \langle \{\hat{J}_y, \hat{J}_z\} \rangle, \quad (18)$$

requiring that  $d\langle \hat{J}_{z'}^2 \rangle / d\beta = 0$  gives the angle of maximal squeezing

$$\tan 2\beta_M = \frac{\langle \{\hat{J}_y, \hat{J}_z\} \rangle}{\langle \hat{J}_y^2 \rangle - \langle \hat{J}_z^2 \rangle}. \quad (19)$$

We will use the notation  $\xi_{S;\beta_M}^2$  and  $\xi_{N;\beta_M}^2$  for the maximal coherent spin squeezing and number squeezing.

It is worth noting the role played by  $\langle \{\hat{J}_y, \hat{J}_z\} \rangle$ . If this term is zero, the maximal squeezing is always found either along

$J_y$  or  $J_z$ . A nonzero value of  $\langle \{\hat{J}_y, \hat{J}_z\} \rangle$  implies that the best squeezing will be found along some other axis.

Equation (19) will allow us to compute at any time during the evolution the direction along which the squeezing is maximal. This will be of special relevance for the case where the initial state is  $|\Psi_{\pi/2,\pi}\rangle$ . As will be shown in Sec. V, in this case the maximal squeezing gets quite sizable in the short-time evolution of the system.

Using the Bose-Hubbard Hamiltonian, in Secs. IV and V we will compute  $\langle \hat{J}_i^2 \rangle(t)$  and the associated squeezing parameters for varying ratios of the tunneling versus atom-atom interaction strength, and present evidence for spin squeezing during the time evolution of the system. To better interpret these numerical results, we will first develop approximate expressions using a semiclassical model.

### III. 1/N APPROXIMATION TO THE BOSE-HUBBARD MODEL

The appearance of spin squeezing in the evolution of the system can be studied numerically by solving the TDSE [Eq. (2)]. It is, however, desirable to find suitable approximations which can expose the physics underneath the process of spin squeezing. In this section, we develop such an approximate model and show that the time evolution of the system can be successfully mapped into the physics of a single fictitious particle evolving on a confining or nonconfining parabolic potential for the  $(\pi/2, 0)$  or  $(\pi/2, \pi)$  states, respectively.

Following [21], we introduce first an auxiliary Hamiltonian defined as

$$H_S = -\frac{2}{N} \hat{J}_x + \frac{U}{NJ} \hat{J}_z^2 = -2h \hat{J}_x + 2\Lambda h^2 \hat{J}_z^2 \quad (20)$$

with  $h = 1/N$  and  $\Lambda = NU/(2J)$ . It differs from  $H_{\text{BH}}$  in Eq. (5) in the suppression of the additive constants and in a factor  $NJ$ , which makes it dimensionless. In the considered regime, the expectation values of the two terms in Eq. (20) are of similar magnitude, so that the factors  $h$  compensate the different  $N$  dependence of the expectation values of the two spin operators.

In Refs. [21,30], a semiclassical approximation to the TDSE has been derived. It uses a systematic expansion in  $1/N$ . Here, we will build on this method and extend it to the expectation values of the quantities required to compute the spin squeezing and the coherence [Eqs. (12) and (14)]. Earlier versions of the same expansion can be found also in Refs. [19,31]. As explained in detail in the Appendix, the expectation values of  $J_x$  and  $J_z^2$  can be computed from the continuous extension of the  $c_k$ 's. To deal with states close to the  $|\Psi(\pi/2, 0)\rangle$  state [Eq. (9)], one assumes that the states of interest are such that their  $c_k$  vary smoothly:  $c_k \sim c_{k\pm 1}$  and that the number of atoms is always large,  $h = 1/N \ll 1$ . This allows us to introduce a continuous variable  $x$  and a continuous function  $\psi(x)$  such that  $\psi(x = k/N) = \sqrt{N} c_k$  [21,30–32]. Next, a new variable  $z \equiv 2x - 1$  is defined, and  $\psi(z)$  ( $-1 \leq z \leq 1$ ) renormalized to  $\int_{-1}^1 dz |\psi(z)|^2 = 1$ . The expressions for the

<sup>1</sup>In the cases we will consider, during the time evolution the wave packet remains at  $\langle \hat{J}_y \rangle = 0$  at all times.

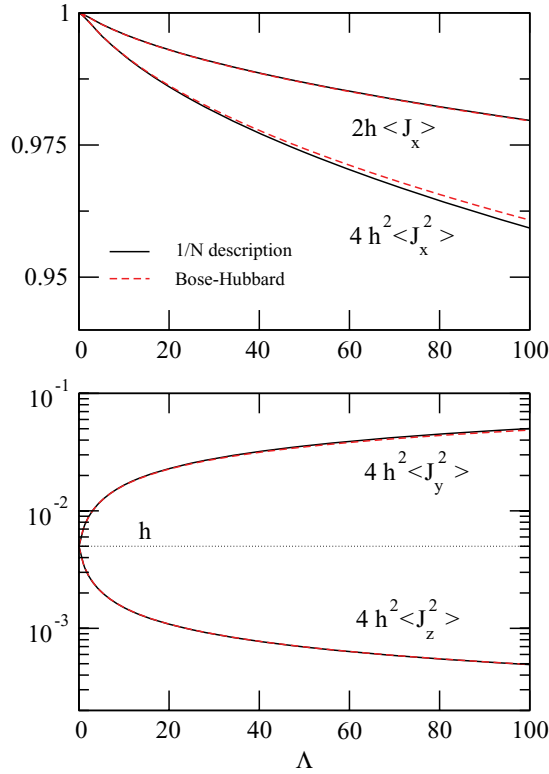


FIG. 2. (Color online) Comparison between the ground-state properties obtained through expressions (A7), solid lines, and the Bose-Hubbard computation for  $N = 200$  (dashed lines) as function of  $\Lambda = NU/(2J)$ .

expectation values are

$$\begin{aligned}
 h \langle \psi | \hat{J}_x | \psi \rangle &\simeq \int_{-1}^1 dz \psi^*(z) \left[ \left( \frac{h^2(-1-z^2)}{4(1-z^2)^{3/2}} \right. \right. \\
 &\quad \left. \left. + \frac{h}{2\sqrt{1-z^2}} + \frac{\sqrt{1-z^2}}{2} \right) \psi(z) \right. \\
 &\quad \left. - \frac{h^2 z}{\sqrt{1-z^2}} \psi'(z) + h^2 \sqrt{1-z^2} \psi''(z) \right], \\
 h^2 \langle \psi | \hat{J}_z^2 | \psi \rangle &= \int_{-1}^1 dz |\psi(z)|^2 \frac{z^2}{4}. \quad (21)
 \end{aligned}$$

As in many other semiclassical expansions, the power series in  $h$  is asymptotic, and one can see above that depending on the behavior of the chosen  $\psi(z)$  as  $z \rightarrow \pm 1$ , divergent contributions will appear already at order  $h^2$ . As usual for asymptotic series, the strategy that we will follow is to truncate those terms that degrade the convergence. We will detail later how this is done. The validity of this  $1/N$  expansion can be seen in Fig. 2, where we show a comparison between our expressions and the exact Bose-Hubbard calculation of the ground-state properties of the system.

