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Equivalent Hermitian Hamiltonians for some non-Hermitian Hamiltonians

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Six years ago, by using operator techniques and path-integral methods, it was shown that the complex non-Hermitian PT-symmetric Hamiltonian $p^2 - gx^4$ is equivalent to a conventional Hermitian Hamiltonian $p^2 + 4gx^4 - 2\hbar\sqrt{g}x$. Further it was revealed that the linear term in the Hermitian Hamiltonian is anomalous in the sense that it has no classical analog. In this paper we show that the complex non-Hermitian PT-symmetric Hamiltonian $p^2 - gx^4 + 4i\hbar\sqrt{g}x$ and the conventional Hermitian Hamiltonian $p^2 + 4gx^4 + 6\hbar\sqrt{g}x$ have the same eigenspectra. In this case, the anomalous terms in both Hamiltonians are different from the previous one and vanish in the semiclassical limit. Further these equivalent Hamiltonians have zero-energy ground states. The exact ground-state wave functions and supersymmetric partner potentials are derived.

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I. INTRODUCTION

This work is motivated by the papers of Jones *et al.* published in 2006 [1–3]. During the last decade there has been an increased interest in non-Hermitian PT-symmetric Hamiltonian systems due to possible applications of non-Hermitian models in particle-physics [4], quantum optics [5], supersymmetric [6], and magnetohydrodynamical [7] models. A Hamiltonian is PT symmetric if it is invariant under space-time reflection: for P, $p \to -p$, $x \to -x$ and for T, $p \to -p$, $x \to x$ and $i \to -i$. More information on PT-symmetric non-Hermitian systems can be found in Refs. [8–11]. When the PT symmetry is not spontaneously broken, non-Hermitian PT-symmetric Hamiltonians produce real and positive energy spectra. However, for a given PT-symmetric Hamiltonian, there is no simple way of finding out whether the PT symmetry is spontaneously broken or not. It was shown by Mostafazadeh [12] that a quantum system possessing an exact (unbroken) PT symmetry is equivalent to a quantum system having a Hermitian Hamiltonian. This was achieved by constructing the unitary operator relating an arbitrary non-Hermitian Hamiltonian with exact PT symmetry to a Hermitian Hamiltonian. Nevertheless, only in a few instances, it was possible to find Hermitian Hamiltonians which possess the same eigenspectra as PT-symmetric non-Hermitian Hamiltonians.

Six years ago, by using operator techniques and path-integral methods, Jones *et al.* [1–3] proved that the eigenspectra of both the complex non-Hermitian PT-symmetric Hamiltonian $H = p^2 - gx^4$ and the conventional Hermitian Hamiltonian $h = p^2 + 4gx^4 - 2\sqrt{g}x$ are the same. Further it was revealed that the linear term in the Hermitian Hamiltonian is anomalous in the sense that it has no classical analog and the linear term has the form $2\hbar\sqrt{g}x$ [2,3]. Further they have shown that the PT-symmetric double-well $H = p^2 + \frac{\mu^2}{2}x^2 - gx^4$ and the Hermitian Hamiltonian $h = p^2 - 2\hbar\sqrt{g}x + 4g(x^2 - \frac{\mu^2}{8g})^2$ have the same spectra. Here we have taken mass $2m = 1$. However, using Rayleigh-Schrödinger perturbation theory, Buslaev and Grecchi [13] had shown this equivalence relation several years earlier.

In this paper we use a completely different approach, based on the quantum action variable expansions (asymptotic energy expansions), to find a Hermitian equivalence of the non-Hermitian PT-symmetric Hamiltonian $p^2 - gx^4 + bx$. It was found in the current study that operator techniques or path-integral methods cannot be directly utilized to determine the equivalent Hermitian Hamiltonians in a manner similar to the case of $H = p^2 - gx^4$. We use the expression for quantum action variables to determine the equivalency instead. Results were verified with eigenvalues determined by direct numerical integration of the Schrödinger equation along appropriate complex contours. The outline of the paper is as follows. In the Sec. II, we describe the quantum action variable method and derive a series expansion of the action variable for $p^2 - gx^4 + bx$ and $p^2 + \alpha x^4 + \beta x$. By imposing the condition that both PT-symmetric non-Hermitian systems and the Hermitian systems should have identical action variable $J(E)$ expansions, we determine g,b,α, and β. The Hermitian Hamiltonians equivalent to the non-Hermitian PT-symmetric Hamiltonians are found in Sec. III. Exact ground-state wave functions and supersymmetric partners of both Hermitian and non-Hermitian Hamiltonians are derived in Sec. IV. Concluding remarks are given in Sec. V.

