Quantum stabilizer codes embedding qubits into qudits

Carlo Cafaro,[1,2,3,*] Federico Maiolini,[1] and Stefano Mancini[1]

[1]*Physics Division, School of Science and Technology, University of Camerino, I-62032 Camerino, Italy*

[2]*Optical Quantum Information Theory Group, Max-Planck Institute for the Science of Light, Gunther-Scharowsky-Str. 1/Bau 26, 91058 Erlangen, Germany*

[3]*Institute of Theoretical Physics, Universitat Erlangen-Nurnberg, Staudtstr. 7/B2, 91058 Erlangen, Germany*

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We study, by means of the stabilizer formalism, a quantum error correcting code which is alternative to the standard block codes since it embeds a qubit into a qudit. The code exploits the noncommutative geometry of discrete phase space to protect the qubit against both amplitude and phase errors. The performance of this code is evaluated on Weyl channels by means of the entanglement fidelity as a function of the error probability. A comparison with standard block codes, like five- and seven-qubit stabilizer codes, shows its superiority.

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I. INTRODUCTION

In order to describe realistic quantum information processes, quantum errors induced by environmental noise must be taken into account. This can be accomplished by introducing the notion of quantum channels, that is, maps on the set of states of the system that are completely positive and trace preserving [1]. At the same time, one would combat quantum errors to avoid their detrimental effect on quantum information processes. To this end, the method of error correcting codes has been borrowed from classical information theory (for a comprehensive introduction to the quantum theory of error correcting codes we refer to [2]). The underlying idea is to exploit redundancy, that is, to encode information in linear subspaces (codes) of the total complex Hilbert space in such a way that errors induced by the interaction with the environment can be detected and corrected.

Usually a logical qubit (a two-dimensional complex Hilbert space $\mathcal{H}_2$) is encoded into n physical qubits (a 2^n dimensional complex Hilbert space $\mathcal{H}_2^{\otimes n}$). This kind of encoding is known as *block encoding*:

$$\mathcal{H}_2 \ni |q_{\text{logical}}\rangle \longmapsto |q_{\text{physical}}\rangle \in \mathcal{H}_2^{\otimes n}.$$

However, there is also the possibility of *embedding* a logical qubit into a d-dimensional quantum physical system, i.e., a *qudit* with complex Hilbert space $\mathcal{H}_d \neq \mathcal{H}_2^{\otimes n}$. We refer to this kind of encoding as *embedding* or *qudit encoding*,

$$\mathcal{H}_2 \ni |q_{\text{logical}}\rangle \longmapsto |q_{\text{physical}}\rangle \in \mathcal{H}_d \neq \mathcal{H}_2^{\otimes n},$$

where $d \neq 2^n$.

For block-coding schemes a powerful formalism, called the stabilizer formalism [3], has been developed describing one of the most important classes of quantum codes, namely, the quantum version of linear codes in classical coding theory. The stabilizer formalism can be extended over nonbinary codes [4] and can also be useful for describing embedding (qudit) codes. Actually, the idea of embedding a qubit into a larger space without resorting to block codes was put forward in [5], where a qubit was encoded into a bosonic mode (infinite-dimensional complex Hilbert space) just using the stabilizer formalism. Later on the possibility of qudit encoding was pointed out in Ref. [6] by using the same formalism. There, being in $\mathcal{H}_d$, the errors were considered as a generalization of Pauli operators, representing shift errors (X type) or phase errors (Z type) or their combination (XZ type). However, in [6] the proposed code was essentially classical since only Z-type errors were taken into consideration.

In this article we upgrade this coding scheme to be fully quantum and, thus, able to correct X-type, Z-type and, XZ-type errors. We then test its effectiveness for $d = 18$ and $d = 50$ on a Weyl quantum noisy channel [7] (an error model characterized by errors of X, Z, and XZ types). We allow for the possibility of considering X and Z errors occurring with both symmetric and asymmetric probabilities. Finally, we compare the performance of such qudit coding schemes to those of the conventional five-qubit stabilizer code [[5, 1, 3]] [8,9] and seven-qubit stabilizer CSS (Calderbank-Shor-Steane) code [[7, 1, 3]] [10,11]. We characterize the performances of these codes by means of the entanglement fidelity [12], rather than the averaged input-output fidelity used in [6]. We show that the use of qudit codes may allow saving of space resources while achieving the same performance of block codes.

The layout of the article is as follows. In Sec. II, we briefly describe block encoding and then introduce qudit encoding. Special focus is devoted to both Pauli groups of n-qubit vectors and generalized Pauli groups of error operators acting on a qudit. In Sec. III, the Weyl noisy quantum channel is discussed together with the entanglement fidelity. In Sec. IV, we study the performance of qudit codes. In Sec. V, for the sake of comparison, we quantify the performance of relevant block codes. Our final remarks appear in Sec. VI.

II. FROM BLOCK CODES TO EMBEDDING CODES

In this section, we briefly recall the block-encoding error correction schemes in terms of the stabilizer formalism. Then, using the same formalism, we introduce the qudit-encoding (embedding qubits into qudits) scheme. In both cases we restrict our attention to a single encoded qubit.

*carlo.cafaro@mpl.mpg.de

A. Pauli group of *n* qubits and block encoding

A qubit is a two-dimensional quantum system with associated complex Hilbert space $\mathcal{H}_2$. Let $\{|0\rangle,|1\rangle\}$ be the canonical basis of this space and consider on it the Pauli operators $X \stackrel{\text{def}}{=} \sigma_x$, $Y \stackrel{\text{def}}{=} i_{\mathbb{C}}\sigma_y$, $Z \stackrel{\text{def}}{=} \sigma_z$ defined in terms of the standard Pauli operators σ_x, σ_y, σ_z realizing the $\mathfrak{su}(2)$ algebra (throughout the paper $i_{\mathbb{C}}$ denotes the imaginary unit of $\mathbb{C}$). They are such that $\{|0\rangle,|1\rangle\}$ are eigenstates of Z,

$$XZ = -ZX,$$

and $Y = XZ$. The Pauli operators so defined suffice to describe all possible errors occurring on a single qubit. Together with the identity operator I (with $I = I_{2\times 2}$), they form a multiplicative group if we allow them to be multiplied by -1, i.e., $\{\pm I, \pm X, \pm Y, \pm Z\}$. We refer to this group as the Pauli group $\mathcal{P}_{\mathcal{H}_2}$. Actually it is a subgroup of the Pauli group realized through the standard σ Pauli operators and it coincides with the discrete version of the Heisenberg-Weyl group.

For n-qubit errors we can then consider the Pauli group $\mathcal{P}_{\mathcal{H}_2^{\otimes n}}$ whose elements result from n-fold direct products (see also [13]):

$$e(\lambda, j_1, \ldots, j_n) = (-)^{\lambda} e_1(j_1) \otimes \ldots \otimes e_n(j_n).$$

The subscripts on the right-hand side label the qubits $1, \ldots, n$, while $j_k = 0,1,2,3$ label the operators I_k, X_k, Y_k, Z_k, respectively, acting on the kth qubit. Furthermore, $\lambda \in \{0,1\}$.

Since $e_k(2) = X_k Z_k$, the elements $e(\lambda, j_1, \ldots, j_n)$ can be rewritten as

$$e(\lambda,a,b) = (-)^{\lambda} X(a) Z(b),$$

where $a = a_1 \ldots a_n$ and $b = b_1 \ldots b_n$ are bit strings of length n, and

$$X(a) \stackrel{\text{def}}{=} (X_1)^{a_1} \otimes \ldots \otimes (X_n)^{a_n},$$
$$Z(b) \stackrel{\text{def}}{=} (Z_1)^{b_1} \otimes \ldots \otimes (Z_n)^{b_n}.$$

Observe that the order of the group $\mathcal{P}_{\mathcal{H}_2^{\otimes n}}$ of n-qubit errors is $|\mathcal{P}_{\mathcal{H}_2^{\otimes n}}| = 2^{2n+1}$. Since the factor $\pm$ in front of an error makes no relevant difference in its action, we can actually assume to work with the quotient group $\mathcal{P}_{\mathcal{H}_2^{\otimes n}}/\{\pm I\}$, with the major exception being the determination whether elements commute or anticommute.

The order of the quotient group is $|\mathcal{P}_{\mathcal{H}_2^{\otimes n}}/\{\pm I\}| = 2^{2n}$. Then there is a *one-to-one* correspondence between $\mathcal{P}_{\mathcal{H}_2^{\otimes n}}/\{\pm I\}$ and the $2n$-dimensional binary vector space $\mathbb{F}_2^{2n}$ whose elements are bit strings of length $2n$ [14]. A vector $v \in \mathbb{F}_2^{2n}$ is denoted $v = (a|b)$, where $a = a_1 \ldots a_n$ and $b = b_1 \ldots b_n$ are bit strings of length n. Scalars take values in the Galois field $\mathbb{F}_2 = \{0,1\}$ and vector addition adds components modulo 2. In short, we have the following correspondence: $e(\lambda,a,b) \in \mathcal{P}_{\mathcal{H}_2^{\otimes n}} \leftrightarrow v_e = (a|b) \in \mathbb{F}_2^{2n}$.