Let us now go back to the Hamiltonian  $H_S$  in Eq. (20). Using the above results, its semiclassical expectation value is

$$\langle \psi | H_S | \psi \rangle = -2h \langle \psi | \hat{J}_x | \psi \rangle + 2\Lambda h^2 \langle \psi | \hat{J}_z^2 | \psi \rangle. \quad (22)$$

When we look for the stationary points of  $\langle \psi | H_S | \psi \rangle - E^{(s)} \langle \psi | \psi \rangle$ , we arrive at

$$\begin{aligned}
 \mathcal{H}_N(z) \psi(z) &\equiv -2h^2 \left( \sqrt{1-z^2} \psi'' - \frac{z}{\sqrt{1-z^2}} \psi' \right) \\
 &\quad + \left( \frac{1}{2} \Lambda z^2 - \sqrt{1-z^2} + \delta \mathcal{V} \right) \psi(z) \\
 &\equiv -2h^2 \partial_z \sqrt{1-z^2} \partial_z \psi + \mathcal{V}(z) \psi = E^{(s)} \psi(z), \quad (23)
 \end{aligned}$$

which is a pseudo-Schrödinger equation similar to the one reported in Ref. [21] except for the additional term  $\delta \mathcal{V}$ :

$$\delta \mathcal{V} = -\frac{h}{\sqrt{1-z^2}} + h^2 \frac{(1+z^2)}{(1-z^2)^{3/2}}, \quad (24)$$

which was neglected in Ref. [21].

Equation (23) can be regarded as a Schrödinger-type equation defined on a compact interval  $z \in [-1, 1]$ . It is expected to provide accurate results provided  $\psi(z)$  vanishes at the boundaries. The equation provides an important insight into the problem, essentially builds on the semiclassical Hamiltonian, which is equal to  $\mathcal{V}(z)$ , and quantizes it, through the effective mass form  $-2h^2 \partial_z \sqrt{1-z^2} \partial_z$ .

In line with the present approximation, the time evolution will then be described via

$$i \hbar \partial_t \psi(z, t) = \mathcal{H}_N \psi(z, t), \quad (25)$$

where now  $t$  is the time measured in units of  $1/J$ . The so-called ‘‘Rabi’’ time of the system is  $t_{\text{Rabi}} = \pi/J$ .

#### IV. DYNAMICAL SQUEEZING AROUND A FIXED STABLE POINT: $\Psi_{\pi/2,0}$ STATE

We consider now the dynamical situation where the condensate is initially prepared in the coherent state  $\Psi(\pi/2, 0)$ , and study the squeezing and coherence of the system as a function of time as it evolves under the action of  $\mathcal{H}_N$ .

In the limit of large  $N$ , small  $h$ , the binomial distribution  $|c_k|^2$  corresponding to the state  $\Psi_{\pi/2,0}$  [see Eq. (10)] approaches the Gaussian distribution in the continuous  $z$  variable

$$\psi_0(z) = \left( \frac{1}{\pi b_0^2} \right)^{1/4} e^{-z^2/(2b_0^2)}, \quad (26)$$

with  $b_0^2 = 2h = 2/N$ . During the time evolution,  $|\psi(z, t)|^2$  will be confined to a fairly narrow region in  $z$  of size  $\simeq \sqrt{2h}$ . For this range of values of  $z$ , we will approximate  $\sqrt{1-z^2} \simeq 1$  in the kinetic energy term of  $\mathcal{H}_N$ , and make a parabolic approximation to  $\mathcal{V}(z)$ :

$$\mathcal{V}(z) \simeq -1 - h + \frac{1}{2} \frac{1}{4} \omega^2 z^2, \quad (27)$$

with effective mass equal to  $1/4$  and frequency given by  $\omega = 2\sqrt{1} + \Lambda - \hbar$ . Thus, the evolution of the  $\Psi_{\pi/2,0}$  state is mapped into the evolution of a centered Gaussian wave packet inside a confining harmonic-oscillator potential. The system will oscillate around the classical stable point, periodically building a certain amount of coherent spin squeezing that we will quantify in the following. The parabolic approximation

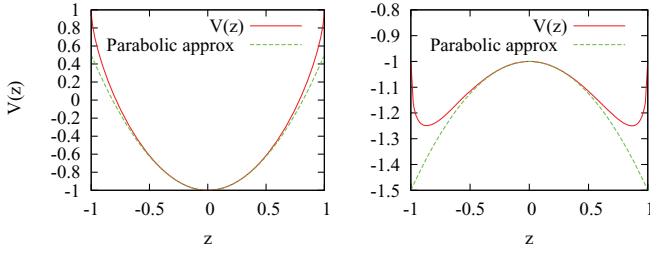


FIG. 3. (Color online) Depiction of  $V(z)$  and its parabolic approximation used in Secs. IV (left) and V (right).

is extremely accurate for our purposes. This is because the initial extent of the packet  $\sqrt{\langle z^2(0) \rangle} = \sqrt{2\hbar} \ll 1$  is always the maximum value attainable during the time evolution.

Under the parabolic approximation for the potential (see Fig. 3), the initial Gaussian wave packet [Eq. (26)] remains Gaussian as it evolves in time. The exact wave function reads as

$$\psi_{\Lambda}(z, t) = \frac{1}{(\pi b^2)^{1/4}} e^{ik} e^{-z^2/(2b^2)} e^{iz^2\phi/(2b^2)}, \quad (28)$$

where

$$\begin{aligned} b^2(t) &= h \left( 1 + \frac{4}{\omega^2} + \left( 1 - \frac{4}{\omega^2} \right) \cos 2\omega t \right), \\ \phi(t) &= \frac{\omega}{4} \left( \frac{4}{\omega^2} - 1 \right) \sin 2\omega t, \\ \kappa(t) &= \frac{1+h}{h}t + \frac{1}{4} \arctan \left( \frac{\omega}{2 \tan \omega t} \right) - \frac{\pi}{8} - \frac{\pi}{4} \left[ \frac{\omega t}{\pi} \right], \end{aligned} \quad (29)$$

where in the last equation  $[x]$  means integer part of  $x$ . Now, we insert the exact  $\psi_{\Lambda}(z, t)$  in the semiclassical expressions for the expectation values of the spin components [Eqs. (21) and (A7)] and replace the denominators by their approximations for small  $z$ , i.e.,  $1/\sqrt{1-z^2} \simeq 1+z^2/2$  or  $1$  depending on the size of their contribution. And finally, we retain terms up to linear in  $\hbar$  (note that  $b^2$  is proportional to  $\hbar$ ):

$$\begin{aligned} 2\hbar \langle \hat{J}_x \rangle &\simeq 1 + \frac{\hbar}{4} \frac{\Lambda^2}{1+\Lambda} (\cos 2\omega t - 1), \\ 4\hbar^2 \langle \hat{J}_x^2 \rangle &\simeq 1 + \frac{\hbar}{2} \frac{\Lambda^2}{1+\Lambda} (\cos 2\omega t - 1), \\ 4\hbar^2 \langle \hat{J}_y^2 \rangle &\simeq \frac{\hbar}{2} (2 + \Lambda - \Lambda \cos 2\omega t), \\ 4\hbar^2 \langle \hat{J}_z^2 \rangle &\simeq \frac{\hbar}{2(1+\Lambda)} (2 + \Lambda + \Lambda \cos 2\omega t), \\ 4\hbar^2 \langle \{\hat{J}_y, \hat{J}_z\} \rangle &\simeq h \frac{\Lambda}{\sqrt{1+\Lambda}} \sin 2\omega t. \end{aligned} \quad (30)$$