II. QUANTUM ACTION VARIABLE EXPANSIONS

In quantum action variable theory [14–16], it was shown that quantum eigenvalues can be obtained by quantizing the quantum action variable (QAV) $J(E)$ [14,15,17]. The exact quantization condition is $J(E) = n\hbar$, where n is a non-negative integer. Details of the quantum action variable theory can be found in Refs. [14–17]. In this section first we will derive quantum action variable expansions for the quartic Hermitian Hamiltonian

$$h(x,p) = p^2 + V(x), \tag{1}$$

where $V(x) = \alpha x^4 + \beta x$. The quantization condition for this potential is

$$J(E) = n\hbar, \tag{2}$$

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where n is a positive integer and QAV $J(E)$ is given by

$$J(E) = \frac{1}{2\pi}\int_{\gamma} P(x,E)dx. \tag{3}$$

$P(x,E)$ satisfies the equation

$$\frac{\hbar}{i}\frac{\partial P(x,E)}{\partial x} + P^2(x,E) = E - V(x) = P_c(x,E), \tag{4}$$

and it is related to the wave function as $P(x,E) = \frac{\hbar}{i}\frac{\partial\Psi/\partial x}{\Psi}$. The contour γ in (3) encloses two physical turning points of $P_c(x,E)$. The details of this method are described elsewhere [18–21]. This method produces series expansions identical to higher-order WKB expansions [22,23] for potentials such as $V(x) = x^{2N}$. However, two methods produce completely different series for potentials such as $V(x) = x^{2N} + bx$ [18]. In order to obtain a series expansion called the asymptotic energy expansion (AEE) of $J(E)$, first $P(x,E)$ is expanded in a series of powers of energy and subsequently obtains recurrence relations. For the above potential, Eq. (4) becomes

$$\frac{\hbar}{i}\frac{\partial P(x,E)}{\partial x} + P^2(x,E) = E - \alpha x^4 - \beta x. \tag{5}$$

Let $\epsilon = E^{-1/4}$ and $y = \alpha^{1/4}\epsilon x$. Then (5) becomes, after simplification,

$$\hat{h}\epsilon^5\frac{\partial P(y,\epsilon)}{\partial y} + \epsilon^4 P^2(y,\epsilon) = 1 - y^4 - Ay\epsilon^3, \tag{6}$$

where $\hat{h} = \frac{\hbar}{i}\alpha^{1/4}$ and $A = \frac{\beta}{\alpha^{1/4}}$.

Now we expand $P(y,\epsilon)$ as a power series in ϵ,

$$P(y,\epsilon) = \epsilon^s\sum_{k=0}^{\infty} a_k(y)\epsilon^k, \tag{7}$$

where a_k and s are determined below. Substituting (7) in Eq. (6) and equating coefficients of ϵ^0, we obtain $s = -2$ and $a_0 = \sqrt{1-y^4}$, and (6) becomes

$$\hat{h}\sum_{k=0}^{\infty}\epsilon^{k+3}\frac{da_k}{dy} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i a_j\epsilon^{i+j} = 1 - y^4 - Ay\epsilon^3, \tag{8}$$

and assuming $a_k = 0$ when $k < 0$ and rearranging terms, we obtain

$$\hat{h}\sum_{k=1}^{\infty}\frac{da_{k-3}}{dy}\epsilon^k + \sum_{k=1}^{\infty}\sum_{i=1}^{k-1} a_i a_{k-i}\epsilon^k + 2a_0\sum_{k=0}^{\infty} a_k\epsilon^k = 1 - y^4 - Ay\epsilon^3 \tag{9}$$

for $k > 0$. Then coefficients a_k are given by

$$a_k = \frac{-1}{2a_0}\left[\sum_{i=1}^{k-1} a_i a_{k-i} + Ay\delta_{k,3} + \hat{h}\frac{da_{k-3}}{dy}\right]. \tag{10}$$