A *quantum stabilizer code* $\mathcal{C}$ is a vector space $\mathcal{C} \subseteq \mathcal{H}_2^{\otimes n}$ stabilized by an Abelian subgroup $\mathcal{S} \subseteq \mathcal{P}_{\mathcal{H}_2^{\otimes n}}$, i.e., such that $\mathcal{S}|\psi\rangle = |\psi\rangle, \forall |\psi\rangle \in \mathcal{C}$. A quantum stabilizer code C with stabilizer generators $g^{(1)}, \ldots, g^{(n-1)}$ is a two-dimensional code space, i.e., a space in which to encode a single qubit. The code words $|0\rangle_L$, $|1\rangle_L$ (basis vectors for such a code space) can be found as orthogonal eigenvectors (corresponding to the eigenvalue $+1$) of any of the generators $g^{(j)}$.

The encoding operation then reads

$$\mathcal{H}_2 \ni |0\rangle \mapsto |0_L\rangle \in \mathcal{H}_2^{\otimes n} \quad \text{and} \quad \mathcal{H}_2 \ni |1\rangle \mapsto |1_L\rangle \in \mathcal{H}_2^{\otimes n}.$$

In view of the correspondence between the Pauli group and the vector space $\mathbb{F}_2^{2n}$, let $v^{(j)} = (a^{(j)}|b^{(j)})$ be the image of the generators $g^{(j)}$ in $\mathbb{F}_2^{2n}$ and introduce the so-called *parity check matrix*:

$$H \stackrel{\text{def}}{=} \begin{pmatrix} (a^{(1)}|b^{(1)}) \\ (a^{(2)}|b^{(2)}) \\ \vdots \\ (a^{(n-1)}|b^{(n-1)}) \end{pmatrix}. \tag{1}$$

Then, for an error $e \in \mathcal{P}_{\mathcal{H}_2^{\otimes n}} \leftrightarrow v_e = (a|b) \in \mathbb{F}_2^{2n}$, the error syndrome $S(e)$ is given by the bit string [13]

$$S(e) = H v_e = l_1 \ldots l_{n-1}, \tag{2}$$

where $l_j = H^T(j) \cdot v_e$.

Errors with nonvanishing error syndrome are *detectable*. They correspond to operators not in $\mathcal{S}$ and not commuting with those in $\mathcal{S}$. That is, a set of error operators $\mathcal{E} \subseteq \mathcal{P}_{\mathcal{H}_2^{\otimes n}}$ is detectable if $\mathcal{E} \notin \mathcal{Z}(\mathcal{S}) - \mathcal{S}$, with $\mathcal{Z}(\mathcal{S})$ the centralizer of the subgroup $\mathcal{S}$. Furthermore, the set of error operators $\mathcal{E} \subseteq \mathcal{P}_{\mathcal{H}_2^{\otimes n}}$ is *correctable* if the set given by $\mathcal{E}^{\dagger}\mathcal{E}$ is in turn detectable [15], i.e., $\mathcal{E}^{\dagger}\mathcal{E} \notin \mathcal{Z}(\mathcal{S}) - \mathcal{S}$.

It would be awfully tedious to identify either detectable errors or sets of correctable errors. However, the quantum stabilizer formalism allows us to simplify this task [3]. This is a consequence of the fact that by means of this formalism it is sufficient to study the effect of the error operators on the generators of the stabilizer and not on the code words themselves. Actually the syndrome extraction corresponds to measuring the stabilizer generators.

Finally, we denote by $[[n,k,d_{\mathcal{C}}]]$ a quantum stabilizer code $\mathcal{C}$ with code parameters n (the length), k (the dimension), and $d_{\mathcal{C}}$ (the distance) encoding k logical qubits into n physical qubits and correcting $\lfloor \frac{d_{\mathcal{C}}-1}{2} \rfloor$-qubit errors ($\lfloor x \rfloor$ denotes the largest integer less than x).

B. Generalized Pauli group and qudit encoding

A qudit is a d-dimensional quantum system with associated complex Hilbert space $\mathcal{H}_d$. In this space we can introduce a generalized version of the Pauli operators X and Z considered in Sec. II A. They can be defined through their action on the canonical basis $\{|k\rangle\}_{k\in\mathbb{Z}_d}$ of $\mathcal{H}_d$ [16],

$$X|k\rangle = |k \oplus 1\rangle \quad \text{and} \quad Z|k\rangle = \omega^k |k\rangle, \quad k \in \mathbb{Z}_d, \tag{3}$$

where "$\oplus$" denotes addition of integers modulo d and $\omega \stackrel{\text{def}}{=} \exp(i_{\mathbb{C}} \frac{2\pi}{d})$ is a primitive dth root of unity ($\omega^d = 1$). The X and Z operators so defined are unitary ($X^{\dagger} = X^{-1}$ and $Z^{\dagger} = Z^{-1}$), but not Hermitian, and satisfy $X^d = Z^d = I$ (with $I = I_{d\times d}$) together with the commutation relations

$$X^a Z^b = \omega^{-ab} Z^b X^a. \tag{4}$$

It is then possible to consider the Pauli group $\mathcal{P}_{\mathcal{H}_d}$ consisting of all operators e of the form

$$e(l,n,m) = \omega^l X^n Z^m,$$

where $l,n,m \in \mathbb{Z}_d$. Similarly to Sec. II A, for errors on a qudit we may refer to the quotient group $\mathcal{P}_{\mathcal{H}_d}/\{\omega^l I | l = 0, \ldots, d-1\}$.

In addition to the reasons stated in Sec. I for passing from block to embedding codes, another motivation is to understand whether or not finite-dimensional versions of the shift-resistant quantum codes of Ref. [5] are effective. Hence, following Ref. [5], we consider $d = 2r_1 r_2$, and introduce a code $\mathcal{C}$ stabilized by the Abelian subgroup $\mathcal{S} \subset \mathcal{P}_{\mathcal{H}_d}$ generated by X^{2r_1}, Z^{2r_2}. The code words (basis for $\mathcal{C}$) are eigenstates of Z^{2r_2} and X^{2r_1} with eigenvalue 1. Hence they only contain $|k\rangle$'s with values of k that are multiples of r_1 and that are invariant under a shift by $2r_1$. They read

$$|0_L\rangle = \frac{1}{\sqrt{r_2}}(|0\rangle + |2r_1\rangle + \cdots + |2(r_2-1)r_1\rangle), \tag{5}$$

$$|1_L\rangle = \frac{1}{\sqrt{r_2}}(|r_1\rangle + \cdots + |(2(r_2-1)+1)r_1\rangle). \tag{6}$$

The encoding operation becomes, in this case,

$$\mathcal{H}_2 \ni |0\rangle \mapsto |0_L\rangle \in \mathcal{H}_d \quad \text{and,} \quad \mathcal{H}_2 \ni |1\rangle \mapsto |1_L\rangle \in \mathcal{H}_d.$$

If states (5) and (6) undergo an amplitude shift, the value of k modulo r_1 is determined by measuring the stabilizer generator Z^{2r_2}, and the shift can be corrected by adjusting k to the nearest multiple of r_1.

The code words in the basis of X's eigenstates can be found by observing that the eigenstates of X and Z operators are connected by the Fourier transform

$$|\widetilde{i}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega^{-ij}|j\rangle,$$

where $X|\widetilde{i}\rangle = \omega^i|\widetilde{i}\rangle$. Then it turns out that

$$|0_L\rangle = \frac{1}{\sqrt{r_1}}(|\widetilde{0}\rangle + |\widetilde{2r_2}\rangle + \cdots + |\widetilde{2(r_1-1)r_2}\rangle), \tag{7}$$

$$|1_L\rangle = \frac{1}{\sqrt{r_1}}(|\widetilde{r_2}\rangle + \cdots + |\widetilde{(2(r_1-1)+1)r_2}\rangle). \tag{8}$$

The code words (7) and (8) have the same form as (5) and (6), but with r_1 and r_2 interchanged. Hence, if they undergo a phase shift, the value of k modulo r_2 is determined by measuring the stabilizer generator X^{2r_1}, and the shift can be corrected by adjusting k to the nearest multiple of r_2.

To understand what the set of correctable errors is according to the condition $\mathcal{E}^\dagger\mathcal{E} \notin \mathcal{Z}(\mathcal{S}) - \mathcal{S}$, we have, with the help of (4),

$$(X^{a'}Z^{b'})^\dagger(X^aZ^b)X^{2r_1} = e^{2\pi i(b-b')/r_2}X^{2r_1}(X^{a'}Z^{b'})^\dagger(X^aZ^b),$$
$$(X^{a'}Z^{b'})^\dagger(X^aZ^b)Z^{2r_2} = e^{-2\pi i(a-a')/r_1}Z^{2r_2}(X^{a'}Z^{b'})^\dagger(X^aZ^b).$$

The phases on the right-hand sides are nontrivial only if

$$|a-a'| < \frac{r_1}{2} \quad \text{and} \quad |b-b'| < \frac{r_2}{2}. \tag{9}$$

Therefore, the code $\mathcal{C}$ can correct all shifts corresponding to these conditions. They amount to $r_1 r_2 = d/2$ and this is also the number of possible error syndromes. Finally, note that other families of qudit codes could be constructed by generalizing the Pauli operators in a different way, e.g., by making them Hermitian.

