Clock synchronization using maximal multipartite entanglement

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We propose a multiparty quantum clock synchronization protocol that makes optimal use of the maximal multipartite entanglement of GHZ-type states. To realize the protocol, different versions of maximally entangled eigenstates of collective energy are generated by local transformations that distinguish among different groupings of the parties. The maximal sensitivity of the entangled states to time differences between the local clocks can then be accessed if all parties share the results of their local time-dependent measurements. The efficiency of the protocol is evaluated in terms of the statistical errors in the estimation of time differences and the performance of the protocol is compared to alternative protocols proposed previously.

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I. INTRODUCTION

Quantum clock synchronization protocols are of fundamental interest in quantum information, since they can illustrate how information about time is encoded in quantum systems. In general, there are two approaches to the problem, one based on the correlations between photon arrival times [1–4] and the other based on the internal time evolution of quantum systems [5–12]. Although the latter approach requires an effective suppression of decoherence and is therefore much more challenging to implement, it might be of greater fundamental interest, since it allows a very general treatment of time in quantum mechanics.

Initially, it was shown that two-party quantum clock synchronization protocols can be used for efficient clock synchronization by using the enhanced sensitivity of bipartite entangled states to small time differences between the measurements performed by the two parties [6]. Later, Krco and Paul [10] extended this idea to a multiparty version, where a W state was used to simultaneously provide bipartite entanglement between a central clock and several other parties. However, the bipartite entanglement obtained from the W state decreases rapidly with an increase in the number of clocks. Ben-Av and Exman [12] pointed out that this is a weakness of the W state that can be overcome by using other Dicke states instead. Specifically, they showed that the optimal bipartite entanglement for this kind of protocol is obtained by using the symmetric Dicke states, where half of the qubits are in the 0 state and half are in the 1 state. Interestingly, none of these protocols uses the specific properties of multipartite entanglement. Here, we consider the question of whether this different type of entanglement could be used for clock synchronization by constructing a protocol that accesses the maximal entanglement of GHZ-type states through an appropriate combination of measurement and communication between the parties.

II. CLOCK SYNCHRONIZATION WITH GHZ STATES

Since clock synchronization should not depend on a knowledge of the time needed for state distribution, the multipartite entangled states used should be energy eigenstates. It is therefore not possible to use GHZ states that are superpositions of the two extremal eigenstates of energy, where all qubits are in the same state of their local energy basis. To obtain an energy eigenstate without changing the multipartite entanglement, half of the local energy eigenstates should be flipped by appropriate local unitary transformations. If the qubits are arranged so that the first half of the qubits is unflipped and the second half of the qubits is flipped, this N-partite entangled energy eigenstate can be given in the energy basis as

$$|\Psi_N\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes\frac{N}{2}}|1\rangle^{\otimes\frac{N}{2}} + |1\rangle^{\otimes\frac{N}{2}}|0\rangle^{\otimes\frac{N}{2}}\right). \qquad (1)$$

Here and in the following, we assume an even number of parties N. States $|0\rangle$ and $|1\rangle$ are local energy eigenstates with energies 0 and $\hbar\omega$, respectively.

The state given by Eq. (1) divides the qubits into two groups. To ensure clock synchronization between all parties, it is necessary that no two parties are always members of the same group. This is achieved by distributing the qubits in different ways, so that each party sometimes receives a qubit from the unflipped group and sometimes receives a qubit from the flipped group. To describe each distribution, we define a sequence $\{f_i\}$, where $i = 1, \ldots, N$. If the qubit of the ith clock owner is a flipped qubit, $f_i = 1$; if not, $f_i = 0$. Since the numbers of flipped and unflipped qubits are equal, the number of possible distributions $\{f_i\}$ is given by the binomial coefficient $N!/[(N/2)!(N/2)!]$. In the most simple version of the protocol, the division into groups can be decided randomly in each run, with equal probabilities for each distribution $\{f_i\}$.

