

Uhlmann fidelity between two-mode Gaussian states

Paulina Marian^{1,2} and Tudor A. Marian¹

¹*Centre for Advanced Quantum Physics, University of Bucharest, P.O. Box MG-11, R-077125 Bucharest-Măgurele, Romania*

²*Department of Physical Chemistry, University of Bucharest, Boulevard Regina Elisabeta 4-12, R-030018 Bucharest, Romania*

(Received 30 November 2011; revised manuscript received 13 May 2012; published 30 August 2012)

We analyze the Uhlmann fidelity of a pair of n -mode Gaussian states of the quantum radiation field. This quantity is shown to be the product of an exponential function depending on the relative average displacement and a factor fully determined by the symplectic spectrum of the covariance matrix of a specific Gaussian state. However, it is difficult to handle our general formula unless the Gaussian states commute or at least one of them is pure. On the contrary, in the simplest cases $n = 1$ and $n = 2$, it leads to explicit analytic formulas. Our main result is a calculable expression of the fidelity of two arbitrary two-mode Gaussian states. This can be applied to build reliable measures of quantum correlations between modes in various branches of quantum physics.

DOI: [10.1103/PhysRevA.86.022340](https://doi.org/10.1103/PhysRevA.86.022340)

PACS number(s): 03.67.Mn, 03.65.Ta, 42.50.Dv

I. INTRODUCTION

One of the most important issues in quantum information is the closeness between input and output states of various quantum protocols [1]. When the input state is pure, a natural figure of merit of any protocol is the input-output quantum-mechanical transition probability. The situation is more complicated in the mixed-state case. Fortunately, an appropriate description in such a case could be connected with a concept introduced long ago by Uhlmann [2]. Let us consider two arbitrary states, $\hat{\rho}'$ and $\hat{\rho}''$, of a given quantum system. When both states are mixed, Uhlmann asserted that a suitable measure of the likeness between their properties is the maximal quantum-mechanical transition probability between their purifications in an enlarged Hilbert space [2,3]. This is an extended notion of transition probability between quantum states which is now called fidelity [3] and has the intrinsic expression [2]

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \left[\text{Tr} \left(\sqrt{\sqrt{\hat{\rho}''} \hat{\rho}' \sqrt{\hat{\rho}''}} \right) \right]^2. \quad (1.1)$$

Indeed, when at least one of the quantum states is pure, the fidelity (1.1) reduces to the familiar overlap between the states: $\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \text{Tr}(\hat{\rho}' \hat{\rho}'')$. Important properties listed and discussed in Refs. [2–8] highly recommended fidelity as a measure of distinguishability of quantum states. Especially useful is the property put forward in Ref. [4] that fidelity equals the minimal squared overlap of the probability distributions for the outcomes of any general measurement. Fidelity-based metrics such as the Bures metric [2,9] and related ones [5–7] have proven to be fruitful in quantum optics and quantum information. For instance, the Bures metric provides insightful distance-type degrees of entanglement [10,11], nonclassicality [12], and polarization [13–16]. Moreover, in the framework of the theory of parameter estimation, Braunstein and Caves [17] established a distinguishability metric which is proportional to the Bures distance between neighboring quantum states. Their findings in Ref. [17] successfully apply in recent developments of quantum estimation theory, as shown in the review [18]. Besides being an unavoidable ingredient in quantum information, fidelity is exploited in various areas of physics where the concept of quantum transition probability is essential. For instance, in condensed matter physics, fidelity is

frequently employed as an indicator of criticality in quantum phase transitions [19,20].

In spite of playing a key role in quantum mechanics due to its operational meaning, fidelity between mixed states has a rather limited use so far. This happens because, in general, Uhlmann's expression (1.1) is not easy to calculate even on finite-dimensional Hilbert spaces [6,7]. Within continuous-variable settings, explicit calculations of fidelity were performed for one-mode Gaussian states (GS's) in Refs. [21,22], for some special two-mode GS's in Refs. [11], or for commuting two-mode states in Refs. [15,16].

In what follows, we concentrate on the GS's of continuous-variable systems, which are important for both quantum optics and quantum information processing [23]. From the experimental side, they are quite accessible and were effectively obtained with light, Bose-Einstein condensates, trapped ions, and Josephson junctions. On the other hand, the GS's of the quantum radiation field are important resources in several quantum information protocols [24,25]. In particular, from the theoretical point of view, the two-mode GS's are a perfect test bed for studying entanglement [11] or other kinds of correlations between the two modes [26,27]. Therefore, an exact analytic formula for the fidelity of an arbitrary pair of two-mode GS's is highly desirable, and its derivation constitutes a main goal of the present paper. Let us mention that we have previously derived the fidelity between two-mode symmetric and squeezed thermal states and used it in evaluating a Gaussian degree of entanglement defined with the Bures metric [11]. Nevertheless, an explicit formula of the fidelity between arbitrary two-mode GS's was not yet found. We acknowledge, however, Ref. [28] as a useful prospect for the multimode Gaussian case.

