Simple asymptotic forms for Sommerfeld and Brillouin precursors

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This article mainly deals with the propagation of step-modulated light pulses in a dense Lorentz medium at distances such that the medium is opaque in a broad spectral region including the carrier frequency. The transmitted field is then reduced to the celebrated precursors of Sommerfeld and Brillouin, far apart from each other. We obtain simple analytical expressions of the first (Sommerfeld) precursor, whose shape only depends on the order of the initial discontinuity of the incident field and whose amplitude rapidly decreases with this order (rise-time effects). We show that, in a strictly asymptotic limit, the second (Brillouin) precursor is entirely determined by the frequency dependence of the medium attenuation and has a Gaussian or Gaussian-derivative shape. We point out that this result applies to the precursor directly observed in a Debye medium at decimetric wavelengths. When attenuation and group-delay dispersion both contribute to its formation, we establish a more general expression of the Brillouin precursor, containing the previous one (dominant-attenuation limit) and that obtained by Brillouin (dominant-dispersion limit) as particular cases. We finally study the propagation of square or Gaussian pulses, and we determine the pulse parameters optimizing the Brillouin precursor. Obtained by standard Laplace-Fourier procedures, our results are explicit and contrast in their simplicity those derived by the uniform saddle-point methods, from which it is difficult to retrieve our asymptotic forms.

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I. INTRODUCTION

More than one century ago, in a short communication [1] made at the 79th congress of the German physicists, Sommerfeld examined the apparent inconsistency between the theory of special relativity and the possibility of superluminal group velocity predicted by the classical wave theory. Considering an incident wave switched on at time $t = -T$ and off at time $t = T$ (square-wave modulation), he mathematically demonstrated that, regardless of the value of the group velocity at the frequency of the optical carrier, no signal can be transmitted by any linear dispersive-attenuative medium before the instant $t = -T + z/c$, where z is the propagation distance and c is the velocity of light in vacuum. In the discussion following Sommerfeld's communication, Voigt proposed a simple physical interpretation of this result. He remarked that the front of the wave encounters a medium that, due to its inertia, seems optically empty and thus that the propagation of the very first beginning of the signal will proceed undisturbed with the velocity of light in vacuum. In other words, local causality implies relativistic causality. The analysis of what happens after the arrival of the wave front was subsequently conducted by Sommerfeld and Brillouin in the case of a step-wave modulation (field switched on at time $t = 0$), the medium being modeled as an ensemble of damped harmonic oscillators with the same resonance frequency ω_0 and the same damping rate γ (Lorentz medium) [2–5]. They found that, in suitable conditions, the transmitted signal consists of two successive transients (that they named "forerunners") preceding the establishment of the steady-state field at the frequency ω_c of the optical carrier (the "main field"). The first and second forerunners, now called the Sommerfeld and Brillouin precursors, were associated with frequencies that are high and low, respectively, compared to the resonance frequency ω_0 of the medium. These results were obtained by means of a spectral approach involving the newly developed saddle-point method [3] and also classical complex analysis [2] and the stationary-phase method [4]. Following these pioneering works, precursors became a canonical problem in electromagnetism and optics [6,7]. Results completing, improving, and even correcting those of Sommerfeld and Brillouin were obtained by means of uniform asymptotic methods [8–11]. The problem was also studied by a purely temporal approach [12]. At the present time, the theoretical study of precursors continues to raise considerable interest. An abundant bibliography can be found in the recent work by Oughstun [13]. Complementary studies on the effects of a finite turn-on time of the incident field on the precursors are reported in [14–17].

From an experimental point of view, the observation of Sommerfeld and Brillouin precursors *in the optical range* raises serious difficulties. Indeed the excitation of the Sommerfeld and Brillouin precursors requires that the corresponding frequencies (respectively high and low compared to ω_0) be present at a significant level in the spectrum of the incident pulse. An experiment intended to observe the Brillouin precursor in water is reported in [18]. Using pulses at a wavelength of 700 nm with a bandwidth of 60 nm, the authors observed pulse breakup in a linear regime as well as a subexponential attenuation with the distance of the new peak. They attributed these features to the formation of a Brillouin precursor. This interpretation has been soundly disputed, in particular because the pulse bandwidth was, in fact, not broad enough to perform the excitation of precursors [19]. Alternative explanations of the observations have been proposed [19,20], and more recent studies [21–23] have confirmed that a subexponential decay of the transmitted energy does not prove the formation of precursors.

