Duplex symmetry and its relation to the conservation of optical helicity

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Helicity is a familiar concept in particle physics and also appears in the physics of fluids and plasmas. In this paper, we present the *optical* helicity in a form readily applicable to both quantum and classical problems. We examine the relationship between the optical helicity and the more familiar optical spin and show that the conservation of helicity is an expression of the electric-magnetic symmetry for light. We show that helicity is distinct from Lipkin's 00-zilch; a simple relationship exists between the two *only* for monochromatic fields. It is only the optical helicity that has the correct dimensions of an angular momentum, thereby accurately describing the helicity of light. To illustrate the physical significance of the optical helicity, we consider a circularly polarized plane wave and the field emitted by a rotating electric dipole.

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I. INTRODUCTION

It is well-established in optics that a beam of light can carry both spin and orbital angular momenta in a direction parallel to the beam axis [1,2]. The spin angular momentum is associated with circular polarization and the orbital angular momentum with the presence of helical phase fronts. This natural separation, however, runs into difficulties when confronted with more fundamental ideas from Maxwellian electromagnetism and from particle physics. In particular, it is known how to separate the total optical angular momentum into spin and orbital parts [3], but these parts are themselves not true angular momenta [4,5]. Particle physics, moreover, would suggest that because photons are massless, the spin angular momentum is not well-defined and that the physically relevant quantity is instead the helicity: the component of spin in the direction of propagation [6,7]. In this paper, we determine the form of the optical helicity and express it in terms of the electric and magnetic fields and of associated potentials so that it may be applied readily to both classical and quantum problems.

Pleasingly, the optical helicity is found to be closely analogous to the forms recognized in fluid mechanics [8,9] and plasma physics [10]. It is subtly different, however, to the forms previously described in electromagnetic theory [7]. In particular, the optical helicity is distinct from Lipkin's 00-zilch [11], referred to recently as the "optical chirality" [12–16]. For a strictly *monochromatic* field, the cycle-averaged optical helicity and 00-zilch are found to be proportional to one another. In general, however, no such proportionality holds, and it is the optical helicity that is the physically meaningful quantity.

Helicity is a scalar (strictly, a pseudoscalar) property of a vector field which is related to the vorticity or twist or angular momentum of the field. In fluid mechanics, helicity is the volume integral of the scalar product of the fluid velocity with its curl [8,9]:

$$\mathcal{H}_{\text{fluid}} = \int d^3r\, \mathbf{v} \cdot (\nabla \times \mathbf{v}). \tag{1}$$

It is used as a measure of the degree of knottedness of vortex lines. The fluid helicity is unchanged if the fluid velocity is modified by the addition of the gradient of an arbitrary scalar field, provided the integral extends over all space. In plasma physics, the magnetic helicity is the volume integral of the scalar product of the vector potential with the magnetic flux density [10,17]:

$$\mathcal{H}_{\text{mag}} = \int d^3r\, \mathbf{A} \cdot \mathbf{B} = \int d^3r\, \mathbf{A} \cdot (\nabla \times \mathbf{A}). \tag{2}$$

It has been used to understand relaxation processes in plasmas [18]. Despite the explicit appearance of the vector potential, the magnetic helicity is *gauge invariant*, provided the integral extends over all space. The formal similarity between the fluid helicity and the magnetic helicity is apparent. In particle physics, the single-particle helicity is the expectation value of the helicity operator, which is the scalar product of the spin with the normalized momentum [6]:

$$\mathcal{H}_{\text{part}} = \frac{\Sigma \cdot \mathbf{p}}{|\mathbf{p}|}. \tag{3}$$

It seems natural, at first sight, to determine the helicity of a photon using this expression, but we then run into difficulties with the form of the photon wave function [19]. It is certainly possible to use the electric and magnetic fields to form the basis of our wave function via the Riemann-Silberstein vector, but the resulting helicity does *not* have the dimensions of an angular momentum [7]. This feature is not specific to the helicity but arises, rather, from the fact that the squares of the electric and magnetic fields do not have the dimensions of a probability density [19].