In order to study the fidelity of two GS's, we find it convenient to introduce the operators:

$$\hat{A} := \hat{\rho}' \hat{\rho}'', \quad \hat{B} := \sqrt{\hat{\rho}''} \hat{\rho}' \sqrt{\hat{\rho}''}, \quad \hat{C} := \sqrt{\hat{\rho}'} \sqrt{\hat{\rho}''}, \quad (1.2)$$

with the properties $\hat{B} = \hat{C}^\dagger \hat{C}$ and $\text{Tr}(\hat{A}) = \text{Tr}(\hat{B}) \geq 0$. When this trace is strictly positive, $\text{Tr}(\hat{A}) > 0$, then a state

$$\hat{\rho}_B := [\text{Tr}(\hat{B})]^{-1} \hat{B} \quad (1.3)$$

exists, so that the fidelity (1.1) reads

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \text{Tr}(\hat{A})[\text{Tr}(\sqrt{\hat{\rho}_B})]^2. \quad (1.4)$$

Equation (1.4) is the starting point of our explicit evaluation of the fidelity between two-mode GS's.

From now on, we assume that $\hat{\rho}'$ and $\hat{\rho}''$ are multimode GS's. The plan of the paper is as follows. In Sec. II we recall the phase-space description of the GS's and mention Williamson's theorem, which is at the heart of our method. Some useful properties of the product of two Hilbert-Schmidt Gaussian operators are collected in Sec. III. Based on the composition rules of such a product, we then proceed in Sec. IV to derive the fidelity (1.4) of a pair of multimode GS's. We express the second factor in the formula (1.4) in terms of the symplectic eigenvalues of the covariance matrix (CM) of the GS $\hat{\rho}_B$. We develop our treatment in the multimode framework as far as possible. Then we restrict ourselves to the workable case of two commuting multimode GS's. In Sec. V, the general expression of the fidelity of two GS's enables us to retrieve a previous explicit formula for one-mode states, in agreement with Refs. [21,22]. Then we reach the main goal of this work: We provide a versatile and workable formula for the fidelity between two-mode GS's. An interesting application of the two-mode formula concludes Sec. V. The final section deals with a discussion of our analytic results and their potential use in studying quantum correlations between modes.

II. PHASE-SPACE DESCRIPTION OF THE GAUSSIAN STATES

An n -mode Hilbert-Schmidt operator \hat{F} is defined by its Weyl expansion:

$$\hat{F} = \frac{1}{(2\pi)^n} \int d^{2n}u f(u) \hat{D}(-u), \quad (2.1)$$

where u is a vector in the Euclidian space \mathbb{R}^{2n} , the weight function $f(u)$ is square-integrable over \mathbb{R}^{2n} , and $\hat{D}(u)$ is an n -mode Weyl displacement operator:

$$\hat{D}(u) := \bigotimes_{j=1}^n \hat{D}_j((u)_j). \quad (2.2)$$

Let \hat{q}_j and \hat{p}_j be the canonical quadrature operators of the j th field mode, and q_j and p_j their eigenvalues. In Eq. (2.2), $(u)_j$ denotes a vector in \mathbb{R}^2 whose components are precisely q_j and p_j :

$$(u)_j := \begin{pmatrix} u_{2j-1} \\ u_{2j} \end{pmatrix} = \begin{pmatrix} q_j \\ p_j \end{pmatrix} \quad (j = 1, 2, \dots, n). \quad (2.3)$$

Further, $\hat{D}_j((u)_j)$ is the corresponding single-mode Weyl displacement operator:

$$\hat{D}_j((u)_j) := \exp[-i(q_j \hat{p}_j - p_j \hat{q}_j)]. \quad (2.4)$$

Recall the standard matrix J of the symplectic form on \mathbb{R}^{2n} , which is block-diagonal and skew-symmetric:

$$J := \bigoplus_{k=1}^n J_k, \quad J_k := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (k = 1, 2, \dots, n). \quad (2.5)$$

Two properties of the Weyl operators (2.2) are useful:

(1) Composition law:

$$\hat{D}(u) \hat{D}(v) = \exp\left(-\frac{i}{2}u^T J v\right) \hat{D}(u+v); \quad (2.6)$$

(2) Orthonormalization relation:

$$\text{Tr}[\hat{D}^\dagger(u)\hat{D}(v)] = (2\pi)^n \delta^{(2n)}(u-v). \quad (2.7)$$

For instance, the latter yields the scalar product of two Hilbert-Schmidt n -mode operators (2.1),

$$\text{Tr}(\hat{F}^\dagger \hat{G}) = \frac{1}{(2\pi)^n} \int d^{2n}u f^*(u) g(u), \quad (2.8)$$

as well as the trace-formula for a weight function,

$$f(u) = \text{Tr}[\hat{D}(u)\hat{F}]. \quad (2.9)$$

Note the value $f(0) = \text{Tr}(\hat{F})$ and that an operator (2.1) is self-adjoint iff $f^*(-u) = f(u)$.