While well-distinguishable Sommerfeld and Brillouin precursors are expected when the medium is opaque in a broad spectral region, coherent transients of another kind are

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obtained in the opposite case where the width of the opacity region is very small compared to the resonance frequency ω_0. They have been naturally named resonant precursors [24] and also Sommerfeld-Brillouin precursors [25]. Indeed they may be seen as resulting from the coalescence of the Sommerfeld and Brillouin precursors, originating a well-marked beat when the optical thickness of the medium is large enough [26]. The conditions required to achieve experimental evidence of these precursors are relatively easy to meet. They have been actually observed in various systems, in particular in a molecular gas [26], in a solid-state sample with a narrow exciton line [25], and in clouds of cold atoms [27,28].

In the present paper we come back to the study of Sommerfeld and Brillouin precursors in a dense Lorentz medium, considering the limit where the medium is opaque in a spectral region with a large width compared to the resonance frequency. We remark that these conditions are met for the parameters considered by Brillouin [29] which are often referred to in the literature. We then succeed in obtaining *simple and explicit analytical expressions* of both precursors. When it is necessary, we determine the range of validity of these analytical solutions by comparing them to exact numerical solutions obtained by fast Fourier transform (FFT). The arrangement of our paper is as follows. In Sec. II, we outline the problem under consideration and give some general results that are useful in the following sections. Section III is devoted to the study of the Sommerfeld precursor. We establish the corresponding expression of the impulse response of the medium and apply it to obtain a general expression of the precursors obtained with causal incident fields. We examine in detail the particular cases where the incident field is discontinuous at the initial time or has the canonical form considered by Brillouin with eventually a finite rise time. We show in Sec. IV that, in a strictly asymptotic limit, the impulse response associated with the Brillouin precursor is Gaussian and that the Brillouin precursor has itself a Gaussian or Gaussian-derivative shape. The precursor obtained in a Debye medium is incidentally examined. A more general expression of the Brillouin precursor in the Lorentz medium is established in Sec. V, containing the previous one and that obtained by Brillouin as particular cases. The propagation in both media of pulses with a square or Gaussian envelope is finally examined in Sec. VI, and we determine the pulse parameters optimizing the Brillouin precursor. We conclude in Sec. VII by summarizing and discussing our main results.

II. GENERAL ANALYSIS

We consider a one-dimensional optical wave propagating in a Lorentz medium in the z direction, with an electric field linearly polarized in the x direction (x,y,z are Cartesian coordinates). We denote $e(0,t)$ as the algebraic amplitude of the field at time t for $z=0$ (inside the medium) and $e(z,t)$ as its value after a propagation distance z through the medium. The incident field $e(0,t)$ being given, the problem is to determine the transmitted field $e(z,t)$. We take for $e(0,t)$ the general form

$$e(0,t) = u(t)\cos(\omega_c t - \varphi), \tag{1}$$

including as particular cases the different forms considered in the literature. ω_c is the frequency of the optical carrier, φ is the phase (eventually time dependent), and $u(t) \geqslant 0$ is the amplitude modulation or field envelope. On the other hand, the medium is fully characterized in the frequency domain by its transfer function $H(z,\omega)$ relating the Fourier transform $E(z,\omega)$ of $e(z,t)$ to that $E(0,\omega)$ of $e(0,t)$ [30].

$$E(z,\omega) = H(z,\omega)E(0,\omega). \tag{2}$$

In the following, we take for t a retarded time equal to the real time minus the luminal propagation time z/c (retarded-time picture). $H(z,\omega)$ then reads

$$H(z,\omega) = \exp\left\{-i\frac{\omega z}{c}[\widetilde{n}(\omega) - 1]\right\}. \tag{3}$$

Here $\widetilde{n}(\omega)$ is the complex refractive index of the medium at frequency ω, which is for the Lorentz medium

$$\widetilde{n}(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 - 2i\gamma\omega}\right)^{1/2}, \tag{4}$$

where ω_0 is the resonance frequency, γ is the damping or relaxation rate, and ω_p is the so-called plasma frequency whose square is proportional to the number density of absorbers. $\mathrm{Re}[\widetilde{n}(\omega)]$ is the usual (real) refractive index $n(\omega)$, and the absorption coefficient $\alpha(\omega)$ *for the amplitude* is given by the relation $\alpha(\omega) = -(\omega/c)\mathrm{Im}[\widetilde{n}(\omega)]$.