II. OPTICAL HELICITY

If we adopt a system of units in which $\mu_0 = \varepsilon_0 = c = 1$, then the magnetic helicity (2) *does* have the dimensions of an angular momentum, and it is then natural to ask whether it might be acceptable as a quantity that describes the helicity of freely propagating electromagnetic waves. That it is not becomes clear when we consider the duplex symmetry [20] due to Heaviside and Larmor [5,21,22]. This symmetry is the invariance of the free-field Maxwell equations under the

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duplex transformation:

$$\mathbf{E} \to \mathbf{E}' = \cos\theta\mathbf{E} + \sin\theta\mathbf{B}, \quad \mathbf{B} \to \mathbf{B}' = \cos\theta\mathbf{B} - \sin\theta\mathbf{E}. \tag{4}$$

Physically important properties of the free electromagnetic field must be invariant under this "rotation" in the space of the electric and magnetic fields [23], a principle referred to recently as electric-magnetic democracy [24]. An important consequence of this is that we cannot tell whether distant galaxies are made from what we call charges or monopoles with suitably chosen properties [25]. We find it necessary to introduce a second (pseudo)vector potential $\mathbf{C}$ such that $\mathbf{E} = -\nabla \times \mathbf{C}$ [26]. This allows us to invoke the duplex transformation (4) by taking

$$\mathbf{A} \to \mathbf{A}' = \cos\theta\mathbf{A} + \sin\theta\mathbf{C}, \quad \mathbf{C} \to \mathbf{C}' = \cos\theta\mathbf{C} - \sin\theta\mathbf{A}, \tag{5}$$

with corresponding transformations for the scalar potential and its counterpart for $\mathbf{C}$. It is clear that the magnetic helicity (2) does not retain its form under a duplex transformation (5) and is, therefore, not an acceptable candidate for an optical helicity. The natural way to proceed is to combine half of the magnetic helicity with half of the corresponding "electric helicity," obtaining

$$\mathcal{H}_{\mathrm{opt}} = \frac{1}{2}\int d^3r(\mathbf{A}\cdot\mathbf{B} - \mathbf{C}\cdot\mathbf{E}). \tag{6}$$

This is, in fact, the optical helicity we seek. It *does* retain its form under a duplex transformation (5) and is a Lorentz pseudoscalar with the dimensions of an angular momentum. We note that the optical helicity is equivalent to the screw action introduced by Candlin [27] and has appeared explicitly in the form above [28,29]. These authors recognized the optical helicity as being a quantity related to the twist of the electromagnetic field and that it can be expressed in terms of the difference of photon numbers for opposite circular polarizations [30]. Zwanziger [31] and Drummond [32] went further and also made the association between helicity conservation and the duplex or Heaviside-Larmor symmetry, which we outline below.

The optical helicity (6) has some unexpected but rather satisfactory properties. First, we note the surprising presence of the vector potentials, which raises the question of gauge dependence. Like the magnetic helicity, the optical helicity is, in fact, *gauge invariant* as the integral over all space of the scalar product of a longitudinal vector field with a transverse vector field is zero [3]. That is, only the gauge-invariant transverse pieces, $\mathbf{A}^{\perp}$ and $\mathbf{C}^{\perp}$, of the vector potentials actually contribute. In addition, the optical helicity is a conserved quantity for all free electromagnetic fields:

$$\frac{d}{dt}\mathcal{H}_{\mathrm{opt}} = 0. \tag{7}$$

Such global conservation laws are the natural consequences of symmetries [33], and it is natural to ask which symmetry of the free electromagnetic field is associated with the conservation of optical helicity. To answer this, we need only employ the optical helicity as the generator of an infinitesimal transformation of the electric and magnetic fields. We find that, for small θ, the quantity $\theta\mathcal{H}_{\mathrm{opt}}$ generates the transformation

$$\mathbf{E} \to \mathbf{E} + \theta\mathbf{B}, \quad \mathbf{B} \to \mathbf{B} - \theta\mathbf{E}, \tag{8}$$

which is the infinitesimal form of the duplex transformation (4). It is clear, therefore, that the conservation of optical helicity is associated with the duplex or Heaviside-Larmor symmetry, a fact that has also been recognized elsewhere [30–32,34]. This connection seems reasonable in light of our observation that the duplex transformation rotates the field vectors of a plane wave about its direction of propagation. We consider this and related ideas in detail elsewhere [35].