After the distribution of the qubits to the locations of the different clocks, each of the parties measures a time-dependent observable $\hat{X}(t)$ on its qubit when their local clock points to a specific time. The observable measured at a time t can be written as

$$\hat{X}(t) = \exp(-i\omega t)|0\rangle\langle 1| + \exp(i\omega t)|1\rangle\langle 0|. \qquad (2)$$

The eigenvalues of the measurement outcomes are ± 1. The eigenstates corresponding to the measurement outcomes are equal superpositions of $|0\rangle$ and $|1\rangle$, where the phase now depends on the time at which the measurement is performed. As a result, this measurement achieves the maximal time sensitivity for local qubit measurements.

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The time sensitivity of the maximal multipartite entanglement of the GHZ-type energy eigenstate given in Eq. (1) originates from the coherence between the components $|0\rangle^{\otimes\frac{N}{2}}|1\rangle^{\otimes\frac{N}{2}}$ and $|1\rangle^{\otimes\frac{N}{2}}|0\rangle^{\otimes\frac{N}{2}}$. This coherence, which represents the full multipartite entanglement of the state, changes the probability of the collective measurement outcome depending on the product of the coherences between the $|0\rangle$ and the $|1\rangle$ components in the eigenstates representing the local measurement outcomes. As a result, the time sensitivity of multipartite entanglement can be represented by the expectation value for the product of all outcomes, $\hat{X}^{\otimes N}$. If the actual measurement times of the parties are given by $\{t_1,t_2,\ldots,t_N\}$, the expectation value of this product is

$$\langle \hat{X}^{\otimes N}\rangle = \cos\left(\sum_{i}^{N}(-1)^{f_i}\,\omega t_i\right). \tag{3}$$

The time dependence of this correlation is maximally sensitive to the characteristic coherence of the GHZ-type state, so we can conclude that the protocol makes optimal use of the maximal multipartite entanglement for clock synchronization.

To access the time sensitivity of the GHZ-type state, all parties must share their measurement results and determine the product of all outcomes. Effectively, the N parties cooperate to measure a single N-particle interference fringe that is sensitive to the collective phase given by ω times the difference between all measurement times of the unflipped qubits ($f_i = 0$) and the measurement times of all flipped qubits ($f_i = 1$). To ensure that all parties are treated equally, it is possible to use a random distribution of qubits, so that every distribution $\{f_i\}$ of flipped and unflipped qubits is equally likely. To keep track of the different distributions, we assign an index j to each, so that the elements of each sequence are given by $f_i(j)$. The total time difference that defines the phase shift in the multipartite interference fringe observed in the $\hat{X}^{\otimes N}$ measurement of the distribution with index j is then

$$T_j = \sum_{i=1}^{N}(-1)^{f_i(j)}t_i. \tag{4}$$

The time differences T_j can be estimated from the outcome statistics of the measurements with an accuracy of $\delta T_j = 1/(\omega\sqrt{k_j})$, where k_j is the number of times that the distribution j is received and measured.

After a sufficiently large number of measurements, all parties have the same estimates for all possible time differences T_j. However, the implications of each T_j are different for each party. Specifically, each party i can obtain the difference between times T_j with $f_i(j) = 0$ and times T_j with $f_i(j) = 1$:

$$\sum_j(-1)^{f_i(j)}T_j = \sum_j\sum_k(-1)^{f_i(j)}(-1)^{f_k(j)}t_k. \tag{5}$$

For $k = i$, the coefficient in the sum is always $+1$, so that the time t_i of the local clock always enters into the sum with a positive value. Since all the other times enter into the sum equally, and since the numbers of $+1$ coefficients and -1 coefficients are exactly equal, the result can be expressed in terms of the difference between the time t_i and the average $\langle t\rangle_{k\neq i}$ of all times other than t_i:

$$\sum_j(-1)^{f_i(j)}T_j = \frac{N!}{(N/2)!(N/2)!}(t_i - \langle t\rangle_{k\neq i}). \tag{6}$$