In the time domain, the medium will be characterized by its impulse response $h(z,t)$, which is the inverse Fourier transform of $H(z,\omega)$, and the transmitted signal $e(z,t)$ is given by the convolution product [30]

$$e(z,t) = h(z,t) \otimes e(0,t). \tag{5}$$

Some general properties of $h(z,t)$ and $e(z,t)$ can be deduced from Eqs. (3)–(5). First, $h(z,t)$ fulfills the condition of relativistic causality, namely, $h(z,t) = 0$ for $t < 0$ [31]. Its area reads as $\int_{-\infty}^{+\infty} h(z,t)dt = H(z,0) = 1$. It remains thus constant and normalized to unity regardless of the propagation distance z. Consequently, $E(z,0) = E(0,0)$; that is,

$$\int_{-\infty}^{+\infty} e(z,t)dt = \int_{-\infty}^{+\infty} e(0,t)dt. \tag{6}$$

The area of the optical field (to distinguish from that of its envelope) is conserved during the propagation. Finally, the fact that $H(z,\infty) = 1$ entails that $h(z,t)$ will start with a Dirac delta function $\delta(t)$. This implies that the propagation of the very beginning of any incident signal $e(0,t)$ will always proceed undisturbed at the velocity c, in agreement with Voigt's remark on Sommerfeld's communication [1]. The previous results are valid whatever the values of the parameters may be.

We examine now under what conditions the medium is opaque in a broad spectral region. To be definite, we will consider that the medium is opaque at the frequency ω when its optical thickness $\alpha(\omega)z$ exceeds 20, the amplitude transmission $|H(z,\omega)| = \exp[-\alpha(\omega)z]$ being then about 2×10^{-9}. Following Sommerfeld [2], we characterize the propagation distance by the parameter $\xi = \omega_p^2 z/2c$, which is homogeneous to a frequency. For large propagation distances $\gamma\xi/(10\omega_0^2) \gg 1$, and it is easily derived from Eq. (4) that the medium will then be opaque in the broad spectral region $\omega_- \leqslant \omega \leqslant \omega_+$, with $\omega_+/\omega_0 \approx \sqrt{\gamma\xi/(10\omega_0^2)}$ and $\omega_-/\omega_0 \approx$

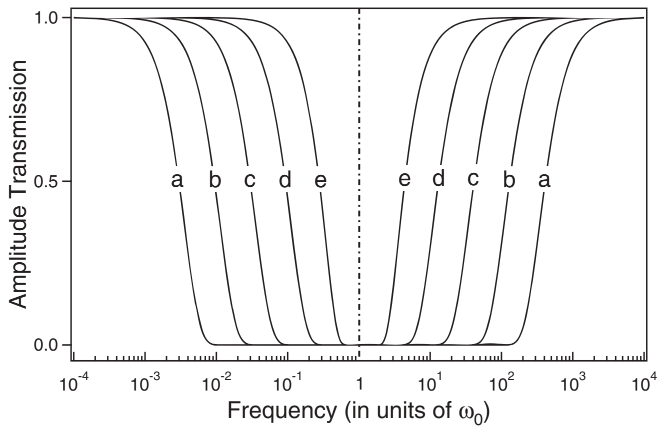

FIG. 1. Amplitude transmission $|H(z,\omega|$ of the medium as a function of the frequency modulus $|\omega|$ (logarithmic scale). Parameters (units of ω_0) : $\omega_p = 1.11$, $\gamma = 0.0707$ and $\xi = 8.31 \times 10^5$ for the curve (a) corresponding to the Brillouin choice ($z = 10^{-2}$ m). The curves (b), (c),(d) and (e) are obtained for propagation distances (and thus ξ) respectively 10, 100, 1 000 and 10 000 times smaller.