We can make contact between the optical helicity (6) and the concept of helicity from particle physics by writing the vector potentials as superpositions of quantized plane waves:

$$\begin{aligned}\hat{\mathbf{A}}^{\perp}(\mathbf{r},t) &= \sum_{\mathbf{k},\lambda}\sqrt{\frac{\hbar}{2V\omega}}\boldsymbol{\epsilon}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathrm{H.c.},\\ \hat{\mathbf{C}}^{\perp}(\mathbf{r},t) &= \sum_{\mathbf{k},\lambda}\sqrt{\frac{\hbar}{2V\omega^3}}\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathrm{H.c.},\end{aligned} \tag{9}$$

where the symbols have their usual meanings. If we choose the basis of circular polarization, the operator representing the optical helicity takes the simple, *exact* form

$$\hat{\mathcal{H}}_{\mathrm{opt}} = \sum_{\mathbf{k}}\hbar(\hat{n}_{\mathbf{k},L} - \hat{n}_{\mathbf{k},R}), \tag{10}$$

where $\hat{n}_{\mathbf{k},L}$ and $\hat{n}_{\mathbf{k},R}$ are the number operators for the left- and right-handed circular polarizations associated with the wave vector $\mathbf{k}$. As the helicity of a photon is $\pm\hbar$ for these polarisations, we see that Eq. (10) simply represents a sum over all modes of the number of photons in each mode multiplied by their helicities. This matches naturally with the concept of helicity from particle physics.

It is natural to enquire as to the relationship between the optical helicity and the optical spin. The spinlike part of the total optical angular momentum is the volume integral of the spin density $\frac{1}{2}(\mathbf{E}\times\mathbf{A}^{\perp} + \mathbf{B}\times\mathbf{C}^{\perp})$ [5]. Similarly, we may think of the quantity $\frac{1}{2}(\mathbf{A}^{\perp}\cdot\mathbf{B} - \mathbf{C}^{\perp}\cdot\mathbf{E})$ as a helicity *density*. The gauge invariance of these quantities is a consequence of the gauge invariance of $\mathbf{A}^{\perp}$ and $\mathbf{C}^{\perp}$. Our helicity and spin densities obey the continuity equation

$$\frac{\partial}{\partial t}\frac{1}{2}(\mathbf{A}^{\perp}\cdot\mathbf{B} - \mathbf{C}^{\perp}\cdot\mathbf{E}) + \nabla\cdot\frac{1}{2}(\mathbf{E}\times\mathbf{A}^{\perp} + \mathbf{B}\times\mathbf{C}^{\perp}) = 0. \tag{11}$$

Thus, the spin density plays the role of a helicity flux density. In this sense, it is analogous to Poynting's vector, which plays the roles of both the linear-momentum density and the energy-flux density [20].

We emphasize that the optical helicity (6) is *distinct* from optical spin although the associated densities are related by the continuity equation (11). Optical helicity is a *pseudoscalar* quantity whereas optical spin is a *pseudovector*. This is illustrated in Fig. 1 in which we consider a left-handed circularly polarized wave incident upon a mirror. The helicity and spin per photon are indicated. Upon reflection, the helicity and wave vector of the wave change sign. The spin, however, does not. The underlying reason for this is that optical helicity is related to the sense of rotation of the field vectors relative to