Within the same approximation, the angle of maximal squeezing [Eq. (19)] can be written as

$$\tan 2\beta_M \simeq \frac{2\sqrt{1+\Lambda}}{2+\Lambda} \frac{1}{\tan \omega t}. \quad (31)$$

These approximate expressions turn out to be very accurate for a broad set of parameters. In Fig. 4, we compare the exact Bose-Hubbard results and those obtained from Eqs. (30). The initial state is  $|\Psi_{\pi/2,0}\rangle$  and is left to evolve in a Hamiltonian

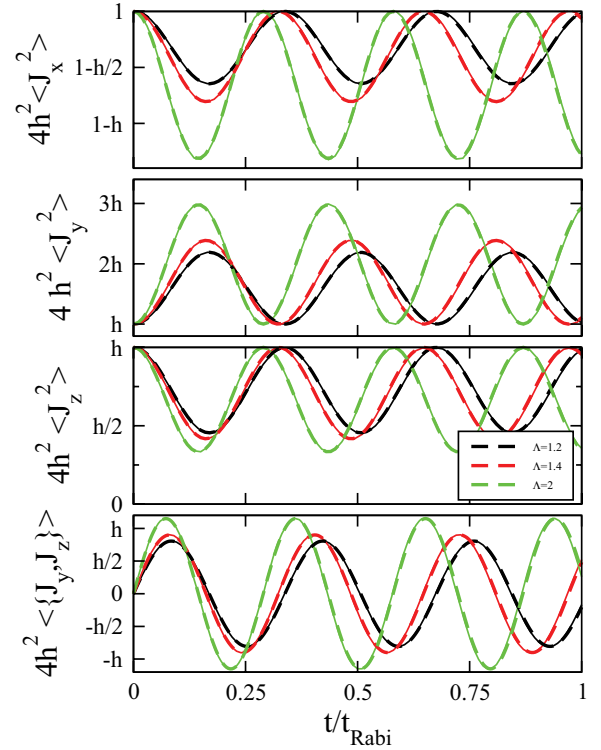


FIG. 4. (Color online) Comparison between the Bose-Hubbard results (dashed lines) and the expressions in Eq. (30) (solid lines) for different values of  $\Lambda = 1.2, 1.4,$  and  $2$ . The number of particles is  $N = 200 = 1/h$ . The initial state is  $\Psi_{\pi/2,0}$ .

with repulsive atom-atom interactions of  $\Lambda = 1.2, 1.4,$  and  $2$ . The expectation value of  $\hat{J}_i^2$  is presented  $i = x, y, z$  together with the expectation value of  $\{\hat{J}_y, \hat{J}_z\}$ . As can be seen, BH predicts periodic oscillations for all the quantities.  $\langle \hat{J}_x^2 \rangle$  is seen to be essentially 1 during the time evolution. The small departure from full coherence is well captured by the term  $\propto \hbar$  in the semiclassical expression.  $\langle \hat{J}_z^2 \rangle$  and  $\langle \hat{J}_y^2 \rangle$  are found to evolve in phase, as predicted in (30).  $\langle \{\hat{J}_y, \hat{J}_z\} \rangle$  is small but nonzero during the evolution, implying the existence of a direction along which the squeezing is maximal.

According to Eqs. (30), the wave packet will squeeze periodically along the  $z$  direction with a frequency  $2\omega$ . The maximal attainable number squeezing takes place when  $2\omega t = n\pi$ , and is

$$\xi_{N,\max}^2 = \frac{1}{1+\Lambda}. \quad (32)$$

Similarly we find that the coherence at maximal squeezing is given by

$$\langle \hat{\alpha} \rangle_{\text{max sq}} = 1 - h \frac{\Lambda^2}{2(1+\Lambda)}. \quad (33)$$

The semiclassical predictions break down when the extent of the wave packet  $\sqrt{\langle z^2 \rangle}$  is of the order of  $h$ . Using Eqs. (30) at the maximum number squeezing yields the condition  $\Lambda \lesssim 1/h = N$ .

As explained in the previous section, a nonzero value of the anticommutator  $\langle \{\hat{J}_y, \hat{J}_z\} \rangle$ , as in Fig. 4, implies that the maximal squeezing is found along an axis  $z'$ , defined

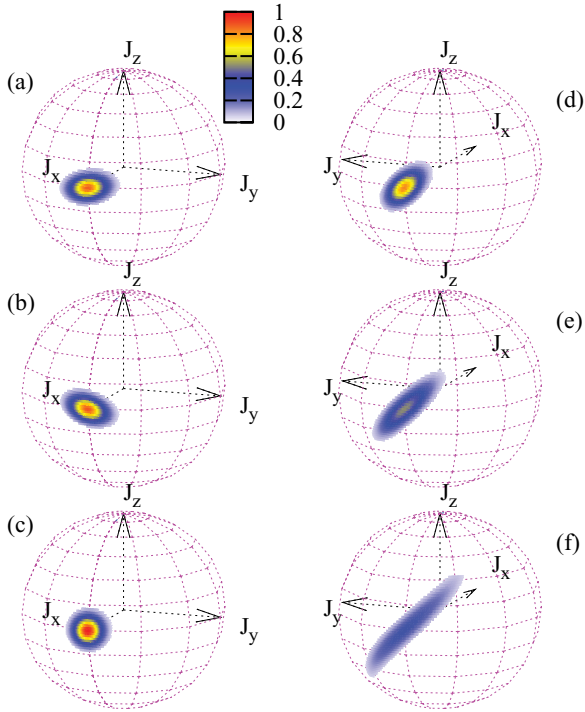


FIG. 5. (Color online) Snapshots of the Husimi distribution  $\rho_H(\theta, \phi)$ . Panels (a), (b), (c) and (d), (e), (f) correspond to initial states  $\Psi_{\pi/2,0}$  and  $\Psi_{\pi/2,\pi}$ , respectively. Panels (a) and (d) are computed at  $t = 0.1t_{\text{Rabi}}$ , (b) and (e) at  $t = 0.2t_{\text{Rabi}}$ , and (c) and (f) at  $t = 0.3t_{\text{Rabi}}$ .  $N = 200$  and  $\Lambda = 2$ .

by an angle  $\beta_M$  [see Eq. (19)]. This also reflects in the Husimi distributions depicted in Fig. 5. In the figure we present three snapshots of the Husimi distributions at different times 0.1, 0.2, and 0.3  $t_{\text{Rabi}}$  computed for  $\Lambda = 2$ . The Husimi distribution is initially symmetric (see Fig. 1), as corresponds to a coherent state. As time evolves, in Figs. 5(a), 5(b), and 5(c) the distribution is seen to be ellipsoidal but noncanonical, i.e., the symmetry axes of the ellipses are not  $y$  and  $z$ . The angle of maximal squeezing is plotted in Fig. 6. The angle varies almost linearly with time, implying that the distribution rotates around the  $x$  direction at an almost constant velocity. This behavior is captured by Eq. (31).