The average of all times $\langle t\rangle$ is obtained by the weighted average of t_i and $N-1$ times $\langle t\rangle_{k\neq i}$. Hence, the difference between the local time t_i and the average time $\langle t\rangle$ can be given by

$$t_i - \langle t\rangle = \left(\frac{N-1}{N}\right)\sum_j(-1)^{f_i(j)}\,\frac{(N/2)!(N/2)!}{N!}\,T_j. \tag{7}$$

After this value is determined by each party, it can be subtracted from each local clock time t_i to adjust the clock times so that they correspond to the average time $\langle t\rangle$.

To determine the efficiency of a clock synchronization protocol, it is necessary to evaluate the precision with which the parties can estimate the adjustment time $t_i - \langle t\rangle$. In general, this precision is limited by the statistical variance of the measurement results. As mentioned above, the estimation errors for the time differences T_j are given by $\delta T_j = 1/(\omega\sqrt{k_j})$, where k_j is the number of times that the distribution j was measured. Since the adjustment times $t_i - \langle t\rangle$ are linear functions of the T_j, it is sufficient to find the sum of the quadratic errors with the appropriate coefficients to obtain the adjustment errors,

$$\delta t_i^2 = \sum_j\left(\frac{N-1}{N}\right)^2\left(\frac{(N/2)!(N/2)!}{N!}\right)^2\delta T_j^2. \tag{8}$$

If each distribution j is measured an equal number of times, k_j can be expressed as the total number of measurements k divided by the number of possible distributions $N!/[(N/2)!(N/2)!]$. Likewise, the sum over j reduces to a simple multiplication with the number of possibilities. In the end, the estimation error for each adjustment time is given by

$$\delta t_i^2 = \left(\frac{N-1}{N}\right)^2\frac{1}{\omega^2 k}. \tag{9}$$

In the limit of large N, this error is simply $\delta t_i = 1/(\omega\sqrt{k})$, independent of the number of parties participating in the clock synchronization. This means that the maximally multipartite entangled states can be used to synchronize N clocks in parallel, without any loss of precision when additional parties are added.

III. COMPARISON WITH PREVIOUS MULTIPARTY PROTOCOLS

Although maximal multipartite entanglement (Fig. 1) clearly improves the efficiency of multiparty clock synchronization, it may be important to put the relative advantage into perspective by considering alternative protocols. Here, we consider the efficiency of multiparty clock synchronization using the parallel distribution of bipartite entanglement for the parallel performance of two-party clock synchronizations [6] and the protocol based on symmetric Dicke states introduced by Ben-Av and Exman [12].

The multiparty protocol for parallel distribution of bipartite entanglement is illustrated in Fig. 2. There are N spatially separated unsynchronized clocks, one of which is the standard clock. The remaining $N-1$ clock owners synchronize their

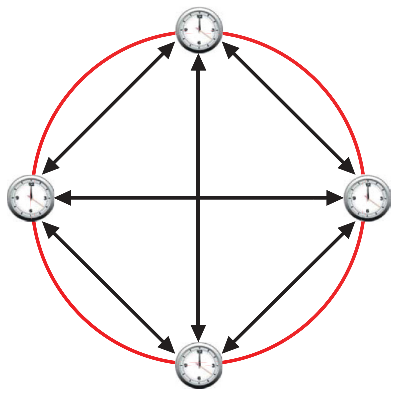

FIG. 1. (Color online) Illustration of clock synchronization with maximal multipartite entanglement. The circle indicates the multipartite entanglement; the arrows indicate classical communication between the parties. Communication among all parties is necessary to make the high sensitivity of multipartite entanglement available for clock synchronization.