$(1 + \omega_p^2/\omega_0^2)^{1/4}\sqrt{10\omega_0^2/(\gamma\xi)}$. The inequality $\gamma\xi/(10\omega_0^2) \gg 1$ is oversatisfied for the parameter values considered by Brillouin [29], namely, $\omega_0 = 4 \times 10^{16}$ s^{-1}, $\omega_p^2 = 1.24\,\omega_0^2$, $\gamma^2 = \omega_0^2/200$, and $z = 10^{-2}$ m. We then get $\xi = 3.0324 \times 10^{21}$ s^{-1} and $\gamma\xi/(10\omega_0^2) \approx 5.87 \times 10^3$. To avoid reducing our study to a particular system or region of the spectrum, all the frequencies (the times) will be referred in the following to their natural unit ω_0 ($1/\omega_0$). Figure 1 shows the profiles of the amplitude transmission $|H(z,\omega)| = \exp[-\alpha(\omega)z]$ as a function of the reduced frequency ω/ω_0 in the Brillouin conditions [curve (a)] and for propagation distances of 10, 100, 1000, and 10 000 times shorter [curves (b) to (e)].

With the medium being opaque for $\omega_- < \omega < \omega_+$, the transfer function may be written as

$$H(z,\omega) = H_S(z,\omega) + H_B(z,\omega), \tag{7}$$

with $H_S(z,\omega) \approx 0$ for $\omega < \omega_+$ and $H_B(z,\omega) \approx 0$ for $\omega > \omega_-$. H_S and H_B are respectively associated with the Sommerfeld and the Brillouin precursors. For $\omega = 0$, $H_S(z,0) \approx 0$ and $H_B(z,0) \approx H(z,0) = 1$. As long as ω_c lies in the opacity region, this implies that the Sommerfeld precursor will have a zero area, while the area of the Brillouin precursor will be equal to that of the incident field.

The formation of the optical precursors is generally governed by combined effects of attenuation (considered above) and dispersion. The dispersion effects can be soundly characterized by the group delay $\tau_g(z,\omega) = -d\Phi/d\omega = z/v_g(\omega) - z/c$, where $\Phi(z,\omega)$ is the argument of $H(z,\omega)$ and $v_g(\omega)$ is the group velocity [31]. We remark that the regions of anomalous dispersion ($dn/d\omega < 0$) or of superluminal group velocity ($\tau_g < 0$) have a width smaller than ω_0 and are entirely comprised inside the opacity region. The corresponding frequencies will thus not directly contribute to the formation of precursors. For the high and low frequencies associated with the Sommerfeld and Brillouin precursors, respectively, we get the asymptotic forms $\tau_g \approx \xi/\omega^2$ [31] and $\tau_g \approx t_B + \omega^2/(\eta b^3)$, where

$$t_B = \frac{[n(0) - 1]z}{c} = \frac{2\xi}{\omega_p^2}\left[\left(1 + \frac{\omega_p^2}{\omega_0^2}\right)^{1/2} - 1\right], \tag{8}$$

$$b = \omega_0\left(3\frac{\xi}{\omega_0}\right)^{-1/3}\left(1 + \frac{\omega_p^2}{\omega_0^2}\right)^{1/6}, \tag{9}$$

$$\frac{1}{\eta} = 1 - \frac{4\gamma^2}{\omega_0^2}\left(1 + \frac{3\omega_p^2}{4\omega_0^2}\right)\Big/\left(1 + \frac{\omega_p^2}{\omega_0^2}\right). \tag{10}$$

$t_B = \tau_g(z,0) - \tau_g(z,\infty)$ is obviously indicative of the time delay of the Brillouin precursor (low frequency) with respect to the Sommerfeld precursor (high frequency). The two precursors will be fully separated when t_B is much larger than the damping time $1/\gamma$. Since $\gamma t_B = O(\gamma\xi/\omega_0^2)$, this condition is automatically fulfilled when the condition of a broad opacity region $[\gamma\xi/(10\omega_0^2) \gg 1]$ holds. Another important point is that τ_g is minimum (stationary) for $\omega \to \infty$ and $\omega \to 0$. As pointed out by Brillouin [4], this ensures that the precursors will not be washed out by the group velocity dispersion.