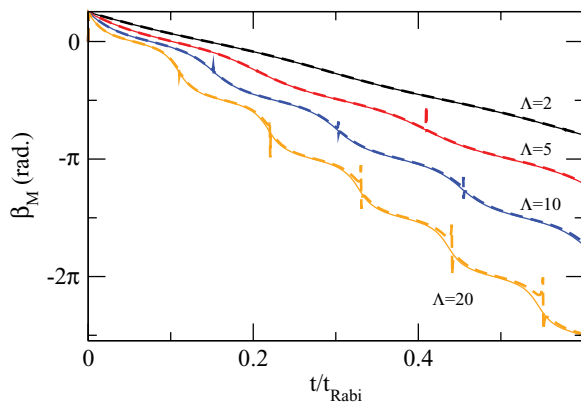


FIG. 6. (Color online) Angle of maximal squeezing [Eq. (19)] computed from the Bose-Hubbard calculation (dashed lines), and using Eq. (31) (solid lines).  $N = 400$ .

## V. EARLY SQUEEZING AROUND AN UNSTABLE POINT: $\Psi_{\pi/2,\pi}$ STATE

When considering the dynamics around the  $(\pi/2, \pi)$  state, in order to make use of the semiclassical model one has to assume that it is the  $(-)^k c_k$  that vary smoothly, and thus introduce a continuous function  $\psi(x = k/N) = \sqrt{N}(-1)^k c_k$  [21,30,32]. As explained in Ref. [21], see also the expressions in our Appendix, the dynamical equation in this case reads as, including only the lowest order in  $\hbar$  terms,

$$i\hbar\partial_t\psi(z,t) = \left(2\hbar^2\partial_z\sqrt{1-z^2}\partial_z + \frac{1}{2}\Lambda z^2 + \sqrt{1-z^2}\right)\psi(z,t), \quad (34)$$

with a negative effective mass. For convenience, we choose to multiply by  $-1$  both sides of the equation and perform complex conjugation, so that

$$i\hbar\partial_t\psi^*(z,t) = (-2\hbar^2\partial_z\sqrt{1-z^2}\partial_z + \mathcal{V}_-(z))\psi^*(z,t), \quad (35)$$

and the evolution of  $\psi^*(z,t)$  is that of an initial wave packet, again of the form of Eq. (26), inside the potential  $\mathcal{V}_-(z) = -(1/2)\Lambda z^2 - \sqrt{1-z^2}$ . When  $\Lambda > 1$ , this is a double-well potential in the  $z$  space [see Fig. 3 (right)] and has a central barrier. Including terms of order  $\hbar$ , we approximate it as

$$\mathcal{V}_-(z) \simeq -1 - \hbar - \frac{1}{4}\bar{\omega}^2 z^2, \quad (36)$$

where  $\bar{\omega} = 2\sqrt{\Lambda - 1 + \hbar}$ . Although this parabolic potential is nonconfining, we still find that the solution of Eq. (35) with  $\mathcal{V}_-(z)$  as in Eq. (36) is formally identical to Eq. (28), so that (up to a phase depending only on  $t$ )

$$\psi_\Lambda^*(z,t) = \frac{1}{[\pi b^2]^{1/4}} e^{-\frac{z^2}{2b^2(t)}} e^{\frac{i\phi(t)z^2}{2b^2(t)}}. \quad (37)$$

However, inserting this  $\psi_\Lambda$  in Eq. (35) with the parabolic approximation for  $\mathcal{V}_-(z)$ , one now finds

$$b^2(t) = \hbar \left[ 1 - \frac{4}{\bar{\omega}^2} + \left( 1 + \frac{4}{\bar{\omega}^2} \right) \cosh 2\bar{\omega}t \right], \quad (38)$$

$$\phi(t) = \frac{\bar{\omega}}{4} \left( \frac{4}{\bar{\omega}^2} + 1 \right) \sinh 2\bar{\omega}t,$$

and correspondingly

$$2\hbar\langle\hat{J}_x\rangle \simeq -1 + \frac{\hbar}{4}\frac{\Lambda^2}{\Lambda-1}(\cosh 2\bar{\omega}t - 1),$$

$$4\hbar^2\langle\hat{J}_x^2\rangle \simeq 1 - \frac{\hbar}{2}\frac{\Lambda^2}{\Lambda-1}(\cosh 2\bar{\omega}t - 1),$$

$$4\hbar^2\langle\hat{J}_y^2\rangle \simeq \frac{\hbar}{2}(2 - \Lambda + \Lambda \cosh 2\bar{\omega}t), \quad (39)$$

$$4\hbar^2\langle\hat{J}_z^2\rangle \simeq \frac{\hbar}{2(\Lambda-1)}[\Lambda(\cosh 2\bar{\omega}t + 1) - 2],$$

$$4\hbar^2\langle\{\hat{J}_y, \hat{J}_z\}\rangle \simeq -\hbar\frac{\Lambda}{\sqrt{\Lambda-1}}\sinh 2\bar{\omega}t.$$

Figure 7 shows that these expressions provide an accurate account of the short-time dynamics of the system: Eqs. (39) predict a fast exponential growth of  $\langle\hat{J}_{y,z}^2\rangle$ , while the system

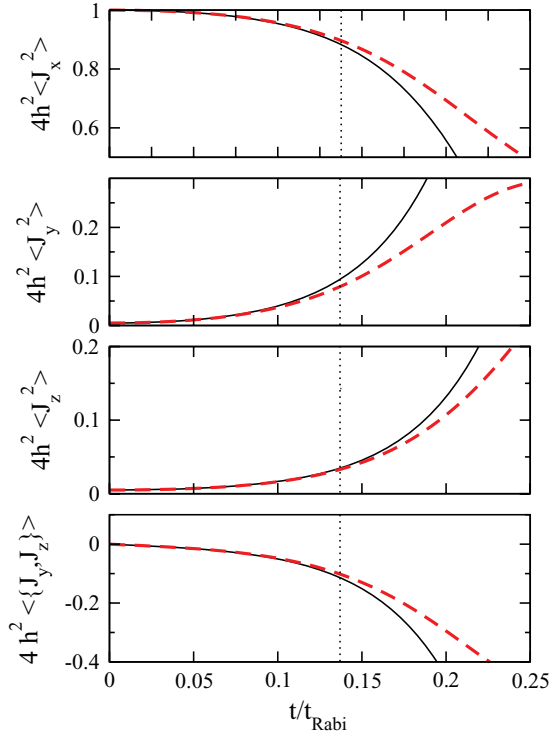


FIG. 7. (Color online) Comparison between the exact Bose-Hubbard (BH) result (dashed lines) and the analytic expressions in Eq. (39) (solid lines) for  $\Lambda = 4$ . The number of particles is  $N = 200$ . The initial state is  $\Psi_{\pi/2,\pi}$ . The dotted line marks the breaking of the parabolic approximation and is given by Eq. (42).