III. THE WEYL QUANTUM CHANNEL AS ERROR MODEL

In this section, we discuss first the Weyl quantum channel for qudit states and for qubit states and then the entanglement fidelity as quantifier of codes' performances.

A. General form of the Weyl quantum channel

Consider a completely positive trace preserving map (CPT map or quantum channel) $\Lambda^{(d)}$,

$$\Lambda^{(d)}: \mathfrak{S}(\mathcal{H}_d) \ni \rho \longmapsto \Lambda^{(d)}(\rho) \in \mathfrak{S}(\mathcal{H}_d),$$

where $\mathfrak{S}(\mathcal{H}_d)$ is the set of positive unit trace linear operators in $\mathcal{H}_d$. Then $\Lambda^{(d)}$ is called bistochastic if

$$\Lambda^{(d)}\left(\frac{1}{d}I\right) \stackrel{\text{def}}{=} \frac{1}{d}I,$$

where I $(= I_{d\times d})$ is the identity operator in $\mathcal{H}_d$. The d-dimensional Weyl channel is a bistochastic quantum channel of the form [7]

$$\Lambda^{(d)}_{\text{Weyl}}(\rho) \stackrel{\text{def}}{=} \sum_{n,m=0}^{d-1}\pi(n,m)\,U_{n,m}\rho U^\dagger_{n,m}, \tag{10}$$

where $\rho \in \mathfrak{S}(\mathcal{H}_d)$ and $\pi(n,m)$ is an arbitrary probability distribution, thus respecting $0 \leqslant \pi(n,m) \leqslant 1$, $\sum_{n,m=0}^{d-1}\pi(n,m) = 1$. The unitary Weyl operators $U_{n,m}$ in (10) are defined as

$$U_{n,m} \stackrel{\text{def}}{=} \sum_{k=0}^{d-1}\exp\left(i_{\mathbb{C}}\frac{2\pi}{d}km\right)|k\oplus n\rangle\langle k|,$$

where "$\oplus$," as specified earlier, denotes the addition of integers modulo d. They also satisfy the (Weyl) commutation relations, (4), i.e.,

$$U_{n,m}U_{n',m'} = \exp\left[i_{\mathbb{C}}\frac{2\pi}{d}(n'm - nm')\right]U_{n',m'}U_{n,m},$$

where $0 \leqslant n,\ n',\ m,\ m' \leqslant d-1$. Note that $U_{n,m}$ may be rewritten as $U_{n,m} = U_{n,0}U_{0,m}$. Furthermore, $U_{n,0} \equiv X^n$ and $U_{0,m} \equiv Z^m$. Therefore in what follows we consider the d-dimensional Weyl channel acting as

$$\Lambda^{(d)}_{\text{Weyl}}(\rho) \stackrel{\text{def}}{=} \sum_{n,m=0}^{d-1}\pi(n,m)X^nZ^m\rho(X^nZ^m)^\dagger. \tag{11}$$

For $d = 2$ we have the most general channel acting on a qubit:

$$\Lambda^{(2)}_{\text{Weyl}}(\rho) = \pi(0,0)\rho + \pi(1,0)X\rho X + \pi(1,1)Y\rho Y + \pi(0,1)Z\rho Z. \tag{12}$$

For instance, if we take $\pi(0,0) \stackrel{\text{def}}{=} 1-p, \pi(1,0) = \pi(1,1) = \pi(0,1) \stackrel{\text{def}}{=} \frac{p}{3}$, the channel $\Lambda^{(2)}_{\text{Weyl}}$ becomes the standard symmetric qubit depolarizing channel.

To justify the choice of Weyl's error model, we point out that the quantum codes employed here are designed to error-correct arbitrary quantum errors such as X errors, Z errors, and combinations of the two (Y errors). Since the Kraus decomposition of the Weyl channel is defined in terms of powers of these aforementioned error operators, it certainly constitutes a natural test bed where quantifying the performance of the selected qudit codes. Such a test bed turns out also to be very general, with the possibility of encompassing physically relevant scenarios.

Hereafter, when we consider the two-dimensional Weyl channel, we mean to take into consideration the following channel parametrization:

$$\pi(n,m) \overset{\text{def}}{=} \pi_X(n)\pi_Z(m) \equiv \pi(n)\pi(m), \tag{13}$$

with

$$\pi(1) = \pi(-1) \overset{\text{def}}{=} p, \pi(0) \overset{\text{def}}{=} 1 - p. \tag{14}$$

In this way we have

$$\Lambda^{(2)}_{\text{Weyl}}(\rho) = (1-p)^2\rho + (1-p)pX\rho X^\dagger + p^2 Y\rho Y^\dagger + p(1-p)Z\rho Z^\dagger. \tag{15}$$

Then we can also consider the possibility of having asymmetric X- and Z-error probabilities [17–22]. In this case the probabilities $\pi(n,m)$ are defined as

$$\pi^{(\text{asymmetric})}(n,m) \overset{\text{def}}{=} \pi_X(n)\pi_Z(m), \tag{16}$$

with π_X and π_Z not identical. In particular, we consider

$$\pi_X(1) = \kappa\pi_Z(1), \tag{17}$$

with $\pi_Z(1) \overset{\text{def}}{=} p$ and $\kappa \in [0,1/p]$ to guarantee that κp is a valid probability value. Note that for $\kappa = 1$ we recover the symmetric case. Finally, when considering block codes on n qubits, the error map simply becomes $\Lambda^{(2)\otimes n}_{\text{Weyl}}$ acting on $\rho \in \mathfrak{S}(\mathcal{H}_2^{\otimes n})$.

B. Entanglement fidelity

Entanglement fidelity is a reliable performance measure of the efficiency of quantum error correcting codes [23]. Suppose a two-dimensional code $\mathcal{C}$ is such that $\mathcal{C} \subset \mathcal{H}$ with $\dim_{\mathbb{C}}\mathcal{H} = N$ (here $\mathcal{H}$ can be either $\mathcal{H}_2^{\otimes n}$ or $\mathcal{H}_d$, hence N is either 2^n or d). Then consider errors taking place through a CPT map $\Lambda : \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{H})$ (which can be either $\Lambda^{(2)\otimes n}$ or $\Lambda^{(d)}$) written, in terms of the Kraus decomposition, as

$$\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger.$$

To recover the errors by means of code $\mathcal{C}$, a *recovery* operation must be applied according to the syndrome extraction. Suppose it is described by a CPT map $\mathcal{R} : \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{H})$,

$$\mathcal{R}(\rho) = \sum_k R_k \rho R_k^\dagger.$$

If the code is effective, we expect that the map resulting from the composition of Λ and $\mathcal{R}$ restricted to the subspace $\mathcal{C}$, namely, $[\mathcal{R} \circ \Lambda]_{|\mathcal{C}}$, will be close to the identity map $\text{id}_{\mathcal{C}}$ on $\mathcal{C}$. In order to evaluate this closeness we can consider a state $\rho = \text{tr}_{\mathcal{C}}|\psi\rangle\langle\psi|$ written in terms of a purification $|\psi\rangle \in \mathcal{C} \otimes \mathcal{C}$ and see how well entanglement (between $\mathcal{C}$ and the reference system identical to $\mathcal{C}$) is preserved by means of

$$\mathcal{F}(\rho,[\mathcal{R} \circ \Lambda]_{|\mathcal{C}}) \overset{\text{def}}{=} \langle\psi|([\mathcal{R} \circ \Lambda^{(N)}]_{|\mathcal{C}} \otimes \text{id}_{\mathcal{C}})(|\psi\rangle\langle\psi|)|\psi\rangle.$$

This is the entanglement fidelity [12] for the map $[\mathcal{R} \circ \Lambda]_{|\mathcal{C}}$.

In terms of the Kraus error operators, $\mathcal{F}$ can be rewritten as [24]

$$\mathcal{F}(\rho,[\mathcal{R} \circ \Lambda]_{|\mathcal{C}}) = \sum_{j,k} |\text{tr}[R_j A_k]_{|\mathcal{C}}|^2.$$

Finally, choosing a purification described by a maximally entangled unit vector $|\psi\rangle \in \mathcal{C} \otimes \mathcal{C}$ for the mixed state $\rho = \frac{1}{\dim_{\mathbb{C}}\mathcal{C}} I_{\mathcal{C}}$, we obtain

$$\mathcal{F}\left(\frac{1}{2}I_{\mathcal{C}},[\mathcal{R} \circ \Lambda]_{|\mathcal{C}}\right) = \frac{1}{2^2}\sum_{j,k} |\text{tr}[R_j A_k]_{|\mathcal{C}}|^2. \tag{18}$$

This is the expression we use in the following.