In the above formula $a_k = 0\;\forall\; k < 0$. Now J can be written as

$$J\,(E) = \sum_{k=0}^{\infty} b_k E^{\frac{-(k-3)}{4}}, \tag{11}$$

where

$$b_k = \frac{1}{2\pi}\int_{\gamma} a_k dy \tag{12}$$

and can be determined analytically in terms of α and β. The contour γ encloses the two branch points of $\sqrt{1-y^4}$ (i.e., $+1$ and -1) on the real axis. The quantization condition $J(E) = n\hbar$ determines the eigenenergies of $V(x) = \alpha x^4 + \beta x$.

For the polynomial potentials, all the integrals $\int_{\gamma} a_k dy$ have the general form of either $\int\frac{x^n}{(1-x^4)^{m+1/2}}dx$ or $\int\frac{x^n}{(1-x^4)^m}dx$, where m is an integer and n is a positive integer [18,20]. However, the second integral, in general, does not contribute to J except when $m = 1$ and $n = 3$.

Using (10) and evaluating integral (12) analytically, we obtain the series expansion. Then the eigenenergy expansion becomes

$$J\,(E) = \sum_{k=0}^{\infty} b_k E^{\frac{-(k-3)}{4}}. \tag{13}$$

Here the first six nonzero b_k's are

$$b_0 = \frac{\Gamma\left[\frac{1}{4}\right]}{3\sqrt{\pi}\alpha^{1/4}\,\Gamma\left[\frac{3}{4}\right]},\quad b_3 = -\frac{\hbar}{2},\quad b_6 = -\frac{(2\hbar^2\alpha - \beta^2)\,\Gamma\left[\frac{3}{4}\right]}{8\sqrt{\pi}\alpha^{3/4}\,\Gamma\left[\frac{1}{4}\right]},\quad b_{12} = \frac{(44\hbar^4\alpha^2 - 60\hbar^2\alpha\beta^2 + 5\beta^4)\,\Gamma\left[\frac{1}{4}\right]}{6144\sqrt{\pi}\alpha^{5/4}\,\Gamma\left[\frac{3}{4}\right]},$$

$$b_{18} = \frac{77(488\hbar^6\alpha^3 - 636\hbar^4\alpha^2\beta^2 + 90\hbar^2\alpha\beta^4 - 3\beta^6)\,\Gamma\left[\frac{3}{4}\right]}{245\,760\sqrt{\pi}\alpha^{7/4}\,\Gamma\left[\frac{1}{4}\right]},$$

$$b_{24} = -\frac{1105(5648\hbar^8\alpha^4 - 6304\hbar^6\alpha^3\beta^2 + 1064\hbar^4\alpha^2\beta^4 - 56\hbar^2\alpha\beta^6 + \beta^8)\,\Gamma\left[\frac{1}{4}\right]}{58\,720\,256\sqrt{\pi}\alpha^{9/4}\,\Gamma\left[\frac{3}{4}\right]}.$$