We focus on multimode Hilbert-Schmidt Gaussian operators (HSGO's). By definition, the weight function (2.9) of such an operator is a shifted Gaussian,

$$f(u) = f(0) \exp\left(-\frac{1}{2}u^T \mathcal{F} u - i\xi^T u\right), \quad (2.10)$$

with a complex shift vector $\xi \in \mathbb{C}^{2n}$ and a symmetric matrix $\mathcal{F} \in M_{2n}(\mathbb{C})$ whose real part is positive definite: $\mathcal{F} + \mathcal{F}^* > 0$. This implies that the square matrix \mathcal{F} is invertible, so that the Gaussian weight function (2.10) is integrable [29]:

$$\begin{aligned} & \int d^{2n}u \exp\left(-\frac{1}{2}u^T \mathcal{F} u - i\xi^T u\right) \\ &= (2\pi)^n [\det(\mathcal{F})]^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\xi^T \mathcal{F}^{-1} \xi\right), \\ & (\text{Re}\{[\det(\mathcal{F})]^{\frac{1}{2}}\} > 0). \end{aligned} \quad (2.11)$$

A special example of an HSGO is the density operator $\hat{\rho}$ of an n -mode GS, which is positive and of unit trace: $\text{Tr}(\hat{\rho}) = 1$. Its weight function (2.9), called the characteristic function (CF) of the state, is fully determined by the first- and second-order moments of all the quadrature operators:

$$\chi(u) = \exp\left[-\frac{1}{2}u^T \mathcal{V} u - i(\langle u \rangle_{\hat{\rho}})^T u\right]. \quad (2.12)$$

In Eq. (2.12), $\mathcal{V} \in M_{2n}(\mathbb{R})$ denotes the symmetric CM of the GS $\hat{\rho}$. It fulfills the concise Robertson-Schrödinger uncertainty relation,

$$\zeta^\dagger \left(\mathcal{V} + \frac{i}{2}J\right) \zeta \geq 0 \quad (\zeta \in \mathbb{C}^{2n}), \quad (2.13)$$

as a necessary and sufficient condition [30,31]. Therefore, the CM \mathcal{V} is positive definite, $\mathcal{V} > 0$, and, according to Williamson's theorem [32], is congruent via a symplectic

matrix $S \in Sp(2n, \mathbb{R})$ to a diagonal matrix:

$$\begin{aligned} \mathcal{V} &= S^T \left(\bigoplus_{j=1}^n \kappa_j I_j \right) S : \quad \det(\mathcal{V}) = \prod_{j=1}^n (\kappa_j)^2 ; \\ \mathcal{V} + \frac{i}{2} J &= S^T \left[\bigoplus_{j=1}^n \left(\kappa_j I_j + \frac{i}{2} J_j \right) \right] S : \\ \det \left(\mathcal{V} + \frac{i}{2} J \right) &= \prod_{j=1}^n \left[(\kappa_j)^2 - \frac{1}{4} \right]. \end{aligned} \quad (2.14)$$

In Eqs. (2.14), the positive numbers κ_j ($j = 1, 2, \dots, n$) are the symplectic eigenvalues of the CM \mathcal{V} [33], and all the single-mode matrices I_j are equal to the 2×2 identity matrix. As a consequence, the n -mode Robertson-Schrödinger uncertainty relation (2.13) is equivalent to the inequalities $\kappa_j \geq \frac{1}{2}$ ($j = 1, 2, \dots, n$).

III. PRODUCT OF TWO HILBERT-SCHMIDT GAUSSIAN OPERATORS

We consider the product $\hat{F} = \hat{F}' \hat{F}''$ of two n -mode HSGO's whose weight functions $f'(u)$ and $f''(u)$ are Gaussians, Eq. (2.10). Making use of Eqs. (2.6), (2.7), and (2.11), one finds that the weight function (2.9) of the product \hat{F} has the same form. Although we do not list the corresponding composition rules [34] here, we just stress that the real part of the resulting symmetric matrix $\mathcal{F} \in M_{2n}(\mathbb{C})$ is positive definite in two special cases we are interested in:

- (1) Both factors \hat{F}' and \hat{F}'' are self-adjoint.
- (2) The factors are adjoints to one another: $\hat{F}' = (\hat{F}'')^\dagger$.