IV. QUDIT CODES FOR WEYL ERRORS

In this section, we analyze in detail how the qudit codes devised in Sec. II work on the Weyl channel for $d = 18$ and $d = 50$ and determine their performance by means of the entanglement fidelity. Our main motivation to use the qudit codes with $d = 18$ and $d = 50$ is that they represent the lowest dimensional perfect qudit systems where a two-dimensional quantum systems (a qubit) can be encoded and protected against arbitrary shift errors of the form $X^n Z^m$ by 1 and 2 units, respectively. Furthermore, the physical dimensionality of these codes is chosen to be, as much as possible, comparable with the physical dimensionality of code spaces characterizing well-known standard stabilizer error correction schemes capable of correcting arbitrary single-qubit errors and, possibly, a few two-qubit errors. For this reason, the five-qubit [8,9] and seven-qubit [10,11] quantum stabilizer codes seem to be a convenient choice. In particular, recalling that a stabilizer code is perfect if all the eigenvalues of the generators constitute valid syndromes for correcting an error, it turns out that both the five and the qudit code with $d = 18$ are perfect and require minimal quantum resources for their task.

A. The $d = 18$ qudit code

1. Encoding

The encoding operation is characterized by

$$\mathcal{H}_2 \ni |0\rangle \mapsto |0_L\rangle \in \mathcal{H}_{18} \quad \text{and} \quad \mathcal{H}_2 \ni |1\rangle \mapsto |1_L\rangle \in \mathcal{H}_{18},$$

where the code words $|0_L\rangle$ and $|1_L\rangle$ are defined according to (5) and (6) as

$$|0_L\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{3}}[|0\rangle + |6\rangle + |12\rangle] \quad \text{and}$$

$$|1_L\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{3}}[|3\rangle + |9\rangle + |15\rangle],$$

respectively. The stabilizer group $\mathcal{S}$ of this code is generated by the two error operators X^6 and Z^6. Here $r_1 = r_2 = 3$.

A simple calculation shows that provided we restrict our focus to the two-dimensional code space $\mathcal{C}^{(d=18)} \subset \mathcal{H}_{18}$, the following identities hold:

$$I = X^6 = X^{12}, \quad X = X^7 = X^{13}, \quad X^2 = X^8 = X^{14},$$
$$X^3 \equiv X^{-3} = X^9 = X^{15}, \quad X^4 \equiv X^{-2} = X^{10} = X^{16},$$
$$X^5 \equiv X^{-1} = X^{11} = X^{17},$$

and

$$I = Z^6 = Z^{12}, \quad Z = Z^7 = Z^{13}, \quad Z^2 = Z^8 = Z^{14},$$
$$Z^3 \equiv Z^{-3} = Z^9 = Z^{15}, \quad Z^4 \equiv Z^{-2} = Z^{10} = Z^{16},$$
$$Z^5 \equiv Z^{-1} = Z^{11} = Z^{17}.$$

Therefore the total number of error operators to be considered amounts to 36 rather than 18^2. Then the Weyl channel (11) may be rewritten as

$$\Lambda^{(d=18)}(\rho) = \sum_{k=0}^{35} A_k \rho A_k^\dagger,$$

where we have relabeled the error operators as follows:

$$A_0 \overset{\text{def}}{=} \sqrt{\pi(0,0)} I, \quad A_1 \overset{\text{def}}{=} \sqrt{\pi(0,1)} Z, \ldots,$$
$$A_{18} \overset{\text{def}}{=} \sqrt{\pi(-1,3)} X^{-1} Z^3, \quad \ldots, A_{34} \overset{\text{def}}{=} \sqrt{\pi(3,-2)} X^3 Z^{-2},$$
$$A_{35} \overset{\text{def}}{=} \sqrt{\pi(3,3)} X^3 Z^3.$$

In the *symmetric case*, the probabilities $\pi(n,m)$ are defined as in Eq. (13), where now

$$\pi(1) = \pi(-1) \overset{\text{def}}{=} p, \quad \pi(2) = \pi(-2) \overset{\text{def}}{=} p^2,$$
$$\pi(3) \equiv \pi(-3) \overset{\text{def}}{=} p^3, \quad \pi(0) \overset{\text{def}}{=} 1 - 2p - 2p^2 - p^3.$$

2. *Correctability*

According to (9) with $r_1 = r_2 = 3$ the set of correctable errors $\mathcal{A}_{\text{correctable}}^{(d=18)}$ is given by the following 9 errors,

$$\mathcal{A}_{\text{correctable}}^{(d=18)} = \{A_0, A_1, A_2, A_6, A_7, A_8, A_{12}, A_{13}, A_{14}\}, \qquad (19)$$

where

$$A_0 \overset{\text{def}}{=} \sqrt{\pi(0,0)} I, \quad A_1 \overset{\text{def}}{=} \sqrt{\pi(0,1)} Z,$$
$$A_2 \overset{\text{def}}{=} \sqrt{\pi(0,-1)} Z^{-1}, \quad A_6 \overset{\text{def}}{=} \sqrt{\pi(1,0)} X,$$
$$A_7 \overset{\text{def}}{=} \sqrt{\pi(1,1)} XZ, \quad A_8 \overset{\text{def}}{=} \sqrt{\pi(1,-1)} XZ^{-1},$$
$$A_{12} \overset{\text{def}}{=} \sqrt{\pi(-1,0)} X^{-1}, \quad A_{13} \overset{\text{def}}{=} \sqrt{\pi(-1,1)} X^{-1} Z,$$
$$A_{14} \overset{\text{def}}{=} \sqrt{\pi(-1,-1)} X^{-1} Z^{-1}.$$

3. *Recovery operators*

Following the recipe provided in [25], it turns out that the two nine-dimensional orthogonal subspaces $\mathcal{V}^{0_L}$ and $\mathcal{V}^{1_L}$ of $\mathcal{H}_{18}$ generated by the action of $\mathcal{A}_{\text{correctable}}^{(d=18)}$ on $|0_L\rangle$ and $|1_L\rangle$ are given by

$$\mathcal{V}^{0_L} = \text{Span}\left\{ |v_k^{0_L}\rangle = \frac{A_k}{\sqrt{\pi_k}} |0_L\rangle \right\}, \quad \text{and}$$
$$\mathcal{V}^{1_L} = \text{Span}\left\{ |v_k^{1_L}\rangle = \frac{A_k}{\sqrt{\pi_k}} |1_L\rangle \right\},$$

with $k \in \mathcal{I}^{(d=18)} \overset{\text{def}}{=} \{0,1,2,6,7,8,12,13,14\}$. The coefficients $\sqrt{\pi_k}$ denote the error amplitudes where, for instance, $\sqrt{\pi_8} \overset{\text{def}}{=} \sqrt{\pi(1,-1)}$. Note that $\langle v_k^{i_L} | v_{k'}^{j_L} \rangle = \delta_{kk'} \delta_{ij}$ with $k,k' \in \mathcal{I}^{(d=18)}$ and $i,j \in \{0,1\}$. Therefore, it follows that $\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L} = \mathcal{H}_{18}$.

The recovery superoperator $\mathcal{R} \leftrightarrow \{R_k\}$ with $k \in \mathcal{I}^{(d=18)}$ is defined by means of [25]

$$R_k = |0_L\rangle\langle v_k^{0_L}| + |1_L\rangle\langle v_k^{1_L}|.$$

In the case under investigation, the entanglement fidelity, (18), reads

$$\mathcal{F}^{(d=18)}(p) \overset{\text{def}}{=} \mathcal{F}^{(d=18)}\left(\frac{1}{2} I_{2\times 2}, \mathcal{R} \circ \Lambda^{(d=18)}\right)$$
$$= \frac{1}{(2)^2} \sum_{l=0}^{35} \sum_{k \in \mathcal{I}^{(d=18)}} |\text{tr}([R_k A_l]_{|\mathcal{C}^{(d=18)}})|^2,$$

where

$$[R_k A_l]_{|\mathcal{C}^{(d=18)}} \overset{\text{def}}{=} \begin{pmatrix} \langle 0_L | R_k A_l | 0_L \rangle & \langle 0_L | R_k A_l | 1_L \rangle \\ \langle 1_L | R_k A_l | 0_L \rangle & \langle 1_L | R_k A_l | 1_L \rangle \end{pmatrix}.$$

After a simple calculation, $\mathcal{F}^{(d=18)}(p)$ becomes

$$\mathcal{F}^{(d=18)}(p) = 1 - 4p^2 - 2p^3 + 4p^4 + 4p^5 + p^6. \qquad (20)$$

We stress that this error correction scheme is effective as long as the failure probability $1 - \mathcal{F}^{(d=18)}(p)$ is strictly smaller than the error probability p. This implies that the $d = 18$–dimensional qudit code is effective only when $0 \leqslant p \lesssim 0.24$. Furthermore, we point out that in its range of effectiveness, $\mathcal{F}^{(d=18)}(p)$ in (20) is a monotonic decreasing function of p.