clocks with this standard clock using bipartite entanglement and classical communication. For this purpose, the owner of the central clock must share $N-1$ maximally entangled two-qubit states with all of the other parties for each measurement. Effectively, each step of the protocol uses a $2(N-1)$ qubit state given by

$$|\text{parallel}\rangle = \left(\frac{1}{\sqrt{2}}\right)^{N-1} (|01\rangle + |10\rangle)^{\otimes(N-1)}. \tag{10}$$

Here, every second qubit is held by the owner of the central clock. At a predetermined time t, the owner of the central clock measures the value of $\hat{X}_{ci}(t)$ on all of her $N-1$ qubits i, where i is the index of the party that holds the qubit entangled with the qubit ci. Likewise, the other parties measure the value of $\hat{X}_{pi}(t_i)$ on their individual qubits according to their local times t_i. The central clock then communicates each result of $\hat{X}_{ci}(t)$ to the party concerned. After a sufficiently large number of measurements k, the owner of clock i can then determine the expectation value of the product,

$$\langle \hat{X}_{pi} \otimes \hat{X}_{ci} \rangle = \cos(\omega(t_i - t)). \tag{11}$$

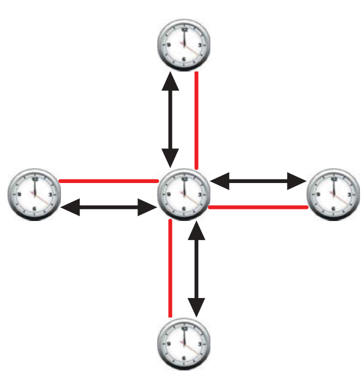

FIG. 2. (Color online) Illustration of clock synchronization using a parallel distribution of bipartite entangled states. Arrows indicate classical communication and lines indicate shared bipartite entanglement. Each of the outer $N-1$ clocks is synchronized separately with the central clock using separate sets of entangled qubits.

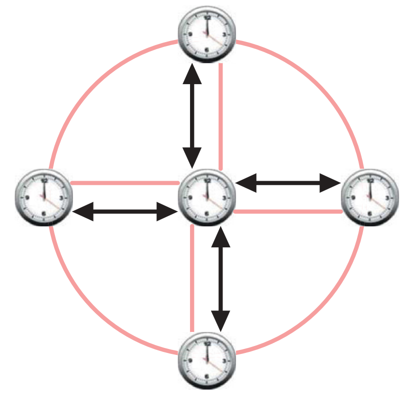

FIG. 3. (Color online) Illustration of clock synchronization using an N-qubit symmetric Dicke state. Arrows represent classical communication; lines indicate the entanglement between the qubits. Since each qubit is entangled with all the other qubits, the owner of the central clock only needs to hold a single qubit per measurement.

The clock owners can then determine $t_i - t$ directly and adjust their clocks accordingly.

The efficiency of clock synchronization can be evaluated by considering the estimation error δt_i for each estimate of adjustment time $t_i - t$. For k measurements, this error is given by $\delta t_i = 1/(\omega\sqrt{k})$. Thus the precision of the time estimates in this protocol is exactly equal to the result for the protocol using multipartite entanglement. However, the parallel distribution of bipartite entanglement requires $2(N-1)$ qubits for each measurement, compared to only N qubits for the multipartite entangled protocol. In terms of the required number of qubits, the use of multipartite entangled states can thus increase the efficiency by a factor of 2. Effectively, the main effect of multipartite entanglement seems to be that the need for multiple reference qubits held by the owner of the central clock is removed by allowing the parties to use the $N-1$ qubits of all the other parties as a collective reference instead.