remains mostly coherent, which agrees well with the full Bose-Hubbard calculation. The results suggest that the evolution of this state will produce much larger squeezing, as we will quantify in the following, than in the case of the  $(\pi/2, 0)$  state, where  $4h^2 \langle \hat{J}_{y,z}^2 \rangle \sim h$ . In fact, it will be during this short-time evolution that the system will build its maximum coherent squeezing. Therefore, the simple analytical predictions provide a powerful tool to characterize the way squeezing is produced in the system.

In contrast with the  $(\pi/2, 0)$  case, now  $|\psi_\Lambda(z, t)|^2$  gets broader in  $z$  space during the time evolution. Thus, the simplified model should break down whenever the extent of the wave packet is comparable to the size of the allowed range for  $z$ :  $\sqrt{\langle z^2 \rangle} \simeq 1$ , or when the momentum  $\hat{p}_z = -i\hbar\partial_z$  is larger than the maximum possible, due to the underlying discretization

$$\sqrt{\langle \hat{p}_z^2 \rangle} \equiv \sqrt{\langle (-\hbar^2 \partial_z^2) \rangle} \simeq 1/2. \quad (40)$$

A good estimate of the time when the parabolic approximation breaks down is obtained from

$$\phi^2(t_{\max}) \simeq 1/h, \quad (41)$$

and thus

$$t_{\max} \simeq \frac{1}{4\bar{\omega}} \ln \left( \frac{8N}{\Lambda} \right). \quad (42)$$

This time predicts correctly why the parabolic approximation breaks down at earlier times as  $\Lambda$  is increased.

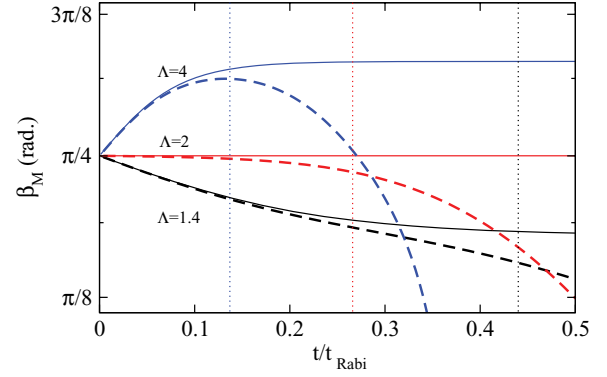


FIG. 8. (Color online) Angle for maximal squeezing obtained from the Bose-Hubbard simulation (dashed lines) and the expression (19) with the ones in Eq. (39) (solid lines). The dotted lines mark the breaking of the parabolic approximation for each  $\Lambda$  [Eq. (42)]. The number of particles is  $N = 200$ .

The evolution of the many-body state is presented in three snapshots of its Husimi distribution in Figs. 5(d), 5(e), and 5(f) for  $\Lambda = 2$ . As seen in Fig. 5, a very different behavior is found in comparison with the evolution of the  $(\pi/2, 0)$  state. In this case, the distribution becomes ellipsoidal, as expected from the nonzero values of  $\langle \{\hat{J}_y, \hat{J}_z\} \rangle$ , but does not rotate with time.

### 1. Squeezing in the initial evolution

As discussed above, in this case there is an exponential growth of  $\langle \hat{J}_{y,z}^2 \rangle$  for  $t \lesssim t_{\max}$ . This feature makes this configuration very relevant for the purpose of producing highly squeezed states along a specific direction. Inserting the semiclassical expressions given in Eq. (30), we get

$$\tan 2\beta_M \simeq -2 \frac{\sqrt{\Lambda - 1}}{\Lambda - 2} \coth(\bar{\omega}t), \quad (43)$$

which for  $t \lesssim t_{\max}$  reproduces the angle obtained with the full Bose-Hubbard calculation, as seen in Fig. 8. The angle at which the squeezing is maximal is initially  $\pi/4$  regardless of the interaction at which the evolution is performed. Different values of  $\Lambda$  produce evolutions in which either the angle grows or decreases at short times. From Eq. (43), retaining contributions linear in  $t$ , we get

$$\beta_M = \frac{\pi}{4} - \frac{1}{2}(2 - \Lambda)t. \quad (44)$$

Two important features seen in Fig. 8 are well captured by these expressions: (a) Eq. (44) predicts the angle to grow (decrease) with time for  $\Lambda < (>)2$ , (b) the value  $\Lambda = 2$  is predicted to have an almost constant angle of maximal squeezing for  $1/4$  of the Rabi time, also confirmed in the Bose-Hubbard calculation.

The usefulness of the squeezing for the improvement of interferometric measurements is characterized by the two squeezing parameters introduced in Eqs. (12) and (13) and their generalizations in Eqs. (15) and (16). In Fig. 9, we depict both  $\xi_{N;\beta_M}^2$  and  $\xi_{S;\beta_M}^2$  computed along the direction of best squeezing defined in Eq. (43). We compare the results obtained with either initial conditions considered in the article,  $\Psi_{\pi/2,0}$  and  $\Psi_{\pi/2,\pi}$ . As can be seen in the figure, starting from the  $\Psi_{\pi/2,\pi}$  the dynamically attainable coherent spin-squeezing parameter