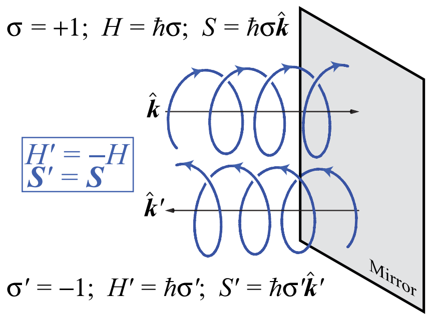

FIG. 1. (Color online) Optical helicity and optical spin are pseudoscalar and pseudovector quantities, respectively. Consider a left-handed circularly polarized wave incident upon a mirror as shown. Following reflection, both the handedness (and therefore helicity) and the direction of propagation of the wave are inverted. This leaves the spin of the wave unchanged.

the direction of propagation whereas optical spin is related to the sense of rotation of the field vectors relative to space.

Optical helicity (6) is distinct from the 00-zilch:

$$\mathcal{Z}^{00} = \frac{1}{2}\int d^3r(\mathbf{E}\cdot\nabla\times\mathbf{E} + \mathbf{B}\cdot\nabla\times\mathbf{B}). \qquad (12)$$

This conserved quantity was introduced by Lipkin [11] and has been referred to recently as the optical chirality [12–16]. The 00-zilch is a higher-order extension of the optical helicity in the sense that the form of the 00-zilch (12) is obtainable from that of the optical helicity (6) by replacing the electric and magnetic fields (and corresponding vector potentials) with their curls. For a strictly *monochromatic* field, the optical helicity and 00-zilch are proportional to one another in a given frame of reference, the proportionality factor being the square of the angular frequency ω. However, in general, the optical helicity and the 00-zilch are distinct quantities, and no proportionality holds between them. Moreover, it is the optical helicity, and not the 00-zilch, that has the correct dimensions of an angular momentum and is the generator of the rotation (4). We suggest, therefore, that it is the optical helicity, rather than the 00-zilch, that faithfully describes the helicity of light.

III. HELICITY OF OPTICAL FIELDS

Optical helicity provides an interesting twist on the familiar paradox of how a circularly polarized plane wave can carry angular momentum [36]. The problem, it will be recalled, is that the component of the angular-momentum density $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ [20] in the direction of propagation vanishes as $\mathbf{E}$ and $\mathbf{B}$ lie in the transverse plane. This suggests, strangely, that a circularly polarized plane wave possesses no angular momentum in the direction of propagation. We can resolve this difficulty by noting that a real beam necessarily has a finite transverse extent. The intensity gradients at the edges of the beam give rise to nonzero field components in the direction of propagation, localizing the angular momentum in these regions. An absorbing particle in the *center* of the beam is able to pick up angular momentum by modifying the beam's transverse profile, generating components of the fields in the direction of propagation [36–38]. In this way, the flux of angular momentum is changed by the absorption [39]. We can approach this problem from a different, and perhaps simpler, perspective by turning to optical helicity. Consider a circularly polarized plane wave propagating in the z direction. The vector potentials assume the forms

$$\mathbf{A}^\perp \pm i\mathbf{C}^\perp = \mathcal{A}(\hat{\mathbf{x}} \mp i\hat{\mathbf{y}})e^{i\omega(z-t)}, \qquad (13)$$

where the two signs correspond to the two circular polarizations. It is straightforward to calculate the densities, and flux densities, of helicity and energy for our wave:

$$\begin{aligned} \tfrac{1}{2}(\mathbf{A}^\perp\cdot\mathbf{B} - \mathbf{C}^\perp\cdot\mathbf{E}) &= \pm|\mathcal{A}|^2\omega, \\ \tfrac{1}{2}(E^2+B^2) &= |\mathcal{A}|^2\omega^2, \\ \tfrac{1}{2}(\mathbf{E}\times\mathbf{A}^\perp + \mathbf{B}\times\mathbf{C}^\perp) &= \pm|\mathcal{A}|^2\omega\hat{\mathbf{z}}, \\ \mathbf{E}\times\mathbf{B} &= |\mathcal{A}|^2\omega^2\hat{\mathbf{z}}. \end{aligned} \qquad (14)$$