Hence, in both above-mentioned cases, the product \hat{F} is itself an n -mode HSGO. In the first case, \hat{F} is not self-adjoint, unless the two factors commute. In the second one, \hat{F} is a positive operator, proportional to the density operator of a GS whose CM is precisely $\mathcal{F} \in M_{2n}(\mathbb{R})$.

We further investigate the product $\hat{A} := \hat{\rho}' \hat{\rho}''$ of two n -mode GS's. First, making use of Eqs. (2.8)–(2.11), we get its trace:

$$\begin{aligned} \text{Tr}(\hat{A}) &= [\det(\mathcal{V}' + \mathcal{V}'')]^{-\frac{1}{2}} \\ &\times \exp \left[-\frac{1}{2} (\delta \langle u \rangle)^T (\mathcal{V}' + \mathcal{V}'')^{-1} \delta \langle u \rangle \right] > 0. \end{aligned} \quad (3.1)$$

We have introduced above the relative average n -mode displacement $\delta \langle u \rangle := \langle u \rangle_{\hat{\rho}'} - \langle u \rangle_{\hat{\rho}''}$.

Other properties of the product operator \hat{A} are found by evaluating its weight function (2.9) via Eq. (2.1) used for both its factors $\hat{\rho}'$ and $\hat{\rho}''$. After finally performing a routine integral of the type (2.11) we are left with the Gaussian weight function $f_A(u)$, Eq. (2.10), which is determined by the following composition rules:

$$\begin{aligned} \mathcal{F}_A &= -\frac{i}{2} J + \left(\mathcal{V}'' + \frac{i}{2} J \right) (\mathcal{V}' + \mathcal{V}'')^{-1} \left(\mathcal{V}' + \frac{i}{2} J \right), \\ \xi_A &= \langle u \rangle_{\hat{\rho}'} - \left(\mathcal{V}' - \frac{i}{2} J \right) (\mathcal{V}' + \mathcal{V}'')^{-1} \delta \langle u \rangle. \end{aligned} \quad (3.2)$$

Similarly, we employ the Hilbert-Schmidt scalar product (2.8) in conjunction with the Gaussian integral (2.11) in order to get

the formula

$$\text{Tr}(\hat{A}^2) = [\text{Tr}(\hat{A})]^2 2^{-n} [\det(\mathcal{F}_A)]^{-\frac{1}{2}}. \quad (3.3)$$

An essential nontrivial step is that the determinant of the matrix \mathcal{F}_A , Eq. (3.2), always factors as follows:

$$\det(\mathcal{F}_A) = \frac{\det[(J\mathcal{V}')(\mathcal{V}'') - \frac{1}{4}I]}{\det(\mathcal{V}' + \mathcal{V}'')}. \quad (3.4)$$

The determinant (3.4) easily emerges when arranging the matrix \mathcal{F}_A in a product form. On the other hand, Eq. (3.2) implies the identity

$$\det \left(\mathcal{F}_A + \frac{i}{2} J \right) = \frac{\det(\mathcal{V}' + \frac{i}{2} J) \det(\mathcal{V}'' + \frac{i}{2} J)}{\det(\mathcal{V}' + \mathcal{V}'')}. \quad (3.5)$$

Both the above determinants (3.4) and (3.5) are manifestly symmetric with respect to the GS's $\hat{\rho}'$ and $\hat{\rho}''$.

We exploit the composition rules (3.1) and (3.2) in three special situations. First, when the GS's coincide, $\hat{\rho}' = \hat{\rho}'' =: \hat{\rho}$, then $\mathcal{F}_A = \mathcal{V}_B =: \mathcal{V}_A$. Hence, Eqs. (3.1) and (3.2) read respectively:

$$\text{Tr}(\hat{\rho}^2) = 2^{-n} [\det(\mathcal{V})]^{-\frac{1}{2}} \leq 1; \quad (3.6)$$

$$\mathcal{V}_A = \frac{1}{2} \left(\mathcal{V} - \frac{1}{4} J \mathcal{V}^{-1} J \right), \quad \xi_A = \langle u \rangle_{\hat{\rho}}. \quad (3.7)$$

Second, Eqs. (3.6), (3.7), and (2.14) enable us to write down two conditions, both of them necessary and sufficient in order that an n -mode GS $\hat{\sigma}$ is pure. They are expressed in terms of its CM \mathcal{V} , as follows:

- (1) Scalar condition:

$$\begin{aligned} \text{Tr}(\hat{\sigma}^2) = 1 &\iff \det(\mathcal{V}) = 2^{-2n} \\ &\iff \kappa_j = \frac{1}{2} \quad (j = 1, 2, \dots, n). \end{aligned} \quad (3.8)$$