In the Dicke state protocol of Ben-Av and Exman [12], bipartite entanglement between a central clock and each party is obtained with only a single qubit at the central clock, as illustrated in Fig. 3. Otherwise, the same measurement and communication procedure as in the parallel distribution of entanglement is used. The symmetric Dicke states is an equal superposition of all energy eigenstates, with half of the qubits in the $|0\rangle$ state and half in the $|1\rangle$ state,

$$|\text{Dicke}(N)\rangle = \frac{\frac{N}{2}!}{\sqrt{N!}}(|11\ldots10\ldots00\rangle + |11\ldots01\ldots00\rangle + \cdots + |00\ldots01\ldots11\rangle). \tag{12}$$

The bosonic symmetry of the state means that the qubits tend to be found in the same superposition states of $|0\rangle$ and $|1\rangle$, resulting in positive correlations between the values of $\hat{X}(t_i)$ obtained by the different parties at the same time t_i. Specifically, the correlation between the measurement at the central clock and the measurement at clock i is given by

$$\langle \hat{X}_{pi} \otimes \hat{X}_{ci} \rangle = \frac{N}{2(N-1)} \cos(\omega(t_i - t)). \tag{13}$$

At the maximal time derivative of the expectation value, the error in the adjustment time $t_i - t$ for k measurements is given by $\delta t_i = 2(N-1)/(N\omega\sqrt{k})$. In the limit of large N, this error is equal to twice the error of our GHZ-state protocol and the parallel distribution protocol. Hence, this protocol requires four times as many qubits to achieve the same accuracy as the GHZ-state protocol, and twice as many qubits as the parallel distribution protocol. The reduction in qubit number over parallel distribution of bipartite states is therefore more than offset by the loss of sensitivity in each individual measurement due to the reduction in the available bipartite entanglement.

We can now summarize our results in terms of the accuracy of clock synchronization achieved with a given number of qubits. Since the time scale is defined by the resonant frequency ω of the qubit dynamics, it is convenient to define the relative accuracy as $1/(\omega\delta t_i)^2$. For the GHZ-type multipartite entanglement, the accuracy of k measurements using $Q = kN$ qubits is then given by

$$\left.\frac{1}{(\omega\delta t_i)^2}\right|_{\text{GHZ}} = \left(\frac{N}{N-1}\right)^2 \frac{Q}{N}. \tag{14}$$

For high N, the accuracy is equal to the number of qubits per party, so the accuracy of the multiparty protocol scales linearly with the ratio of qubits and parties, Q/N.

Significantly, the straightforward extension of the bipartite protocol by parallel distribution of entangled qubit pairs performs only half as well. Specifically, the accuracy of k measurements using $Q = 2k(N-1)$ qubits is

$$\left.\frac{1}{(\omega\delta t_i)^2}\right|_{\text{pairs}} = \frac{1}{2}\left(\frac{N}{N-1}\right) \frac{Q}{N}. \tag{15}$$

The need for extra reference qubits held by the owner of the central clock therefore rapidly reduces the efficiency of each qubit to half the value achieved by the protocol using maximal multipartite entanglement.

Finally, the protocol using the simultaneous bipartite entanglement between a single central qubit and $N-1$ others achieves a sensitivity reduced by a factor of $N/(2(N-1))^2$ due to the reduction in bipartite entanglement associated with the increase in entangled partners for each qubit. The accuracy of k measurements using $Q = kN$ qubits is therefore

$$\left.\frac{1}{(\omega\delta t_i)^2}\right|_{\text{Dicke}} = \frac{1}{4}\left(\frac{N}{N-1}\right)^2 \frac{Q}{N}. \tag{16}$$

In the limit of high N, this is a reduction to one-quarter of the GHZ-type protocol, twice as much as the reduction in accuracy due to the additional reference qubits in the parallel distribution protocol.

IV. CONCLUSIONS

We have shown how the maximal N-partite entanglement of GHZ-type states can be used for multiparty clock synchronization by randomly dividing the parties into two groups during each run and sharing the measurement results with all other parties to determine the adjustments necessary to set each local clock to the average time of all clocks. The accuracy of clock synchronization corresponds to the accuracy achieved by $N-1$ bipartite protocols in parallel, but the number of qubits used is reduced by half. Oppositely, the previously proposed use of symmetric Dicke states uses the same number of qubits, but the accuracy is only one-quarter due to the reduced amount of bipartite entanglement.

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