4. *Asymmetric errors*

By repeating the steps in the previous subsection using Eqs. (16) and (17), we get

$$\mathcal{F}_{\text{asymmetric}}^{(d=18)}(p) = 1 - 2p^2 - p^3 + \kappa^2(-2p^2 + 4p^4 + 2p^5)$$
$$+ \kappa^3(-p^3 + 2p^5 + p^6). \qquad (21)$$

Equations (20) and (21) become, to the leading order in p with $p \ll 1$,

$$\mathcal{F}^{(d=18)}(p) \overset{p \ll 1}{\approx} 1 - 4p^2, \quad \mathcal{F}_{\text{asymmetric}}^{(d=18)}(p) \overset{p \ll 1}{\approx} 1 - 2(1 + \kappa^2) p^2.$$

It results that the presence of asymmetric Weyl errors with $\kappa < 1$ increases the performance of the correction scheme. This can be understood by noting that as soon as $\kappa \to 0$ the noise model becomes classical-like and the errors become of a single type, namely, the Z type, hence more easy to correct (this limiting case is similar to that investigated in Ref. [6]). On the contrary, for $\kappa > 1$ the performance of the code is lowered by error asymmetries.

B. The $d = 50$ qudit code

The encoding operation in this case is characterized by

$$\mathcal{H}_2 \ni |0\rangle \mapsto |0_L\rangle \in \mathcal{H}_{50} \quad \text{and} \quad \mathcal{H}_2 \ni |1\rangle \mapsto |1_L\rangle \in \mathcal{H}_{50},$$

where the code words $|0_L\rangle$ and $|1_L\rangle$ are defined according to (5) and (6) as

$$|0_L\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{5}}[|0\rangle + |10\rangle + |20\rangle + |30\rangle + |40\rangle] \quad \text{and}$$
$$|1_L\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{5}}[|5\rangle + |15\rangle + |25\rangle + |35\rangle + |45\rangle],$$

respectively. The stabilizer group $\mathcal{S}$ of this code is generated by the two error operators X^{10} and Z^{10}. Here $r_1 = r_2 = 5$.

Indeed, it can be shown that provided we restrict our focus to the two-dimensional code space $C^{(d=50)} \subset H_{50}$, the set of all (50^2) non-normalized errors can be reduced to 100 operators. Following the same line of reasoning of the previous subsection, it is possible to find the recovery superoperator. After some algebraic calculations, we arrive at the following expression for the entanglement fidelity:

$$\mathcal{F}^{(d=50)}(p) = 1 - 4p^3 - 4p^4 - 2p^5 + 4p^6 + 8p^7 + 8p^8 + 4p^9 + p^{10}. \quad (22)$$

We emphasize that this error correction scheme is effective as long as the failure probability $1 - \mathcal{F}^{(d=50)}(p)$ is strictly smaller than the error probability p. This implies that the $d = 50$–dimensional qudit code is effective only when $0 \leqslant p \lesssim 0.43$. This p range of effectiveness is larger than that of the $d = 18$–dimensional qudit code. Furthermore, comparing the p expansions of (20) and (22) to the leading orders for $p \ll 1$, it follows that

$$\mathcal{F}^{(d=18)}(p) \overset{p\ll 1}{\approx} 1 - 4p^2 \leqslant 1 - 4p^3 \overset{p\ll 1}{\approx} \mathcal{F}^{(d=50)}(p).$$

From the above equation, it follows that the $d = 50$–dimensional qudit code outperforms the $d = 18$–dimensional qudit code in the p range where both error correction schemes are effective as expected. Moreover, $\mathcal{F}^{(d=50)}(p)$ in (22) is a monotonic decreasing function of the error probability parameter belonging in its range of effectiveness.

1. *Asymmetric errors*

By taking probabilities as in Eqs. (16) and (17) we obtain, for the $d = 50$ qudit code,

$$\begin{aligned}\mathcal{F}^{(d=50)}_{\text{asymmetric}}(p) = {} & (1 - 2p^3 - 2p^4 - p^5) \\ & + \kappa^3(-2p^3 + 4p^6 + 4p^7 + 2p^8) \\ & + \kappa^4(-2p^4 + 4p^7 + 4p^8 + 2p^9) \\ & + \kappa^5(-p^5 + 2p^8 + 2p^9 + p^{10}). \quad (23)\end{aligned}$$

Equations (22) and (23) become, to the leading order in p with $p \ll 1$,

$$\mathcal{F}^{(d=50)}(p) \overset{p\ll 1}{\approx} 1 - 4p^3, \quad \mathcal{F}^{(d=50)}_{\text{asymmetric}}(p) \overset{p\ll 1}{\approx} 1 - 2(1+\kappa^3)p^3.$$

Also in this case, for $\kappa < 1$ the presence of asymmetric Weyl errors increases the performance of the correction scheme, while for $\kappa > 1$ the performance of the code is lowered.

Furthermore, comparing (21) and (23) we get, to the leading order in p with $p \ll 1$,

$$\mathcal{F}^{(d=18)}_{\text{asymmetric}}(p) \overset{p\ll 1}{\approx} 1 - 2(1 + \kappa^2)p^2,$$

$$\mathcal{F}^{(d=50)}_{\text{asymmetric}}(p) \overset{p\ll 1}{\approx} 1 - 2(1 + \kappa^3)p^3.$$

That is, the $d = 50$ qudit code outperforms the $d = 18$ one for any $\kappa \in [0,1/p]$.

V. COMPARISON WITH BLOCK (STABILIZER) CODES

In this section, for the sake of comparison, we quantify the performance of the standard five-qubit stabilizer code [[5, 1, 3]] [8,9] and the seven-qubit stabilizer CSS code [[7,1,3]] [10,11] on the tensor product of Weyl channels (12) by means of the entanglement fidelity [12,24].

A. The five-qubit stabilizer code

1. *Encoding*

The [[5,1,3]] code is the smallest single-error correcting quantum code. Of all quantum codes that encode one qubit and correct all single-qubit errors, the [[5,1,3]] is the most efficient, saturating the quantum Hamming bound. It encodes one qubit in five qubits. The cardinality of its stabilizer group $\mathcal{S}$ is $|\mathcal{S}| = 2^{5-1} = 16$ and the $5 - 1 = 4$ group generators are given by [13]

$$\{X_1Z_2Z_3X_4, X_2Z_3Z_4X_5, X_1X_3Z_4Z_5, Z_1X_2X_4Z_5\}.$$

The distance of the code is $d_{\mathcal{C}} = 3$ and therefore the weight of the smallest error $A_l^\dagger A_k$ that cannot be detected by the code is 3. Finally, we recall that it is a nondegenerate code since the smallest weight for elements of $\mathcal{S}$ (other than identity) is 4 and therefore it is greater than the distance $d = 3$. The encoding operation for the [[5,1,3]] code is characterized by

$$\mathcal{H}_2 \ni |0\rangle \mapsto |0_L\rangle \in \mathcal{H}_2^{\otimes 5} \quad \text{and} \quad \mathcal{H}_2 \ni |1\rangle \mapsto |1_L\rangle \in \mathcal{H}_2^{\otimes 5},$$

where the code words $|0_L\rangle$ and $|1_L\rangle$ are defined as [13]

$$|0\rangle \to |0_L\rangle \overset{\text{def}}{=} \frac{1}{4}\begin{bmatrix} |00000\rangle + |11000\rangle + |01100\rangle + |00110\rangle + |00011\rangle + |10001\rangle - |01010\rangle - |00101\rangle + \\ -|10010\rangle - |01001\rangle - |10100\rangle - |11110\rangle - |01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle \end{bmatrix}$$

and

$$|1\rangle \to |1_L\rangle \overset{\text{def}}{=} \frac{1}{4}\begin{bmatrix} |11111\rangle + |00111\rangle + |10011\rangle + |11001\rangle + |11100\rangle + |01110\rangle - |10101\rangle - |11010\rangle + \\ -|01101\rangle - |10110\rangle - |01011\rangle - |00001\rangle - |10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle \end{bmatrix},$$

respectively.

Then, the action of channel $\Lambda^{(2)\otimes 5}$ on $\rho \in \mathfrak{S}(\mathcal{H}_2^{\otimes 5})$ can be written as

$$\Lambda^{(2)\otimes 5}(\rho) = \sum_{k=0}^{2^{10}-1} A_k \rho A_k^\dagger, \tag{24}$$

where, according to the notation in Sec. II, $A_k \propto X(a_1a_2a_3a_4a_5)Z(b_1b_2b_3b_4b_5)$.

2. Correctability

The 16 weight 0 and one quantum error operators in (24) are given by

$$A_0 = \sqrt{\tilde{p}_0} I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5,$$
$$A_1 = \sqrt{\tilde{p}_1} X_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5, \ldots,$$
$$A_{15} = \sqrt{\tilde{p}_{15}} I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Z_5,$$

where the coefficients $\tilde{p}_l$ with $l = 0,\ldots,15$ can be easily deduced from the above distribution of errors and Eq. (13), $\tilde{p}_0 = (1-p)^{10}$, $\tilde{p}_1 = p(1-p)^9, \ldots,$ $\tilde{p}_6 = p^2(1-p)^8, \ldots,$ $\tilde{p}_{15} = p(1-p)^9$.