- (2) Matrix condition:

$$\mathcal{V}_A = \mathcal{V} \iff (J\mathcal{V})^2 = -\frac{1}{4} I. \quad (3.9)$$

Third, the square root $\sqrt{\hat{\rho}}$ of a given n -mode GS $\hat{\rho}$ with the CF (2.12) is proportional to an equally displaced n -mode GS: its CM $\tilde{\mathcal{V}}$ determines the proportionality factor

$$\text{Tr}(\sqrt{\hat{\rho}}) = \sqrt[4]{\det(2\tilde{\mathcal{V}})} \quad (3.10)$$

and satisfies Eq. (3.7):

$$\mathcal{V} = \frac{1}{2} \left(\tilde{\mathcal{V}} - \frac{1}{4} J \tilde{\mathcal{V}}^{-1} J \right). \quad (3.11)$$

Accordingly, its symplectic eigenvalues $\tilde{\kappa}_j$ are determined by those of the CM \mathcal{V} , hereafter denoted κ_j :

$$\tilde{\kappa}_j = \kappa_j + \sqrt{(\kappa_j)^2 - \frac{1}{4}} \quad (j = 1, 2, \dots, n). \quad (3.12)$$

IV. FIDELITY OF TWO MULTIMODE GAUSSIAN STATES

We successively apply Eqs. (3.10), (2.14), and (3.12) to the GS $\hat{\rho}_B$, Eq. (1.3), for getting a general expression of the

fidelity (1.4):

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \text{Tr}(\hat{\rho}' \hat{\rho}'') \times 2^n \prod_{j=1}^n \left\{ (\kappa_B)_j + \sqrt{[(\kappa_B)_j]^2 - \frac{1}{4}} \right\}. \quad (4.1)$$

In Eq. (4.1), the set $\{(\kappa_B)_j, (j = 1, 2, \dots, n)\}$ is the symplectic spectrum of the CM \mathcal{V}_B of the n -mode GS $\hat{\rho}_B$. Some expected properties of the fidelity can be read in the above expression. For instance, let us note the inequalities

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') \geq \text{Tr}(\hat{\rho}' \hat{\rho}'') > 0, \quad (4.2)$$

with saturation of the first one when at least one of the GS's $\hat{\rho}'$ and $\hat{\rho}''$ is pure. The second inequality is displayed by Eq. (3.1): the Hilbert-Schmidt scalar product of any pair of n -mode GS's is strictly positive. Another property of the fidelity (4.1) is its symplectic invariance:

$$\mathcal{F}(\hat{U}(S)\hat{\rho}'\hat{U}^\dagger(S), \hat{U}(S)\hat{\rho}''\hat{U}^\dagger(S)) = \mathcal{F}(\hat{\rho}', \hat{\rho}''), \quad (4.3)$$

where $S \in Sp(2n, \mathbb{R})$ and $\hat{U}(S)$ are the unitary operators of the metaplectic representation on the Hilbert space of the n -mode states.

There is a special case when the multimode fidelity (4.1) can be readily evaluated, namely, that of commuting GS's. As a prototype of such states, we mention the n -mode thermal states. We concentrate on a pair of commuting n -mode GS's, $\hat{\rho}'$ and $\hat{\rho}''$, whose CM's \mathcal{V}' and \mathcal{V}'' have the symplectic spectra $\{\kappa'_j\}$ and $\{\kappa''_j\}$, respectively. The commutation relation $[\hat{\rho}', \hat{\rho}''] = \hat{0}$ implies, via the resulting equality $\mathcal{V}_B = \mathcal{F}_A$, that the diagonalizable matrices $J\mathcal{V}'$ and $J\mathcal{V}''$ commute, so that they have a complete system of common eigenvectors. Since the matrix $J\mathcal{V}_B$ has the same eigenvectors, we find the composition law of the symplectic eigenvalues:

$$(\kappa_B)_j = \frac{\kappa'_j \kappa''_j + \frac{1}{4}}{\kappa'_j + \kappa''_j} \quad (j = 1, 2, \dots, n). \quad (4.4)$$

Accordingly, the fidelity formula (4.1) simplifies to

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \prod_{j=1}^n \frac{2}{(\kappa'_j + \kappa''_j)^2} \times \left\{ \kappa'_j \kappa''_j + \frac{1}{4} + \sqrt{\left[(\kappa'_j)^2 - \frac{1}{4} \right] \left[(\kappa''_j)^2 - \frac{1}{4} \right]} \right\}. \quad (4.5)$$

Equation (4.5) displays two additional properties of the fidelity, namely, its symmetry, $\mathcal{F}(\hat{\rho}'', \hat{\rho}') = \mathcal{F}(\hat{\rho}', \hat{\rho}'')$, and the saturable inequality