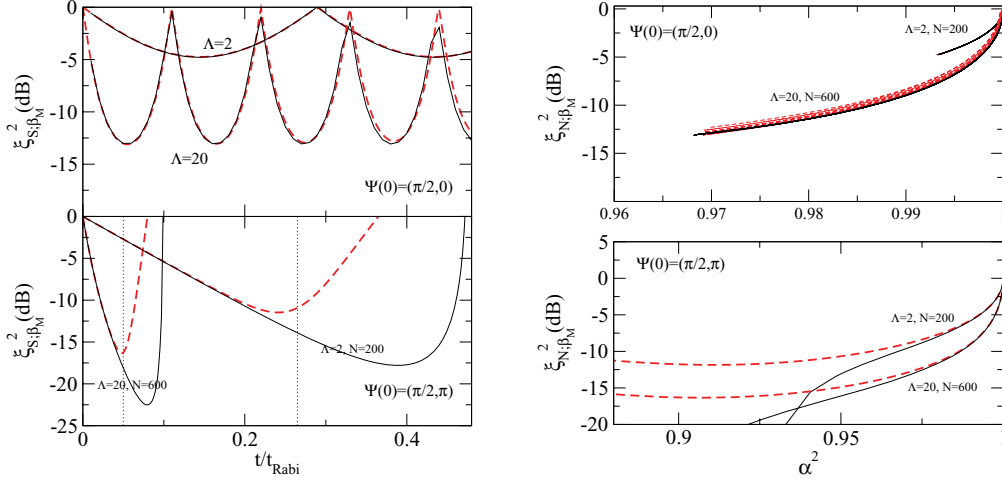


FIG. 9. (Color online) (Left) Coherent spin-squeezing parameter  $\xi_{S;\beta_M}^2$  computed in the direction of maximal squeezing as a function of time. The dotted lines mark the breaking of the parabolic approximation [Eq. (42)]. (Right) Number-squeezing parameter  $\xi_{N;\beta_M}^2$  computed in the direction of maximal squeezing as a function of the spin coherence  $\alpha^2$ . The upper and lower panels correspond to the initial states  $\Psi_{\pi/2,0}$  and  $\Psi_{\pi/2,\pi}$ , respectively. Dashed lines are Bose-Hubbard calculations, while the solid lines are obtained using Eqs. (30) and (39).

is much smaller than the attainable one from the  $\Psi_{\pi/2,0}$  state.  $\xi_S^2$  remains smaller than one for up to  $0.4 t_{\text{Rabi}}$  for  $\Lambda = 2$ . The speed of coherent spin squeezing  $\partial \xi_{S;\beta_M}^2 / \partial t$  at the angle of best squeezing is seen to be equal when starting from any of the two states

$$\frac{\partial \xi_{S;\beta_M}^2}{\partial t} = -2\Lambda. \quad (45)$$

The maximal coherent squeezing obtained for the  $(\pi/2,\pi)$  case is obtained at the time when the parabolic approximation breaks down, as seen clearly in Fig. 9. At this time scale, we have  $\tan 2\beta_M \simeq -2\sqrt{\Lambda - 1}/(\Lambda - 2)$  and

$$\xi_{S;\beta_M}^2(t_{\text{max}}) = 2\sqrt{\frac{2}{N\Lambda}}. \quad (46)$$

## VI. COMPARISON TO STANDARD SQUEEZING PROCEDURES

In Secs. IV and V, we have presented two methods of producing spin-squeezed states. The first builds on the evolution of the initial state in the vicinity of a semiclassical stable point. The second one profits from the presence of a bifurcation in the semiclassical description. In both cases, we have presented simple formulas which quantify how the coherent spin squeezing evolves with time. In this section, we will compare these two methods to standard ones: adiabatic squeezing and diabatic Kitagawa-Ueda [11] one-axis twisting.

### A. Adiabatic spin squeezing

This is the maximum spin squeezing that can be obtained in the ground states by adiabatically varying the parameters of the Bose-Hubbard Hamiltonian. Experimentally, one is limited in the variation of the atom-atom interaction but can vary the linear coupling between the two wells by ramping the potential barrier [6]. In our model, the ground states are determined by the Schrödinger equation in Eq. (23). For the range of values of  $\Lambda$  to be considered, the parabolic approximation is again

sufficient, so that for a given  $\Lambda$  the ground state is

$$\psi_{\text{GS}}(z) = \frac{1}{[\pi b_{\text{GS}}^2]^{1/4}} e^{-z^2/(2b_{\text{GS}}^2)} \quad (47)$$

with  $b_{\text{GS}}^2 = 4h/\omega = (2h)/\sqrt{1 + \Lambda - h}$ . Retaining terms linear in  $h$ ,

$$\alpha \simeq 1 + h - \frac{h}{2\sqrt{1 + \Lambda}}, \quad (48)$$

$$\xi_{N;\text{GS}}^2 \simeq \frac{1}{\sqrt{1 + \Lambda}},$$

and thus

$$\xi_{S;\text{GS}}^2(\Lambda) = \frac{1}{\sqrt{1 + \Lambda}} \left[ 1 - 2h + \frac{h}{\sqrt{1 + \Lambda}} \right]. \quad (49)$$

### B. One-axis twisting

One-axis twisting (OAT) was proposed by Kitagawa and Ueda [11]. Their Hamiltonian is  $H_{\text{KU}} = \hbar\chi \hat{J}_z^2$ . Compared to Bose-Hubbard, this implies that their  $J = 0$  and  $\chi = U$ . They worked with time  $t_{\text{KU}}$  in “time units,” whereas here we express time  $t$  in units of  $1/J$ . To have more compact expressions, they introduced  $\mu \equiv 2\chi t_{\text{KU}}$ : in our notation

$$\mu = 2Ut_{\text{KU}} = 2U \frac{1}{J} t = \frac{4}{N} \Lambda t. \quad (50)$$

Since we are here studying squeezings for times of the order of the Rabi time, and  $N \gg 1$ , this means that in our applications  $\mu$  will always be small.

The initial state considered was  $\Psi(\pi/2,0)$  [similar results are obtained for the  $\Psi(\pi/2,\pi)$ ] so that the spin remains aligned along the  $x$  axis:  $\langle \hat{J}_y \rangle = \langle \hat{J}_z \rangle = 0$ , while

$$\langle \hat{J}_x \rangle = N/2 \cos^{N-1}(\mu/2). \quad (51)$$

For small times, this simplifies to

$$\alpha = 2h \langle \hat{J}_x \rangle \simeq 1 - 2h\Lambda^2 t^2, \quad (52)$$



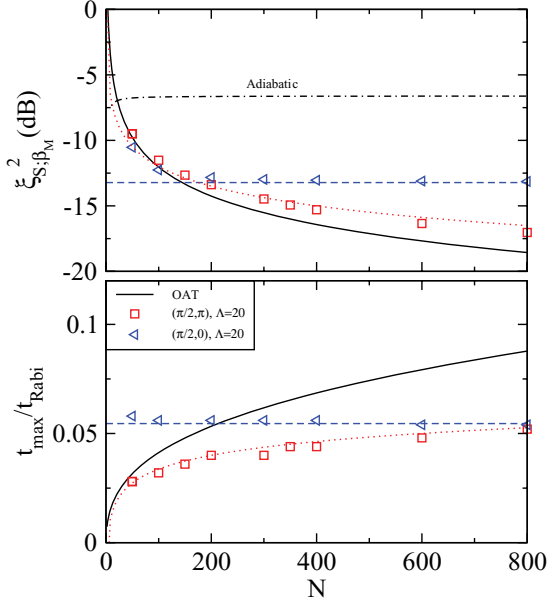


FIG. 10. (Color online) Maximum attainable coherent spin squeezing (upper panel) and the time when this maximum value is obtained (lower panel) as a function of the number of atoms  $N$ . We compare the methods described in Secs. IV and V with the adiabatic squeezing [Eq. (49)] (dotted-dashed lines) and the one-axis twisting of Ref. [11] by means of Eqs. (54) and (50) (solid lines). The exact Bose-Hubbard calculations corresponding to the initial states  $(\pi/2, 0)$  and  $(\pi/2, \pi)$  are plotted as triangles and squares, respectively. Analytic formulas obtained for the  $(\pi/2, \pi)$  [Eqs. (46) and (42)] are plotted in dotted lines. Analytic expressions for the  $(\pi/2, 0)$  case, the ratio of Eqs. (32) and (33) and the relation above Eq. (32) which defines the corresponding time, are plotted as dashed lines. Note that the plots are made for a fixed value  $\Lambda = 20$ .