Next we derive the series expansion of $J(E)$ for the non-Hermitian PT-symmetric "wrong-sign" quartic Hamiltonian:

$$H\,(x,p) = p^2 - gx^4 + bx. \tag{14}$$

The wrong-sign quartic potential is handled in a slightly different manner from the quartic potential. In this case we let $\epsilon = E^{-1/4}$ and $y = g^{1/4}\epsilon x$. Then (4) becomes, after simplification,

$$\hat{h}\epsilon^5\frac{\partial P\,(y,\epsilon)}{\partial y} + \epsilon^4 P^2\,(y,\epsilon) = 1 + y^4 - By\epsilon^3, \tag{15}$$

where $\hat{h}=\frac{\hbar}{i}g^{1/4}$ and $B=\frac{b}{g^{1/4}}$. Next we expand $P(y,\epsilon)$ as a power series in ϵ,

$$P\,(y,\epsilon)=\epsilon^{s}\sum_{k=0}^{\infty}a_k\,(y)\,\epsilon^{k}, \tag{16}$$

where a_k and s are determined below. Substituting (16) in Eq. (15) and equating coefficients of ϵ^0, we obtain $s=-2$ and $a_0=\sqrt{1+y^4}$, and Eq. (15) becomes

$$\hat{h}\sum_{k=0}^{\infty}\epsilon^{k+3}\frac{da_k}{dy}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_ia_j\epsilon^{i+j}=1+y^4-By\epsilon^3, \tag{17}$$

and assuming $a_k=0$ when $k<0$ and rearranging terms, we obtain

$$\hat{h}\sum_{k=1}^{\infty}\frac{da_{k-3}}{dy}\epsilon^k+\sum_{k=1}^{\infty}\sum_{i=1}^{k-1}a_ia_{k-i}\epsilon^k+2a_0\sum_{k=0}^{\infty}a_k\epsilon^k$$
$$=1+y^4-By\epsilon^3 \tag{18}$$

for $k>0$. Then coefficients a_k are given by

$$a_k=\frac{-1}{2a_0}\left[\sum_{i=1}^{k-1}a_ia_{k-i}+By\delta_{k,3}+\hat{h}\frac{da_{k-3}}{dy}\right] \tag{19}$$

for $k>0$. In the above formula $a_k=0\ \forall\ k<0$. Now J can be written as

$$J\,(E)=\sum_{k=0}^{\infty}b_kE^{\frac{-(k-3)}{4}}, \tag{20}$$

where

$$b_k=\frac{1}{2\pi}\int_{\gamma}a_k dy \tag{21}$$

and can be determined analytically in terms of g and b. The contour γ encloses the two branch points of $\sqrt{1+y^4}$ (i.e., $e^{i\pi/4}$ and $e^{3i\pi/4}$) in the complex plane. The quantization condition $J(E)=n\hbar$ determines the eigenenergies of $V(x)=-gx^4+bx$.

Using (19) and evaluating integral (21) analytically, we obtain the series expansion of $J(E)$. Then the eigenenergy expansion becomes

$$J\,(E)=\sum_{k=0}^{\infty}b_kE^{\frac{-(k-3)}{4}}. \tag{22}$$

The first six nonzero b_k's are

$$b_0=\frac{\Gamma\left[\frac{1}{4}\right]}{3g^{1/4}\sqrt{2\pi}\ \Gamma\left[\frac{3}{4}\right]},\quad b_3=-\frac{\hbar}{2},\quad b_6=-\frac{(b^2+2g\hbar^2)\,\Gamma\left[\frac{3}{4}\right]}{8g^{3/4}\sqrt{2\pi}\ \Gamma\left[\frac{1}{4}\right]},\quad b_{12}=-\frac{(5b^4+60b^2g\hbar^2+44g^2\hbar^4)\,\Gamma\left[\frac{1}{4}\right]}{6144g^{5/4}\sqrt{2\pi}\ \Gamma\left[\frac{3}{4}\right]}$$

$$b_{18}=-\frac{77(3b^6+90b^4g\hbar^2+636b^2g^2\hbar^4+488g^3\hbar^6)\,\Gamma\left[\frac{3}{4}\right]}{245\,760g^{7/4}\sqrt{2\pi}\ \Gamma\left[\frac{1}{4}\right]},$$

$$b_{24}=-\frac{1105(b^8+56b^6g\hbar^2+1064b^4g^2\hbar^4+6304b^2g^3\hbar^6+5648g^4\hbar^8)\,\Gamma\left[\frac{1}{4}\right]}{58\,720\,256g^{9/4}\sqrt{2\pi}\ \Gamma\left[\frac{3}{4}\right]}.$$

In order to obtain equivalent Hamiltonians we impose the condition that expressions of action variable $J(E)$ in Eqs. (13) and (22) must be identical.