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') \leq 1 : \quad \mathcal{F}(\hat{\rho}', \hat{\rho}'') = 1 \quad \text{iff} \quad \hat{\rho}'' = \hat{\rho}'. \quad (4.6)$$

We come back to the general case of two noncommuting n -mode GS's: Writing an explicit formula of their fidelity (4.1) seems to be a hard task. Indeed, evaluation of the CM \mathcal{V}_B of the n -mode GS $\hat{\rho}_B$, Eq. (1.3), requires a three-time application of the composition rule (3.2); therefore the CM $\mathcal{V}_B \in M_{2n}(\mathbb{R})$ has a complicated structure. However, we succeeded to calculate its symplectic invariants $\det(\mathcal{V}_B)$ and $\det(\mathcal{V}_B + \frac{i}{2}J)$. Indeed, taking advantage once again of Eqs. (2.8) and (2.11), we find

a formula which is similar to Eq. (3.3):

$$\text{Tr}(\hat{\mathcal{B}}^2) = [\text{Tr}(\hat{\mathcal{B}})]^2 2^{-n} [\det(\mathcal{V}_B)]^{-\frac{1}{2}}. \quad (4.7)$$

Their comparison gives, via Eq. (1.2), the expression (3.4):

$$\det(\mathcal{V}_B) = \frac{\det[(J\mathcal{V}') (J\mathcal{V}'') - \frac{1}{4}I]}{\det(\mathcal{V}' + \mathcal{V}'')} \geq 2^{-2n}. \quad (4.8)$$

On the other hand, a rather involved calculation makes use of Eqs. (3.2) and (3.11) to yield the determinant

$$\det\left(\mathcal{V}_B + \frac{i}{2}J\right) = \frac{\det(\mathcal{V}' + \frac{i}{2}J) \det(\mathcal{V}'' + \frac{i}{2}J)}{\det(\mathcal{V}' + \mathcal{V}'')}, \quad (4.9)$$

which coincides with the non-negative determinant (3.5).

When at least one of the GS's $\hat{\rho}'$ and $\hat{\rho}''$, say $\hat{\rho}'$, is pure, then insertion of the appropriate conditions (3.8) and (3.9) into Eq. (4.8) leads to the equality $\det(\mathcal{V}_B) = 2^{-2n}$, showing that the GS $\hat{\rho}_B$ is also pure.

For later convenience, let us introduce the following notations:

$$\begin{aligned} \Delta &:= \det(\mathcal{V}' + \mathcal{V}'') \geq 1, \\ \Gamma &:= 2^{2n} \det\left[(J\mathcal{V}') (J\mathcal{V}'') - \frac{1}{4}I\right] \geq \Delta, \\ \Lambda &:= 2^{2n} \det\left(\mathcal{V}' + \frac{i}{2}J\right) \det\left(\mathcal{V}'' + \frac{i}{2}J\right) \geq 0. \end{aligned} \quad (4.10)$$

The inequality $\Delta \geq 1$ originates in the feature of the positive definite matrix $\check{\mathcal{V}} := \frac{1}{2}(\mathcal{V}' + \mathcal{V}'')$ of being the CM of a n -mode GS determined up to its average displacement. Let us remark that our main results in this section are the explicit expressions of the symplectic invariants (4.8) and (4.9) for GS's with any n . However, only for one- and two-mode states are these invariants sufficient to get analytic formulas for the fidelity via Eq. (4.1).

V. EXPLICIT RESULTS

A. One-mode Gaussian states

For the sake of completeness we begin with the one-mode case. The symplectic spectrum of the CM \mathcal{V}_B consists of a single eigenvalue:

$$\kappa_B = \frac{1}{2} \sqrt{\frac{\Lambda}{\Delta} + 1}.$$

According to Eqs. (3.2) and (4.10), the fidelity (4.1) reads

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \exp\left[-\frac{1}{2}(\delta\langle u \rangle)^T (\mathcal{V}' + \mathcal{V}'')^{-1} \delta\langle u \rangle\right] \times (\sqrt{\Delta + \Lambda} - \sqrt{\Lambda})^{-1}. \quad (5.1)$$

The symmetric formula (5.1) was derived and analyzed gradually long ago in Refs. [22]. In addition, by using the structure of the one-mode GS's, which are displaced squeezed thermal states [35], we succeeded in proving explicitly in the present work the property (4.6) of their fidelity (5.1).