It is straightforward, though tedious, to check that, for the above given errors of weight 0 and 1, we have

$$S(A_l^\dagger A_k) \neq 0, \quad \text{with} \quad l,k \in \{0,1,\ldots,15\},$$

where $S(A_l^\dagger A_k)$ is the error syndrome of the error operator $A_l^\dagger A_k$ defined, according to (2), as $S(A_l^\dagger A_k) \overset{\text{def}}{=} H^{[[5,1,3]]} v_{A_l^\dagger A_k}$. The quantity $H^{[[5,1,3]]}$ is the parity check matrix for the five-qubit code, while $v_{A_l^\dagger A_k}$ is the vector in the 10-dimensional binary vector space F_2^{10} corresponding to the error operator $A_l^\dagger A_k$.

Hence, the set of correctable error operators is given by

$$\mathcal{A}_{\text{correctable}} = \{A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}\} \subseteq \mathcal{A},$$

where the cardinality of $\mathcal{A}$ defining the channel in (24) equals 2^{10}.

3. Recovery operators

All weight 0 and 1 error operators satisfy the error correction conditions [15]

$$P_{\mathcal{C}} A_l^\dagger A_k P_{\mathcal{C}} \propto P_{\mathcal{C}}, \quad l,k \in \{0,1,\ldots,15\},$$

where $P_{\mathcal{C}}$ is the orthogonal projector operator ($P_{\mathcal{C}} = P_{\mathcal{C}}^2$ and $P_{\mathcal{C}} = P_{\mathcal{C}}^\dagger$) on the code space $\mathcal{C}^{([[5,1,3]])}$ defined as

$$P_{\mathcal{C}} \overset{\text{def}}{=} |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|.$$

The two 16-dimensional orthogonal subspaces $\mathcal{V}^{0_L}$ and $\mathcal{V}^{1_L}$ of $\mathcal{H}_2^{\otimes 5}$ generated by the action of $\mathcal{A}_{\text{correctable}}^{[[5,1,3]]}$ on $|0_L\rangle$ and $|1_L\rangle$ are given by

$$\mathcal{V}^{0_L} = \text{Span}\left\{ \left|v_k^{0_L}\right\rangle = \frac{A_k}{\sqrt{\tilde{p}_k}} |0_L\rangle \right\} \quad \text{and}$$
$$\mathcal{V}^{1_L} = \text{Span}\left\{ \left|v_k^{1_L}\right\rangle = \frac{A_k}{\sqrt{\tilde{p}_k}} |1_L\rangle \right\},$$

with $k = 0, 1, \ldots, 15$. Note that $\langle v_l^{i_L} | v_{l'}^{j_L} \rangle = \delta_{ll'} \delta_{ij}$ with $l, l' \in \{0,1,\ldots,15\}$ and $i,j \in \{0,1\}$. Therefore, it follows that $\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L} = \mathcal{H}_2^{\otimes 5}$. The recovery superoperator $\mathcal{R} \leftrightarrow \{R_l\}$ with $l = 1,\ldots,16$ is defined by means of [25]

$$R_l = |0_L\rangle\langle v_l^{0_L}| + |1_L\rangle\langle v_l^{1_L}|.$$

Finally, the composition of this recovery operation $\mathcal{R}$ with the map $\Lambda^{(2)\otimes 5}(\rho)$ in (24) yields

$$\mathcal{R} \circ \Lambda^{(2)\otimes 5}(\rho) = \sum_{k=0}^{2^{10}-1} \sum_{l=1}^{16} (R_l A_k) \rho (R_l A_k)^\dagger. \tag{25}$$

4. Entanglement fidelity

Here we want to describe the action of $\mathcal{R} \circ \Lambda^{(2)\otimes 5}$ in (25) restricted to the code subspace $\mathcal{C}^{[[5,1,3]]}$. Note that the recovery operators can be expressed as

$$R_{l+1} = R_1 \frac{A_l}{\sqrt{\tilde{p}_l}} = (|0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|) \frac{A_l}{\sqrt{\tilde{p}_l}},$$

with $l \in \{0,\ldots,15\}$. Recalling that in this case $A_l = A_l^\dagger$, it turns out that

$$\langle i_L | R_{l+1} A_k | j_L \rangle = \frac{1}{\sqrt{\tilde{p}_l}} \langle i_L | 0_L \rangle \langle 0_L | A_l^\dagger A_k | j_L \rangle + \frac{1}{\sqrt{\tilde{p}_l}} \langle i_L | 1_L \rangle \langle 1_L | A_l^\dagger A_k | j_L \rangle.$$

We now need to compute the 2×2 matrix representation $[R_l A_k]_{|\mathcal{C}}$ of each $R_l A_k$ with $l = 0,\ldots,15$ and $k = 0,\ldots,2^{10}-1$, where

$$[R_{l+1} A_k]_{|\mathcal{C}} \overset{\text{def}}{=} \begin{pmatrix} \langle 0_L | R_{l+1} A_k | 0_L \rangle & \langle 0_L | R_{l+1} A_k | 1_L \rangle \\ \langle 1_L | R_{l+1} A_k | 0_L \rangle & \langle 1_L | R_{l+1} A_k | 1_L \rangle \end{pmatrix}.$$

For l, $k = 0,\ldots,15$, we note that $[R_{l+1} A_k]_{|\mathcal{C}}$ becomes

$$[R_{l+1} A_k]_{|\mathcal{C}} = \begin{pmatrix} \langle 0_L | A_l^\dagger A_k | 0_L \rangle & 0 \\ 0 & \langle 1_L | A_l^\dagger A_k | 1_L \rangle \end{pmatrix} = \sqrt{\tilde{p}_l} \delta_{lk} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

while for any pair (l,k) with $l, = 0,\ldots,15$ and $k > 15$, it follows that

$$\langle 0_L | R_{l+1} A_k | 0_L \rangle + \langle 1_L | R_{l+1} A_k | 1_L \rangle = 0.$$

We conclude that the only matrices $[R_l A_k]_{|\mathcal{C}}$ with nonvanishing trace are given by

$$[R_s A_{s-1}]_{|\mathcal{C}} = \sqrt{\tilde{p}_{s-1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with $s = 1,\ldots,16$. Therefore, the entanglement fidelity, (18), can be written as

$$\mathcal{F}^{[[5,1,3]]}(p) \overset{\text{def}}{=} \mathcal{F}^{[[5,1,3]]}\left(\frac{1}{2} I_{2\times 2}, \mathcal{R} \circ \Lambda^{(2)\otimes 5}\right) = \frac{1}{(2)^2} \sum_{k=0}^{2^{10}-1} \sum_{l=1}^{16} |\text{tr}([R_l A_k]_{|\mathcal{C}})|^2$$

and results in

$$\mathcal{F}^{[[5,1,3]]}(p) = 1 - 40p^2 + 200p^3 - 490p^4 + 728p^5 - 700p^6 + 440p^7 + 175p^8 + 40p^9 - 4p^{10}. \quad (26)$$

We remark that this error correction scheme is effective as long as the failure probability $1 - \mathcal{F}^{[[5,1,3]]}(p)$ is strictly smaller than the error probability p. This implies that the five-qubit code is effective only when $0 \leqslant p \lesssim 2.9 \times 10^{-2}$. Finally, we emphasize that this block-encoding scheme is less efficient than the previously mentioned qudit-encoding schemes as shown in Fig. 1.

Asymmetric errors. By taking probabilities as in Eqs. (16) and (17), we obtain for the five-qubit code

$$\begin{aligned}\mathcal{F}^{[[5,1,3]]}_{\text{asymmetric}}(p) = {} & 1 - 10p^2 + 20p^3 - 15p^4 + 4p^5 + \kappa(-20p^2 + 80p^3 - 120p^4 + 80p^5 - 20p^6) \\ & + \kappa^2(-10p^2 + 80p^3 - 220p^4 + 280p^5 - 170p^6 + 40p^7) \\ & + \kappa^3(20p^3 - 120p^4 + 280p^5 - 320p^6 + 180p^7 - 40p^8) \\ & + \kappa^4(-15p^4 + 80p^5 - 170p^6 + 180p^7 - 95p^8 + 20p^9) + \kappa^5(4p^5 - 20p^6 + 40p^7 - 40p^8 + 20p^9 - 4p^{10}).\end{aligned} \quad (27)$$

We stress that, unlike the finding uncovered in [17], asymmetries in the considered Weyl noisy channel do affect the performance of the five-qubit code quantified in terms of the entanglement fidelity. This difference is ultimately a consequence of the fact that while in [17] it is assumed that error probabilities p_X, p_Y, and p_Z are all *linear* in the error probability p (although weighted with different coefficients α_X, α_Y, and α_Z with $\alpha_X + \alpha_Y + \alpha_Z = 1$), here we assume that p_Y is *quadratic*in p while keeping both p_X and p_Z *linear* in p.