which is the same result found when we expand the semiclassical approximation to  $\langle \hat{J}_x \rangle$  given in Eq. (30). For longer times, in OAT, the angle for maximal squeezing was found to be [11]  $\beta_{M,\text{OAT}} = \frac{1}{2} \arctan(B/A)$  with  $A = 1 - (\cos \mu)^{N-2}$  and  $B = 4 \sin \frac{\mu}{2} (\cos \frac{\mu}{2})^{N-2}$ . The minimum variance in the  $(y, z)$  plane is given by

$$V_- = \frac{N}{4} \left\{ \left[ 1 + \frac{1}{4}(N-1)A \right] - \frac{1}{4}(N-1)\sqrt{A^2 + B^2} \right\} \quad (53)$$

so that

$$\xi_{S,\text{OAT}}^2 = \frac{(4/N)V_-}{\cos^{2(N-1)}(\mu/2)}. \quad (54)$$

The comparison with the OAT is especially relevant as it corresponds to the limit  $U \gg J$  of the BH Hamiltonian (1).

### C. Maximal squeezing and scaling properties

In Fig. 10, we compare the maximum attainable coherent spin squeezings according to the different methods, considering a fixed value of  $\Lambda$ . First, we note that the  $N$  scaling of the maximum attainable squeezing starting from the  $\Psi(\pi/2, 0)$  state saturates to  $\xi_S^2 \simeq 1/(1 + \Lambda)$ , with small  $1/N$  corrections

as predicted in Eq. (32). This is similar to the adiabatic case, which also saturates, albeit to a higher value  $\xi_S^2 \simeq 1/\sqrt{1 + \Lambda}$ .

The large- $N$  behavior of the coherent spin squeezing achieved from the  $\Psi(\pi/2, \pi)$  state is, however, different. The large- $N$  scaling of the maximum coherent spin squeezing in this case is closer to the one obtained from the one-axis twisting method  $\xi_S^2 \sim N^{-2/3}$ , as seen in Fig. 10 for  $\Lambda = 20$ . In this case, the falloff predicted by Eq. (46) is  $\xi_S^2 \propto (N\Lambda)^{-1/2}$ , in good agreement with the BH results. Two important differences appear, however. The first one is that these large squeezings are achieved at very early times in the evolution of the system (see lower panel of Fig. 10). Second, as shown in Eq. (45), the parameter  $\Lambda$  provides control on the speed of coherent spin squeezing in the system. As seen in Fig. 10, the time for maximal squeezing obtained from the BH calculation is well reproduced by Eq. (42), showing that the source of coherent squeezing in the systems is essentially the inflationary parabolic evolution described in Sec. V.

Finally, let us note that the present results for the time evolution of the  $(\pi/2, 0)$  and  $(\pi/2, \pi)$  initial states are for moderate  $\Lambda = NU/(2J)$  values, i.e., with  $J \neq 0$ . In the  $\Lambda \gg 1$  limit, the dynamics is the same in both cases and, as expected, agrees with that of the OAT. Thus, our results are relevant as they quantify the effects of the linear coupling  $J$  on the maximum coherent spin squeezing achievable with the considered states.

## VII. SUMMARY AND CONCLUSIONS

We have studied the formation of squeezed states in the quenched evolution of coherent initial states of ultracold atoms trapped in double-well potentials. The system is initially prepared in either the  $(\pi/2, 0)$  or  $(\pi/2, \pi)$  coherent states, which in turn correspond to the ground state of the noninteracting system or its highest excited state, respectively.

Simple analytical formulas have been derived which correctly describe (a) the dynamics of the system for a broad range of repulsive interactions and (b) the formation of squeezed states in the initial time evolution. Expressions are given for the angle of maximal squeezing and the magnitude of the squeezing. The semiclassical model provides a mapping relating the dynamical evolution of the many-body states considered to the dynamics of a particle evolving on a parabolic potential in the Fock space. Within this picture, the evolution of the  $\Psi_{(\pi/2, 0)}$  state corresponds to that of a Gaussian wave packet in the presence of a confining parabolic potential, and simple periodic formulas describe the time evolution of the relevant magnitudes. The evolution of the  $\Psi_{(\pi/2, \pi)}$  state is mapped, for short times, onto the motion of a wave packet in a repulsive parabolic potential. In the second case, we have shown that the squeezing of the many-body state can be much larger than the maximum squeezing obtained in the first case, thus providing a promising experimental resource for coherent spin squeezing. We have compared the maximum attainable squeezing to the adiabatic and to the Kitagawa-Ueda OAT. We find that the large- $N$  scaling of the maximum coherent squeezing in the  $\Psi_{(\pi/2, \pi)}$  case is similar to OAT, but with the advantage that the linear coupling  $\Lambda$  allows us to control the speed at which the squeezing develops in the system. In the experimentally relevant situation where one is limited by the nonlinearity

in the system, this allows us to accelerate the generation of squeezing in the system. The two initial conditions considered are within reach experimentally in internal bosonic Josephson junctions [8]. We therefore expect that the findings reported here will be checked against new experiments soon.

### ACKNOWLEDGMENTS

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### APPENDIX: EXPECTATION VALUES OF $\hat{J}_i$ AND $\hat{J}_i^2$

First note that the action of the spin operators  $\hat{J}_i$  and  $\hat{J}_i^2$  on the general state  $|\Psi\rangle$  of Eq. (3) gives

$$\begin{aligned} \langle k, N - k | \hat{J}_x | \Psi \rangle &= \frac{1}{2} [b_k c_{k+1} + b_{k-1} c_{k-1}], \\ \langle k, N - k | \hat{J}_y | \Psi \rangle &= \frac{i}{2} [b_{k-1} c_{k-1} - b_k c_{k+1}], \\ \langle k, N - k | \hat{J}_z | \Psi \rangle &= \frac{1}{2} [2k - N] c_k \end{aligned} \quad (\text{A1})$$

and

$$\begin{aligned} \langle k, N - k | \hat{J}_x^2 | \Psi \rangle &= \frac{1}{4} [b_k b_{k+1} c_{k+2} + [b_k^2 + b_{k-1}^2] c_k \\ &\quad + b_{k-1} b_{k-2} c_{k-2}], \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \langle k, N - k | \hat{J}_y^2 | \Psi \rangle &= -\frac{1}{4} [b_k b_{k+1} c_{k+2} - [b_k^2 + b_{k-1}^2] c_k \\ &\quad + b_{k-1} b_{k-2} c_{k-2}], \end{aligned}$$