III. SOMMERFELD PRECURSOR

A. Transfer function $H_S(z,\omega)$ and impulse response

In the limit considered here $\omega^2 \geqslant \omega_+^2 \gg \omega_0^2$, and $H_S(z,\omega)$ takes the following asymptotic form, accounting for both dispersion (main contribution) and attenuation:

$$H_S(z,\omega) \approx \exp\left[-\frac{\xi}{i\omega + 2\gamma}\right]. \tag{11}$$

The corresponding impulse response $h_S(z,t)$ is easily determined by using standard results of Laplace transforms [32]. We get

$$h_S(z,t) = \delta(t) - \sqrt{\frac{\xi}{t}}J_1(2\sqrt{\xi t})e^{-2\gamma t}u_H(t), \tag{12}$$

where $J_n(s)$ and $u_H(t)$ respectively designate the Bessel function of the first kind of index n and the Heaviside unit-step function. Except for their very first oscillation, the Bessel functions $J_n(s)$ are perfectly approximated by their asymptotic form,

$$J_n(s) \approx \sqrt{\frac{2}{\pi s}}\cos\left(s - n\frac{\pi}{2} - \frac{\pi}{4}\right), \tag{13}$$

and the impulse response $h_S(z,t)$ can be characterized by an instantaneous frequency $\omega \approx d(2\sqrt{\xi t})/dt = \sqrt{\xi/t}$. The range of validity of Eq. (12) may be estimated by determining the change $\delta H_S(z,\omega)$ of $H_S(z,\omega)$ due to the first term neglected in the asymptotic expansion of $\ln[H_S(z,\omega)]$ used to obtain Eq. (11). We find $\delta H_S(z,\omega)/H_S(z,\omega) = O(\xi\omega_0^2/\omega^3)$, which is negligible when $\omega^3 \gg \xi\omega_0^2$, i.e., when $\xi^{1/2} \gg \omega_0^2 t^{3/2}$. In fact, Eq. (12) fits very well the exact impulse response as soon as $\xi^{1/2}$ exceeds $\omega_0^2 t^{3/2}$ by a factor $\sqrt{10}$ (half an order of magnitude). This is achieved as long as $t \leqslant t_S$, with

$$\omega_0 t_S = \sqrt[3]{\frac{\xi}{10\omega_0}}. \tag{14}$$

In a strict asymptotic limit ($z \to \infty$), $t_S \to \infty$ and $\exp(-2\gamma t_S) \to 0$. As expected, the entirety of the impulse response is then reproduced by Eq. (12).

B. Precursor originated by a causal incident field

The Sommerfeld precursor $e_S(z,t)$ is obtained by convoluting $h_S(z,t)$ with the incident field $e(0,t) = u(t)\cos(\omega_c t - \varphi)$ introduced in the general analysis [Eq. (1)]. We are mainly interested here in the physical case where the incident field is causal [$e(0,t) = 0$ for $t < 0$], with $u(t)$ being either a unit step $u_H(t)$ or a function *monotonously* rising from 0 to 1 with a rate $r \lesssim \omega_c$ for $t > 0$ (step or steplike modulation). The convolution product of Eq. (5) takes the form

$$e_S(z,t) = \int_{-\infty}^{t} h_S(z,\theta)e(0,t-\theta)d\theta, \tag{15}$$

which can be transformed by repeated integrations per parts to yield

$$e_S(z,t) = \sum_{n=0}^{\infty} d_n h_S^{(n+1)}(z,t). \tag{16}$$

Here d_n is the discontinuity of the nth derivative of $e(0,t)$ at the initial time [33] and $f^{(n)}(t)$ is a short-hand notation for $\int_{-\infty}^{t}\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_{n-1}} f(t_n)dt_n\cdots dt_2 dt_1$. In a frequency description, the previous result can be retrieved by expanding the Fourier transform $E(0,\omega)$ of $e(0,t)$ in powers of $1/i\omega$ and exploiting the equivalence between multiplication by $1/i\omega$ in the frequency domain and integration in the time domain [30]. Writing the impulse response under the form $h_S(z,t) = k_S(z,t)\exp(-2\gamma t)$, we easily show by means of standard Laplace procedures [32] that $k_S^{(n+1)}(z,t) = (t/\xi)^{n/2} J_n(2\sqrt{\xi t})u_H(t)$. Insofar as $k_S(z,t)$ is very rapidly varying compared to $\exp(-2\gamma t)$, $h_S^{(n+1)}(z,t) \approx k_S^{(n+1)}(z,t)\exp(-2\gamma t)$, and we finally get

$$e_S(z,t) \approx \sum_{n=0}^{\infty} d_n \left(\frac{t}{\xi}\right)^{n/2} J_n(2\sqrt{\xi t})\exp(-2\gamma t)u_H(t). \tag{17}$$