From Eqs. (26) and (27) we obtain, to the leading order in p with $p \ll 1$,

$$\mathcal{F}^{[[5,1,3]]}(p) \overset{p\ll 1}{\approx} 1 - 40p^2, \quad \mathcal{F}^{[[5,1,3]]}_{\text{asymmetric}}(p) \overset{p\ll 1}{\approx} 1 - 10(1+\kappa)^2 p^2.$$

Hence, also in this case, the presence of asymmetric errors increases the performance of the correction scheme for $\kappa < 1$, while for $\kappa > 1$ the performance of the code is lowered.

B. The seven-qubit stabilizer code

The [[7,1,3]] CSS code encodes one qubit in seven qubits. The cardinality of its stabilizer group $\mathcal{S}$ is $|\mathcal{S}| = 2^{7-1} = 64$ and the set of $7 - 1 = 6$ group generators is given by [13]

$$\{X_4X_5X_6X_7, X_2X_3X_6X_7, X_1X_3X_5X_7, Z_4Z_5Z_6Z_7, Z_2Z_3Z_6Z_7, Z_1Z_3Z_5Z_7\}.$$

The distance of the code is $d_{\mathcal{C}} = 3$ and therefore the weight of the smallest error $A_l^\dagger A_k$ that cannot be detected by the code is 3. Finally, we recall that it is a nondegenerate code since the smallest weight for elements of $\mathcal{S}$ (other than identity) is 4 and therefore it is greater than the distance $d_{\mathcal{C}} = 3$. The encoding for the [[7,1,3]] code is given by [13]

$$|0\rangle \to |0_L\rangle = \frac{1}{(\sqrt{2})^3}\begin{bmatrix} |0000000\rangle + |0110011\rangle + |1010101\rangle + |1100110\rangle \\ +|0001111\rangle + |0111100\rangle + |1011010\rangle + |1101001\rangle \end{bmatrix}$$

and

$$|1\rangle \to |1_L\rangle = \frac{1}{(\sqrt{2})^3}\begin{bmatrix} |1111111\rangle + |1001100\rangle + |0101010\rangle + |0011001\rangle \\ +|1110000\rangle + |1000011\rangle + |0100101\rangle + |0010110\rangle \end{bmatrix}.$$

Following the same line of reasoning as in the previous subsection we can compute the correctable errors by means of $H^{[[7,1,3]]}$, the parity check matrix for the seven-qubit code. Finally, after determining the recovery operators, it can be shown that the entanglement fidelity reads

$$\begin{aligned}\mathcal{F}^{[[7,1,3]]}(p) = {} & 1 - 42p^2 + 140p^3 + 231p^4 - 2772p^5 + 9240p^6 - 18\,216p^7 + 24\,255p^8 - 22\,792p^9 \\ & + 15\,246p^{10} - 7140p^{11} + 2233p^{12} - 420p^{13} + 36p^{14}.\end{aligned} \quad (28)$$

Observe that the seven-qubit code is effective for $0 \leqslant p \lesssim 2.6 \times 10^{-2}$. Comparing the p expansions of (26) and (28) to the leading orders for $p \ll 1$, it follows that

$$\mathcal{F}^{[[5,1,3]]}(p) \overset{p\ll 1}{\approx} 1 - 40p^2 \geqslant 1 - 42p^2 \overset{p\ll 1}{\approx} \mathcal{F}^{[[7,1,3]]}(p),$$

and in addition, the p range of applicability of the five-qubit code is larger than that of the seven-qubit code. Thus, for the symmetric Weyl channel considered, the five-qubit code outperforms the seven-qubit code. However, we shall see that this ordering does not hold when considering asymmetric scenarios.

1. Asymmetric errors

In the asymmetric scenario of Eqs. (16) and (17) we obtain for the seven-qubit code

$$\begin{aligned}\mathcal{F}^{[[7,1,3]]}_{\text{asymmetric}}(p) = {} & 1 - 21p^2 + 70p^3 - 105p^4 + 84p^5 - 35p^6 + 6p^7 \\ & + \kappa(-21p^2 + 126p^3 - 315p^4 + 420p^5 - 315p^6 + 126p^7 - 21p^8) \\ & + \kappa^2(-21p^3 + 126p^4 - 315p^5 + 420p^6 - 315p^7 + 126p^8 - 21p^9) \\ & + \kappa^3(-35p^3 + 420p^4 - 1785p^5 + 3850p^6 - 4725p^7 + 3360p^8 - 1295p^9 + 210p^{10}) \\ & + \kappa^4(105p^4 - 1050p^5 + 4095p^6 - 8400p^7 + 9975p^8 - 6930p^9 + 2625p^{10} - 420p^{11}) \\ & + \kappa^5(-126p^5 + 1155p^6 - 4284p^7 + 8505p^8 - 9870p^9 + 6741p^{10} - 2520p^{11} + 399p^{12}) \\ & + \kappa^6(70p^6 - 609p^7 + 2184p^8 - 4235p^9 + 4830p^{10} - 3255p^{11} + 1204p^{12} - 189p^{13}) \\ & + \kappa^7(-15p^7 + 126p^8 - 441p^9 + 840p^{10} - 945p^{11} + 630p^{12} - 231p^{13} + 36p^{14}). \end{aligned} \tag{29}$$

From Eqs. (28) and (29) it follows, to the leading order in p with $p \ll 1$, that

$$\mathcal{F}^{[[7,1,3]]}(p) \overset{p\ll 1}{\approx} 1 - 42p^2,$$
$$\mathcal{F}^{[[7,1,3]]}_{\text{asymmetric}}(p) \overset{p\ll 1}{\approx} 1 - 21(1+\kappa)p^2.$$

Once again, for $\kappa < 1$ it results that the presence of asymmetric errors increases the performance of the correction scheme, while for $\kappa > 1$ the performance of the code is lowered.

Furthermore, by comparing (27) and (29) it follows, to the leading order in p with $p \ll 1$, that

$$\begin{aligned}\mathcal{F}^{[[5,1,3]]}_{\text{asymmetric}}(p) & \overset{p\ll 1}{\approx} 1 - 10(1+\kappa)^2 p^2 \\ & \quad > 1 - 21(1+\kappa)p^2 \overset{p\ll 1}{\approx} \mathcal{F}^{[[7,1,3]]}_{\text{asymmetric}}(p), \\ \kappa & < 1.1; \\ \mathcal{F}^{[[5,1,3]]}_{\text{asymmetric}}(p) & \overset{p\ll 1}{\approx} 1 - 10(1+\kappa)^2 p^2 \\ & \quad < 1 - 21(1+\kappa)p^2 \overset{p\ll 1}{\approx} \mathcal{F}^{[[7,1,3]]}_{\text{asymmetric}}(p), \\ \kappa & > 1.1.\end{aligned}$$

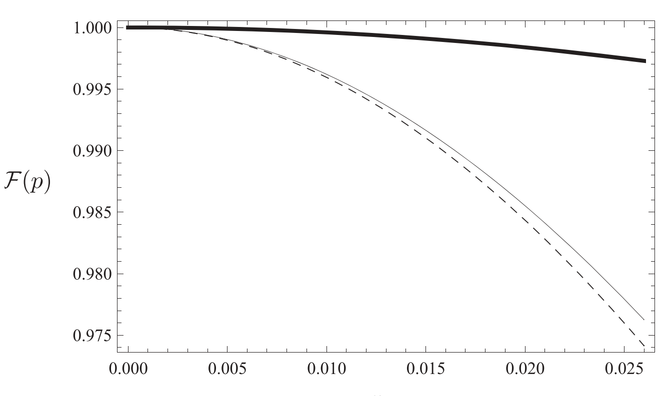

FIG. 1. The quantities $\mathcal{F}^{(18)}$ (thick solid line), $\mathcal{F}^{[[5,1,3]]}$ (thin solid line), and $\mathcal{F}^{[[7,1,3]]}$ (dashed line) are plotted vs the error probability p with $0 \leqslant p \lesssim 2.6 \times 10^{-2}$. Within the scale resolution of the graph the curve for $\mathcal{F}^{(50)}$ appears to coincide with the top horizontal axis.

Thus, with respect to the noise model discussed in Ref. [17], we conclude that here the comparison between the five-qubit code and the seven-qubit code is slightly more involved.

VI. FINAL REMARKS

In this article, we discussed how to protect a qubit embedded in a qudit from both amplitude and phase errors occurring in the discrete phase space. A code has been devised using stabilizer formalism and its performances compared with those of common block codes for a general Weyl noisy quantum channel allowing symmetric and asymmetric error probabilities.

Specifically we have considered the $d = 18$ and $d = 50$ qudit stabilizer codes together with the five-qubit and the CSS seven-qubit quantum stabilizer codes. The performances of these codes were quantified by means of the entanglement fidelity as a function of the error probability.

We have revealed that qudit codes have an enormously wider (by approximately an order of magnitude) range of applicability in the error probability. Furthermore, already the $d = 18$ qudit code outperform- and seven-qubit block codes for symmetric errors (see Fig. 1). Our theoretical analysis leads to the conclusion that the qudit codes with $d = 18$ and $d = 50$ outperform the common five- and CSS seven-qubit stabilizer codes. This, in principle, allows one to save space resources (since $d = 18 < \dim_{\mathbb{C}} H_2^{\otimes 5} = 32 \ll \dim_{\mathbb{C}} H_2^{\otimes 7} = 128$), however, one should also account for the difficulties in implementing qudit systems, an issue that seems to be nontrivial and not quite settled yet. For an overview of the experiments performed for producing quantum optical qudits, we refer the reader to [26].