Note that these expressions are exact and that no cycle averaging need be applied to obtain them. The similarity of these results to the 00-zilch density and flux density of a plane wave (see, e.g., Eqs. (16) and (17) of Ref. [14], respectively) is a direct consequence of the monochromaticity of the plane wave. The physical significance of the frequency dependences in Eq. (14) becomes clear when we take the ratio of the first two quantities and multiply numerator and denominator trivially by $\hbar$:

$$\frac{\tfrac{1}{2}(\mathbf{A}^\perp\cdot\mathbf{B} - \mathbf{C}^\perp\cdot\mathbf{E})}{\tfrac{1}{2}(E^2+B^2)} = \frac{\pm\hbar}{\hbar\omega}. \qquad (15)$$

As the energy of a photon is $\hbar\omega$, we see that our wave possesses a helicity of $\pm\hbar$ per photon as it should. In contrast, the wave possesses a 00-zilch of $\pm\omega^2\hbar$ per photon—a frequency dependence noted as unusual by Lipkin [11]. Comparing the flux densities of helicity and energy, we infer that the rate at which helicity is *transported* by the wave is similarly proportional to the rate at which energy is transported. We can conclude that the absorption of a photon by a body removes a helicity of $\pm\hbar$ from the beam and that this is converted into intrinsic angular momentum of the body about the direction of propagation. We should emphasize that this analysis, based on optical helicity, in no way invalidates earlier arguments based on optical angular momentum [36–39] but may be seen as resolving the plane-wave paradox more directly.

Finally, we consider how helicity is generated in the emission of light by matter. We examine here the simplest situation: emission of light by an electric dipole placed at the coordinate origin, rotating about the z axis with angular frequency ω. The electric field of a dipole generally has both transverse and longitudinal parts [40]. However, as helicity is a property only of the free and fully transverse field, we need to work in the far field. For $r = |\mathbf{r}|$ sufficiently large, the dipole fields are [20]

$$\mathbf{B} = \mathrm{Re}\left[\frac{\omega^2}{4\pi r^2}\mathbf{r}\times\mathbf{p}e^{i\omega(r-t)}\right], \quad \mathbf{E} = \frac{1}{r}\mathbf{B}\times\mathbf{r}, \qquad (16)$$

where $\mathbf{p} = p(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})/\sqrt{2}$. It is then straightforward to calculate $\mathbf{A}^\perp$ and $\mathbf{C}^\perp$ and to find that the flux of helicity passing through a large sphere centered upon the dipole is zero. This is because the angular momentum radiated by the dipole carries equal and opposite helicity through antipodal points on the sphere. If we consider the localized fluxes of helicity and energy, we find that the helicity carried away by the field amounts to $\pm\hbar$ per photon near the north pole of the sphere but is very small near the equator. This is because the emitting dipole appears to be rotating when viewed from the north pole but appears to be oscillating in the manner of a linear dipole when viewed from the equator. The total helicity radiated in the northern hemisphere amounts to $\pm 3\hbar/8$ per photon with an exactly opposite amount for the southern hemisphere. Interestingly, similar behavior is found when applying the magnetic helicity to the solar atmosphere where the observed local helicities in the upper and lower hemispheres are predominantly opposite [17,41].

IV. CONCLUSION

We have expressed the helicity of light in terms of the electric and magnetic fields and their corresponding vector potentials. This form satisfies the duplex, or Heaviside-Larmor, symmetry [20–24]. Its global conservation, for freely propagating, fully transverse optical fields, is an expression of this symmetry. In the presence of charges, the electric-magnetic symmetry is broken, and helicity is no longer an absolutely conserved quantity in general. We have demonstrated the relation of the optical helicity to other quantities: the optical spin and the 00-zilch. As simple applications of optical helicity, we have revisited the question of the angular momentum carried by a plane wave and calculated the rate at which helicity is radiated by a rotating electric dipole in the far field.

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