B. Two-mode Gaussian states

The CM \mathcal{V}_B has two symplectic eigenvalues: $(\kappa_B)_1$ and $(\kappa_B)_2$. Their squares are the roots of a monic quadratic binomial

whose coefficients are expressed in terms of the determinants (4.10). A straightforward calculation exploiting Eqs. (3.2) and (4.1) leads to the symmetric formula:

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \exp \left[-\frac{1}{2} (\delta \langle u \rangle)^T (\mathcal{V}' + \mathcal{V}'')^{-1} \delta \langle u \rangle \right] \times \left[(\sqrt{\Gamma} + \sqrt{\Lambda}) - \sqrt{(\sqrt{\Gamma} + \sqrt{\Lambda})^2 - \Delta} \right]^{-1}. \quad (5.2)$$

Previous results reported in Refs. [11] for special states (squeezed thermal and symmetric) are seen to be particular cases of Eq. (5.2).

The determinants (4.10) of 4×4 matrices occurring in Eq. (5.2) are easily computable. As a salient example, we evaluate them for a pair of undisplaced two-mode GS's whose CM's \mathcal{V}' and \mathcal{V}'' are both in an unscaled standard form. This means

that they are partitioned into 2×2 diagonal submatrices of the type

$$\mathcal{V} = \begin{pmatrix} b_1 I & \mathcal{C} \\ \mathcal{C} & b_2 I \end{pmatrix}; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}. \quad (5.3)$$

Here $b_1 \geq \frac{1}{2}$, $b_2 \geq \frac{1}{2}$, $c \geq |d|$. A matrix (5.3) has two symplectic invariants:

$$\det(\mathcal{V}) = (b_1 b_2 - c^2)(b_1 b_2 - d^2) > 0, \\ \det \left(\mathcal{V} + \frac{i}{2} J \right) = \det(\mathcal{V}) - \frac{1}{4} (b_1^2 + b_2^2 + 2cd) + \frac{1}{16} \geq 0. \quad (5.4)$$

We denote b'_1, b'_2, c', d' and b''_1, b''_2, c'', d'' the standard-form parameters of the CM's \mathcal{V}' and \mathcal{V}'' , respectively. Then, Eqs. (4.10), (5.3), and (5.4) allow us to factor both determinants Δ and Γ and to visualize the symplectic invariant Λ :

$$\Delta = [(b'_1 + b''_1)(b'_2 + b''_2) - (c' + c'')^2][(b'_1 + b''_1)(b'_2 + b''_2) - (d' + d'')^2]; \\ \Gamma = 16 \{ [b'_1 b'_2 - (d')^2][b''_1 b''_2 - (d'')^2] + \frac{1}{4} (b'_1 b''_1 + b'_2 b''_2 + 2d' c'') + \frac{1}{16} \} \{ [b'_1 b'_2 - (c')^2][b''_1 b''_2 - (d'')^2] \\ + \frac{1}{4} (b'_1 b''_1 + b'_2 b''_2 + 2c' d'') + \frac{1}{16} \}; \\ \Lambda = 16 \{ \det(\mathcal{V}') - \frac{1}{4} [(b'_1)^2 + (b'_2)^2 + 2c' d'] + \frac{1}{16} \} \{ \det(\mathcal{V}'') - \frac{1}{4} [(b''_1)^2 + (b''_2)^2 + 2c'' d''] + \frac{1}{16} \}. \quad (5.5)$$

We are left to substitute formulas (5.5) into Eq. (5.2) written for undisplaced two-mode GS's.

VI. CONCLUSIONS

To sum up, in this work we have tackled a long-standing problem in quantum information with continuous variables: the Uhlmann fidelity of two multimode GS's. Our simple phase-space approach led to the important formulas (3.3)–(3.5) and (4.7)–(4.9), which are valid for any number of modes and could be useful for future research. However, we stress that an analytic result for n -mode GS's seems hard to be found when $n \geq 3$: indeed, one then needs to solve a higher-degree characteristic equation that cannot be easily obtained.

While in the single-mode case an explicit formula was found many years ago, we derived here a computable analytic expression in the two-mode case, Eq. (5.2). We expect that this formula will open a productive research in quantifying

bipartite quantum correlations (entanglement and discord) in the Gaussian scenario. An example at hand is the already performed evaluation of a degree of entanglement for some important classes of GS's (squeezed thermal states, symmetric states) in Refs. [11]. The analysis of an arbitrary two-mode GS requires our present result. Use of fidelity in evaluating the Gaussian geometric discord defined with the Bures metric is currently one of our main interests. Note also that fidelity in mixed-state estimation or reconstruction could be a potential important tool. It appears that applications of the fidelity to the emerging field of quantum information and to other interesting areas of quantum physics are far from being exhausted.

ACKNOWLEDGMENTS

This work was supported by the Romanian National Authority for Scientific Research through Grant No. IDEI-1012/2011 for the University of Bucharest.

[1] B. Schumacher, *Phys. Rev. A* **51**, 2738 (1995).

[2] A. Uhlmann, *Rep. Math. Phys.* **9**, 273 (1976).