III. DERIVATION OF EQUIVALENCE CONDITION

Before we impose conditions on $J(E)$ to obtain equivalent Hermitian Hamiltonians corresponding to non-Hermitian systems, it would be useful to examine the accuracy of the above expansions. For this purpose we examine expressions of $J(E)$ for non-Hermitian Hamiltonian $H_1=p^2-gx^4$ and the conventional Hermitian Hamiltonian $h_1=p^2+4gx^4-2\hbar\sqrt{g}x$ with exact energies obtained by numerical methods. By substituting $b=0$ in $J(E)$ expansion of H and $\alpha=4g$ and $\beta=-2\sqrt{g}\hbar$ in $J(E)$ expansion of h we find both $J(E)$ expansions are identical (up to the first 100 terms) and are given by (22) with

$$b_0=\frac{\Gamma\left[\frac{1}{4}\right]}{3g^{1/4}\sqrt{2\pi}\ \Gamma\left[\frac{3}{4}\right]},\quad b_3=-\frac{\hbar}{2},\quad b_6=-\frac{g^{1/4}\hbar^2\ \Gamma\left[\frac{3}{4}\right]}{4\sqrt{2\pi}\ \Gamma\left[\frac{1}{4}\right]},$$

$$b_{12}=-\frac{11g^{3/4}\hbar^4\ \Gamma\left[\frac{1}{4}\right]}{1536\sqrt{2\pi}\ \Gamma\left[\frac{3}{4}\right]},\quad b_{18}=-\frac{4697g^{5/4}\hbar^6\ \Gamma\left[\frac{3}{4}\right]}{30\,720\sqrt{2\pi}\ \Gamma\left[\frac{1}{4}\right]},$$

$$b_{24}=-\frac{390\,065g^{7/4}\hbar^8\ \Gamma\left[\frac{1}{4}\right]}{3\,670\,016\sqrt{2\pi}\ \Gamma\left[\frac{3}{4}\right]}.$$

Next we apply the quantization condition $J(E)=n\hbar$ to obtain the eigenspectrum. Table I shows the eigenvalues obtained by quantizing $J(E)$. For comparison purposes eigenspectra of the above system were obtained numerically by diagonalizing the Hamiltonian in the Harmonic oscillator basis. Results are shown in the Table I.

TABLE I. Comparison between calculated energy eigenvalues with $J(E)$ expansion and E_{Exact}, which is obtained by the matrix diagonalization method for the Hamiltonian $H=p^2+4gx^4-2\hbar\sqrt{g}x$. The calculation was carried out for $g=1.0$. Units of energy is chosen such that $\hbar=1$ and $m=1/2$.

n	$E_{J(E)}$	E_{Exact}
0	1.6590745	1.4771498
1	6.003393	6.0033861
2	11.802434	11.802434
3	18.458819	18.458819
4	25.791792	25.791792
5	33.694280	33.694280
6	42.093808	42.093808
7	50.937404	50.937404
8	60.184331	60.184331
9	69.802098	69.802098
10	79.764086	79.764086

Except for the ground state and the first excited state, all the eigenvalues obtained by quantizing $J(E)$ agree well with corresponding numerical values over eight significant figures. The accuracy of this expansion method has been tested for various systems, and for more details please refer to [18–21]. Having shown the accuracy of the action variable method, now we obtain the main result of this paper.