$$\begin{aligned} \langle k, N - k | \hat{J}_z^2 | \Psi \rangle &= \frac{1}{4} (2k - N)^2 c_k \\ \times \langle k, N - k | \hat{J}_y \hat{J}_z + \hat{J}_z \hat{J}_y | \Psi \rangle \\ &= i \left( \frac{2k - N - 1}{2} b_k c_{k+1} - \frac{2k - N + 1}{2} b_{k-1} c_{k-1} \right), \end{aligned} \quad (\text{A3})$$

where  $b_k = \sqrt{(k+1)(N-k)}$ . We will assume that the states  $\Psi$  are such that either their  $c_k$  vary smoothly (when the initial state is  $\Psi_{\pi/2,0}$ ), or it is their  $(-)^k c_k$  that vary smoothly (when the initial state is  $\Psi_{\pi/2,\pi}$ ). Also, we assume that the number of atoms is large,  $h = 1/N \ll 1$ . This allows us to introduce a continuous variable  $x$  and a continuous function  $\psi(x)$  such that  $\psi(x = k/N) = \sqrt{N} c_k$  or  $\sqrt{N} (-)^k c_k$  [21,30–32]. The factor  $\sqrt{N}$  guarantees that  $\sum_0^N |c_k|^2 = 1$  becomes  $\int_0^1 dx |\psi(x)|^2 = 1$  in the large- $N$  limit. With these notations and using  $b(x) = \sqrt{(x+h)(1-x)}$ ,

$$\begin{aligned} c_k^* \langle k, N - k | \hat{J}_x | \Psi \rangle &= \pm \frac{1}{2} \psi^*(x) [b(x) \psi(x+h) + b(x-h) \psi(x-h)], \\ c_k^* \langle k, N - k | \hat{J}_y | \Psi \rangle &= \pm \frac{i}{2} \psi^*(x) [b(x-h) \psi(x-h) - b(x) \psi(x+h)], \\ c_k^* \langle k, N - k | \hat{J}_z | \Psi \rangle &= \frac{1}{2} \psi^*(x) (2x - 1) \psi(x), \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} c_k^* \langle k, N - k | \hat{J}_x^2 | \Psi \rangle &= \frac{N}{4} \psi^*(x) \{ b(x) b(x+h) \psi(x+2h) + [b(x)^2 + b(x-h)^2] \psi(x) + b(x-h) b(x-2h) \psi(x-2h) \}, \\ c_k^* \langle k, N - k | \hat{J}_y^2 | \Psi \rangle &= -\frac{N}{4} \psi^*(x) \{ b(x) b(x+h) \psi(x+2h) - [b(x)^2 + b(x-h)^2] \psi(x) + b(x-h) b(x-2h) \psi(x-2h) \}, \\ c_k^* \langle k, N - k | \hat{J}_z^2 | \Psi \rangle &= \frac{N}{4} \psi^*(x) (2x - 1)^2 \psi(x), \end{aligned} \quad (\text{A5})$$

$$c_k^* \langle k, N - k | \hat{J}_y \hat{J}_z + \hat{J}_z \hat{J}_y | \Psi \rangle = \pm \frac{iN}{2} \psi^*(x) [(2x - h - 1) b(x) \psi(x+h) - (2x + h - 1) b(x-h) \psi(x-h)], \quad (\text{A6})$$

where the sign is  $+$  ( $-$ ) for states close to  $\Psi_{\pi/2,0}$  ( $\Psi_{\pi/2,\pi}$ ). No approximation has yet been made. Now, we expand these expressions in powers of  $h = 1/N$  up to order  $h^2$ , introduce the variable  $z = 2x - 1$ , and change  $\psi(x) \rightarrow \sqrt{2} \psi(z)$  to fulfill  $\int_{-1}^1 dz |\psi(z)|^2 = 1$ . In the large- $N$  limit, by replacing the sum over  $k$  by an integration over  $z$  times  $N/2$  one finds

$$\begin{aligned} h \langle \Psi | \hat{J}_x | \Psi \rangle &\simeq \pm \int_{-1}^1 dz \psi^*(z) \left[ \left( \frac{h^2(-1-z^2)}{4(1-z^2)^{3/2}} + \frac{h}{2\sqrt{1-z^2}} + \frac{\sqrt{1-z^2}}{2} \right) \psi(z) - \frac{h^2 z}{\sqrt{1-z^2}} \psi'(z) + h^2 \sqrt{1-z^2} \psi''(z) \right], \\ h \langle \Psi | \hat{J}_y | \Psi \rangle &\simeq \pm \int_{-1}^1 dz \psi^*(z) \left[ \left( \frac{ih^2 z}{2(1-z^2)^{3/2}} - \frac{ihz}{2\sqrt{1-z^2}} \right) \psi(z) - 2 \left( -\frac{ih^2}{2\sqrt{1-z^2}} + \frac{ih(-1+z^2)}{2\sqrt{1-z^2}} \right) \psi'(z) \right], \\ h \langle \Psi | \hat{J}_z | \Psi \rangle &= \int_{-1}^1 dz \psi^*(z) z \psi(z), \end{aligned}$$

$$\begin{aligned}
h^2 \langle \Psi | \hat{J}_x^2 | \Psi \rangle &\simeq \int_{-1}^1 dz \psi^*(z) \left[ \left( \frac{h}{2} + \frac{1}{4}(1-z^2) + \frac{2h^2(-2+z^2)}{8-8z^2} \right) \psi(z) - 2h^2 z \psi'(z) + h^2(1-z^2) \psi''(z) \right], \\
h^2 \langle \Psi | \hat{J}_y^2 | \Psi \rangle &\simeq \int_{-1}^1 dz \psi^*(z) \left[ \frac{h^2(-2+z^2)}{4(-1+z^2)} \psi(z) + 2h^2 z \psi'(z) + h^2(-1+z^2) \psi''(z) \right], \\
h^2 \langle \Psi | \hat{J}_z^2 | \Psi \rangle &= \int_{-1}^1 dz |\psi(z)|^2 \frac{z^2}{4}, \\
h^2 \langle \Psi | \{ \hat{J}_y, \hat{J}_z \} | \Psi \rangle &\simeq \pm \int_{-1}^1 dz \psi^*(z) \left[ \left( \frac{ih}{2\sqrt{1-z^2}} - \frac{ih^2(-1+2z^2)}{2(1-z^2)^{3/2}} \right) \psi(z) - 2 \left( \frac{ih^2 z}{2\sqrt{1-z^2}} + \frac{1}{2} ihz \sqrt{1-z^2} \right) \psi'(z) \right].
\end{aligned} \tag{A7}$$

Note that this approximation still fulfills

$$\langle \Psi | \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 | \Psi \rangle = \frac{N}{2} \left( \frac{N}{2} + 1 \right). \tag{A8}$$

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