[3] R. Jozsa, *J. Mod. Opt.* **41**, 2315 (1994).

[4] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, *Phys. Rev. Lett.* **76**, 2818 (1996).

[5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge,

England, 2000), pp. 409–416; A. Gilchrist, N. K. Langford, and M. A. Nielsen, *Phys. Rev. A* **71**, 062310 (2005).

[6] P. E. M. F. Mendonça, R. d. J. Napolitano, M. A. Marchioli, C. J. Foster, and Y.-C. Liang, *Phys. Rev. A* **78**, 052330 (2008).

[7] J. A. Miszczak, Z. Puchała, P. Horodecki, A. Uhlmann, and K. Życzkowski, *Quant. Inf. Comp.* **9**, 103 (2009).

- [8] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, England, 2006).
- [9] D. Bures, *Trans. Am. Math. Soc.* **135**, 199 (1969).
- [10] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, *Phys. Rev. Lett.* **78**, 2275 (1997).
- [11] Paulina Marian, T. A. Marian, and H. Scutaru, *Phys. Rev. A* **68**, 062309 (2003) for squeezed thermal states; Paulina Marian and T. A. Marian, *ibid.* **77**, 062319 (2008) for symmetric states; *Eur. Phys. J. Special Topics* **160**, 281 (2008) also for symmetric states.
- [12] Paulina Marian, T. A. Marian, and H. Scutaru, *Phys. Rev. Lett.* **88**, 153601 (2002).
- [13] A. B. Klimov, L. L. Sánchez-Soto, E. C. Yustas, J. Söderholm, and G. Björk, *Phys. Rev. A* **72**, 033813 (2005); L. L. Sánchez-Soto, E. C. Yustas, G. Björk, and A. B. Klimov, *ibid.* **76**, 043820 (2007).
- [14] A. Luis, *Opt. Commun.* **273**, 173 (2007).
- [15] G. Björk, J. Söderholm, L. L. Sánchez-Soto, A. B. Klimov, Iulia Ghiu, Paulina Marian, and T. A. Marian, *Opt. Commun.* **283**, 4440 (2010).
- [16] Iulia Ghiu, G. Björk, Paulina Marian, and T. A. Marian, *Phys. Rev. A* **82**, 023803 (2010).
- [17] S. L. Braunstein and C. M. Caves, *Phys. Rev. Lett.* **72**, 3439 (1994).
- [18] M. G. A. Paris, *Int. J. Quantum Inf.* **7**, 125 (2009).
- [19] P. Zanardi, M. G. A. Paris, and L. C. Venuti, *Phys. Rev. A* **78**, 042105 (2008).
- [20] S.-J. Gu, *Int. J. Mod. Phys. B* **24**, 4371 (2010).
- [21] J. Twamley, *J. Phys. A: Math. Gen.* **29**, 3723 (1996).
- [22] H. Scutaru, *J. Phys. A: Math. Gen.* **31**, 3659 (1998); Gh.-S. Păraoanu and H. Scutaru, *Phys. Rev. A* **58**, 869 (1998).
- [23] A. Ferraro, S. Olivares, and M. G. A. Paris, *Gaussian States in Quantum Information* (Bibliopolis, Naples, Italy, 2005).
- [24] S. L. Braunstein and P. Van Loock, *Rev. Mod. Phys.* **77**, 513 (2005).
- [25] C. Weedbrook, S. Pirandola, R. Garcia-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, *Rev. Mod. Phys.* **84**, 621 (2012).
- [26] P. Giorda and M. G. A. Paris, *Phys. Rev. Lett.* **105**, 020503 (2010).
- [27] G. Adesso and A. Datta, *Phys. Rev. Lett.* **105**, 030501 (2010).
- [28] Gh.-S. Păraoanu and H. Scutaru, *Phys. Rev. A* **61**, 022306 (2000).
- [29] G. B. Folland, *Harmonic Analysis in Phase Space* (Princeton University Press, Princeton, New Jersey, 1989), pp. 256–257, Appendix A, Theorem 1.
- [30] R. Simon, E. C. G. Sudarshan, and N. Mukunda, *Phys. Rev. A* **36**, 3868 (1987).
- [31] H. Scutaru, *Phys. Lett. A* **141**, 223 (1989); **167**, 326 (1992).
- [32] J. Williamson, *Am. J. Math.* **58**, 141 (1936). See also Ref. [29], p. 177, Proposition (4.22).
- [33] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
- [34] These composition rules, albeit general, have the same structure as the specialized formulas written in our Eq. (3.2). However, they are equivalent to those of the oscillator semigroup treated in Chapter 5 of Ref. [29].
- [35] Paulina Marian and T. A. Marian, *Phys. Rev. A* **47**, 4474 (1993).