In order to find equivalent Hermitian Hamiltonians corresponding to non-Hermitian systems, we find b,α, and β in terms of g and $\hbar$ by imposing the condition that the terms of $J(E)$ expansion in Eqs. (13) and (22) are equal. The first four terms of the expansion produce the following conditions:

$$\alpha = 4g, \quad b(b^2 + 16g\hbar^2) = 0, \quad 2b^2 + \beta^2 = 4\hbar^2 g. \tag{23}$$

There are two solutions for α,b and β:

$$\alpha = 4g, \quad b = 0, \quad \beta = \pm 2\hbar\sqrt{g} \tag{24}$$

or

$$\alpha = 4g, \quad b = \pm 4i\hbar\sqrt{g}, \quad \beta = \pm 6\hbar\sqrt{g}. \tag{25}$$

The first solution (24) corresponds to the equivalent Hamiltonians $H_1 = p^2 - gx^4$ and $h_1 = p^2 + 4gx^4 - 2\hbar\sqrt{g}x$ as found by Jones *et al.* [1–3]. The second solution corresponds to new equivalent Hamiltonians $H_2 = p^2 - gx^4 \pm 4i\hbar\sqrt{g}x$ and $h_2 = p^2 + 4gx^4 \pm 6\hbar\sqrt{g}x$. Note that linear terms in H_2 and h_2 are also anomalous. For $\alpha = 4g$, $b = \pm 4i\hbar\sqrt{g}$, and $\beta = \pm 6\hbar\sqrt{g}$, the individual terms in the $J(E)$ expansions of Hamiltonians H_2 and h_2 are found to be identical (over hundreds of terms). Further these two Hamiltonians were studied by using numerical methods. In Table II we present the first ten eigenenergies of H_2 and h_2, which have been determined by direct numerical integration of the Schrödinger equation. The Schrödinger equation for the Hamiltonian h_2 was solved along the real axis while the contour in Refs. [1–3] was used for integrating the Schrödinger equation for Hamiltonian H_2.

It is evident from Table II that both Hamiltonians H_2 and h_2 have the same eigenspectra for the first ten eigenstates. On the other hand, expansion of $J(E)$ is very accurate for large eigenvalues, and both Hamiltonians have identical $J(E)$ expansions. Therefore we claim that both H_2 and h_2 have the same eigenspectra.

TABLE II. The first ten exact eigenenergy values of $h_2 = p^2 + 4x^4 + 6x$ and $H_2 = p^2 - x^4 + 4ix$ and approximate eigenvalues E_J obtained by the $J(E)$ expansion method with nine terms. Units of energy is chosen such that $\hbar = 1$ and $m = 1/2$.

n	E_{h_2}	E_{H_2}	E_J
0	$<10^{-10}$	$<10^{-9}$	1.806738380
1	5.624339342	5.624339342	5.624445087
2	11.53474946	11.53474946	11.53474922
3	18.24541905	18.24541904	18.24541904
4	25.61148849	25.61148849	25.61148849
5	33.53663215	33.53663215	33.53663215
6	41.95281550	41.95281549	41.95281550
7	50.80926445	50.80926444	50.80926445
8	60.06646405	60.06646403	60.06646405
9	69.69266253	69.69266252	69.69266254
10	79.66170147	79.66170147	79.66170149

Further it was found that the equivalent Hamiltonians $H = p^2 + \frac{\mu^2}{2}x^2 - gx^4$ and $h = p^2 - 2\hbar\sqrt{g}x + 4g(x^2 - \frac{\mu^2}{8g})^2$ produced identical $J(E)$ expansions. Further we claim that there are *no other* non-Hermitian PT-symmetric Hamiltonians with a $-x^4$ term and a linear term equivalent to a Hermitian Hamiltonian with x^4 and a linear term ($h = p^2 + 4gx^4 + \lambda x, \lambda \in \mathbb{C}$). The reason for this claim is that except for the parameters in Eqs. (24) and (25), no other parameter values can make $J(E)$ of both Hamiltonians identical.

By direct numerical integration of the Schrödinger equation, it was revealed that H_2 and h_2 have zero-energy ground states (Table II). This will enable us to obtain exact wave functions of the ground states of both Hamiltonians. In the next section we obtain exact ground-state wave functions as well as the supersymmetric (SUSY) partners of these Hamiltonians.