The performance of the qudit code is also robust against asymmetries in error probabilities. In fact, restricting our analysis to $\kappa > 1.1$, it results that the $d = 18$ qudit code outperforms the seven-qubit code until strong asymmetries come into play, as can be seen by comparing (21) with (29) to the leading order in p with $p \ll 1$:

$$\begin{aligned}\mathcal{F}^{[[7,1,3]]}_{\text{asymmetric}}(p) & \overset{p\ll 1}{\approx} 1 - 21(1+\kappa)p^2 < 1 - 2(1+\kappa^2)p^2 \\ & \overset{p\ll 1}{\approx} \mathcal{F}^{(d=18)}_{\text{asymmetric}}(p), \quad \kappa < \frac{21+\sqrt{593}}{4} \approx 11.34;\end{aligned}$$

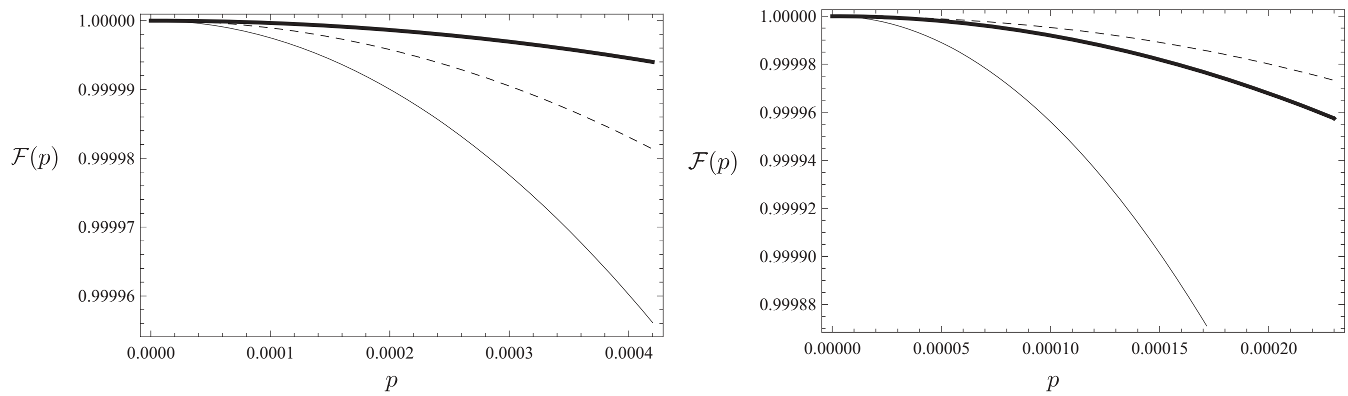

FIG. 2. The quantities $\mathcal{F}_{\text{asymmetric}}^{(18)}$ (thick solid line), $\mathcal{F}_{\text{asymmetric}}^{[[5,1,3]]}$ (thin solid line), and $\mathcal{F}_{\text{asymmetric}}^{[[7,1,3]]}$ (dashed line) are plotted vs the error probability p for $\kappa = 4 < 11.34$ (left) and for $\kappa = 20 > 11.34$ (right). Within the scale resolution of the graphs the curves for $\mathcal{F}_{\text{asymmetric}}^{(50)}$ appear to coincide with the top horizontal axes.

$$\mathcal{F}_{\text{asymmetric}}^{[[7,1,3]]}(p) \overset{p\ll 1}{\approx} 1 - 21(1+\kappa)p^2 > 1 - 2(1+\kappa^2)p^2$$
$$\overset{p\ll 1}{\approx} \mathcal{F}_{\text{asymmetric}}^{(d=18)}(p), \quad \kappa > \frac{21+\sqrt{593}}{4} \approx 11.34.$$

Comparative results for the various codes' performances in the presence of asymmetries are graphically represented in Fig. 2. The different uncovered behaviors in the four error correcting schemes employed in this article can be ascribed to the fact that the errors in $\mathcal{P}_{\mathcal{H}_2^{\otimes 5}}$ (or $\mathcal{P}_{\mathcal{H}_2^{\otimes 7}}$) are fundamentally different from those in $\mathcal{P}_{\mathcal{H}_{18}}$ and $\mathcal{P}_{\mathcal{H}_{50}}$.

Finally, it would be interesting to consider the presence of correlations between X and Z errors in the qudit code. These can be introduced as follows:

$$\pi(n,m) \overset{\text{def}}{=} (1-\mu)\pi(n)\pi(m) + \mu\delta_{n,m}\pi(m),$$

where μ with $0 \leqslant \mu \leqslant 1$ represents the degree of correlation. Following the very same line of reasoning provided in Sec. IV, it can be shown that, for instance, the entanglement fidelity becomes

$$\mathcal{F}_{\text{corr}}^{(d=18)}(p) = (1 - 4p^2 - 2p^3 + 4p^4 + 4p^5 + p^6)$$
$$+ \mu(2p^2 + p^3 - 4p^4 - 4p^5 - p^6).$$

It then results that $(\partial \mathcal{F}_{\text{corr}}^{(d=18)}/\partial\mu)_{p=\text{const.}} \overset{p\ll 1}{\approx} 2p^2 \geqslant 0$; that is, memory effects lead to better performances. The reason is that, in the limit of very strong correlations $\mu \to 1$, only one type of error (namely, $Y = XZ$) takes place. As such, this case shows similarities to the case of asymmetric errors with $\kappa \to 0$.

In conclusion, we are strongly motivated by our investigation to believe that encoding a qubit into a qudit can be a useful approach in quantum coding. We are aware of the difficulties in realizing and controlling qudit systems even of low dimensionality, however, we have witnessed a lot of progress in this direction recently. Quantum optical qudits can be generated by means of experimental schemes based on interferometric setups, orbital angular momentum entanglement, and biphoton polarization [26]. For instance, in the interferometric scheme employed in [27], high symmetry and maximally entangled qutrits are realized with a fidelity of up to 0.985 as the superposition state of the three possible paths of a single photon in a three-arm interferometer. Therefore, the realization of the discussed qudit codes seems possible in the future.

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[1] D. Petz, *Quantum Information Theory and Quantum Statistics* (Springer-Verlag, Berlin, 2008).
[2] D. Gottesman, in *Quantum Information Science and Its Contributions to Mathematics, Proceedings of Symposia in Applied Mathematics 68* (American Mathematical Society, Providence, RI, 2010), pp. 13–58.
[3] D. Gottesman, Ph.D. thesis, California Institute of Technology, Pasadena, 1997.
[4] A. Ketkar *et al.*, IEEE Trans. Info. Theory **52**, 4892 (2006).
[5] D. Gottesman *et al.*, Phys. Rev. A **64**, 012310 (2001).
[6] S. Pirandola, S. Mancini, S. L. Braunstein, and D. Vitali, Phys. Rev. A **77**, 032309 (2008).
[7] G. G. Amosov, J. Math. Phys. **48**, 2104 (2007).
[8] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, Phys. Rev. Lett. **77**, 198 (1996).
[9] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[10] A. M. Steane, Proc. R. Soc. London A **452**, 2551 (1996).
[11] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).
[12] B. Schumacher, Phys. Rev. A **54**, 2614 (1996).
[13] F. Gaitan, *Quantum Error Correction and Fault Tolerant Quantum Computing* (CRC Press, Boca Raton, FL, 2008).
[14] R. A. Calderbank *et al.*, IEEE Trans. Info. Theory **44**, 1369 (1998).

[15] E. Knill *et al.*, Los Alamos Science **27**, 188 (2002).
[16] D. Gottesman, Chaos Solitons Fractals **10**, 1749 (1999).
[17] C. Cafaro and S. Mancini, Phys. Rev. A **82**, 012306 (2010).
[18] L. Ioffe and M. Mezard, Phys. Rev. A **75**, 032345 (2007).
[19] Z. W. E. Evans *et al.*, arXiv:0709.3875 [quant-ph] (2007).
[20] A. M. Stephens, Z. W. E. Evans, S. J. Devitt, and L. C. L. Hollenberg, Phys. Rev. A **77**, 062335 (2008).
[21] P. K. Sarvepalli *et al.*, in *IEEE International Symposium on Information Theory, 2008* (2008), pp. 305–309.
[22] S. A. Aly, arXiv:0803.0764 [quant-ph] (2008).
[23] C. Cafaro *et al.*, Open Syst. Info. Dyn. **18**, 1 (2011).
[24] M. A. Nielsen, arXiv:quant-ph/9606012 (1996).
[25] E. Knill and R. Laflamme, Phys. Rev. A **55**, 900 (1997).
[26] M. Genovese and P. Traina, Adv. Sci. Lett. **1**, 153 (2008).
[27] R. Thew *et al.*, Quantum Info. Comput. **4**, 93 (2004).