IV. GROUND-STATE WAVE FUNCTIONS AND SUSY PARTNERS

In this section we find the exact wave functions of the zero ground state of Hamiltonians H_2 and h_2. First we consider the Hermitian Hamiltonian h_2. For the ground state, the Schrödinger equation becomes (assume $\hbar = 1$ and $2m = 1$)

$$-\frac{d^2\psi_0}{dx^2} + (4x^4 + 6x)\psi_0(x) = 0.$$

Let $\psi_0(x) = u_0(x)\, e^{-\frac{2}{3}|x|^3}$. Then the solution $u_0(x)$ is found by the series expansion method:

$$u_0(x) = {}_1F_1\left(\frac{5}{2},\frac{2}{3},\frac{4x^3}{3}\right),$$

where ${}_1F_1$ is the confluent hypergeometric function and has the series expansion ${}_1F_1(a,b,z) = \sum_{n=0}^{\infty} \frac{a^{(n)}z^n}{b^{(n)}n!}$ and $a^{(n)} = a(a+1)(a+2)\cdots(a+n-1)$ and $b^{(n)} = b(b+1)(b+2)\cdots(b+n-1)$.

The ground-state wave function of non-Hermitian Hamiltonian H_2 is obtained by using the method developed by Dorey *et al.*[24]. The ground-state wave function $\Phi_0(x)$ is found as

$$\Phi_0(x) = ixe^{\frac{i}{3}x^3},$$

where the quantization contour starts and ends at $|x| = \infty$, joining sectors S_{-1} and S_1, where

$$S_k = \left\{x : \left|\arg(x) - \frac{\pi k}{3}\right| < \frac{\pi}{6}\right\}.$$

Having found the exact wave functions, now we determine the SUSY partners of both Hamiltonians, which will also be isospectral. First we find the superpotential $W_H(x)$ [25] of H_2.

$$W_{H2}(x) = -\frac{1 + ix^3}{x},$$

and hence the supersymmetric partner potential V_{H2} is

$$V_{H2}(x) = -x^4 + \frac{2}{x^2}.$$

However, supersymmetric partners for the potential $4x^4 + 6x$ cannot be found as a simple expression as above. The superpotential for this problem is

$$W_{h2}(x) = x^2[2 - 5\phi(x)],$$

where $\phi(x) = \frac{{}_1F_1(\frac{11}{6},\frac{5}{3},\frac{4x^3}{3})}{{}_1F_1(\frac{5}{6},\frac{2}{3},\frac{4x^3}{3})}$. Hence the supersymmetric partner potential V_{h2} is

$$V_{h2}(x) = 4x^4 - 6x + 10x^4\phi(x)[5\phi(x) - 4].$$

Hence the Hamiltonian $h_3 = p^2 + V_{h2}(x)$ is the isospectral Hermitian equivalent of the non-Hermitian Hamiltonian $H_3 = p^2 + V_{H2}(x)$.

V. SUMMARY AND CONCLUDING REMARKS

In this paper we showed that the complex non-Hermitian PT-symmetric Hamiltonian $p^2 - gx^4 + 4i\hbar\sqrt{g}x$ and the conventional Hermitian Hamiltonian $p^2 + 4gx^4 + 6\hbar\sqrt{g}x$ have the same eigenspectra. We utilized a method based on series expansion of the quantum action variables $J(E)$ in rational powers of reciprocal of energy. The $J(E)$ expansions of these two Hamiltonians were found to be identical. Moreover, the Schrödinger equation for these Hamiltonians was solved numerically to determine the quantum eigenspectrum. Both Hamiltonians produced almost the same eigenvalues (up to ten significant figures) for the first ten states. The $J(E)$-based quantization method produces accurate eigenvalues for the higher eigenstates. Therefore we claim that Hamiltonians $p^2 + 4gx^4 + 6\hbar\sqrt{g}x$ and $p^2 - gx^4 + 4i\hbar\sqrt{g}x$ have the same spectra and they have zero-energy ground states. We have obtained the exact ground-state wave functions and hence proved analytically that the ground states of these two Hamiltonians are in fact zero-energy ground states. Although we did not present a rigorous proof for the equivalence, we believe that identical $J(E)$ and numerical evidence strongly support our claim.

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