The nth term of the series has a maximal amplitude $a_0 = |d_0|$ at $t = t_0 = 0$ for $n = 0$ and

$$a_n = \frac{1}{\sqrt{\pi}}|d_n|\left(\frac{2n-1}{8e}\right)^{(2n-1)/4}\left(\frac{\gamma}{\xi}\right)^{1/4}(\gamma\xi)^{-n/2}, \tag{18}$$

at $t \approx t_n = (2n-1)/8\gamma$ for $n > 0$. Since $\xi \propto z$, Eq. (18) shows that, for large propagation distance, a_n rapidly decreases with n, so that a good approximation of the exact result is obtained by keeping only the first term $n = p$ of the series for which $d_p \neq 0$. In the frequency description, this amounts to restricting the asymptotic expansion of $E(0,\omega)$ to its first nonvanishing term [7]. We then get

$$e_S(z,t) \approx d_p \left(\frac{t}{\xi}\right)^{p/2} J_p(2\sqrt{\xi t})\exp(-2\gamma t)u_H(t). \tag{19}$$

Denoting q as the next integer following p for which $d_q \neq 0$, Eq. (19) is exact when $\varepsilon = a_q/a_p \approx 0$, and $\exp(-2\gamma t_S) \approx 0$. These conditions are met in the strict asymptotic limit and are closely approached for the propagation distance considered by Brillouin. At distances that may be 1000 times smaller (simple asymptotic limit), we shall see that Eq. (19) enables us to correctly reproduce the essential features of the precursor originated by representative incident fields.

C. Precursor originated by a discontinuous incident field

We consider first the instructive case where $e(0,t) = u_H(t)\cos(\omega_c t)$ for which $p = 0$ with $d_0 = 1$ [33] and $q = 2$ with $d_2 = -\omega_c^2$. Equation (19) then reads as

$$e_S(z,t) \approx J_0(2\sqrt{\xi t})\exp(-2\gamma t)u_H(t), \tag{20}$$

with $\varepsilon \approx 0.13\omega_c^2\gamma^{-3/4}\xi^{-5/4}$ [see Eq. (18)]. The precursor does not depend on ω_c, and the initial discontinuity of the incident field is integrally transmitted, in agreement with the general analysis. For $\omega_c < \omega_+ = \sqrt{\gamma\xi/10}$ (opacity condition), ε is always smaller than $0.013(\gamma/\xi)^{1/4}$, which is about 2.2×10^{-4} in the Brillouin conditions and 1.2×10^{-3} for a propagation distance 1000 times smaller (simple asymptotic limit). In the first case, $\omega_0 t_S = 44$ and $\exp(-2\gamma t_S) \approx 2 \times 10^{-3}$. As previously indicated, we are then close to the strict asymptotic limit, and the precursor is perfectly reproduced by its asymptotic form at any time where it has a significant amplitude. This remark also holds for the cases considered in the following sections. In the simple asymptotic limit $\omega_0 t_S = 4.4$, and as expected, Eq. (20) perfectly fits the exact solution for $\omega_0 t \leqslant 4.4$. For larger times, the fit remains very good except for a slight drift of the instantaneous frequency of the oscillations whose envelope is very well reproduced at any time (Fig. 2).

D. Precursor originated by the canonical incident field of Sommerfeld and Brillouin

Following Sommerfeld and Brillouin, most authors have considered an incident field of the *canonical form* $e(0,t) = u_H(t)\sin(\omega_c t)$ for which $p = 1$ with $d_1 = \omega_c$ and $q = 3$ with $d_3 = -\omega_c^3$. We then get

$$e_S(z,t) \approx \omega_c\sqrt{\frac{t}{\xi}} J_1(2\sqrt{\xi t})\exp(-2\gamma t)u_H(t), \tag{21}$$

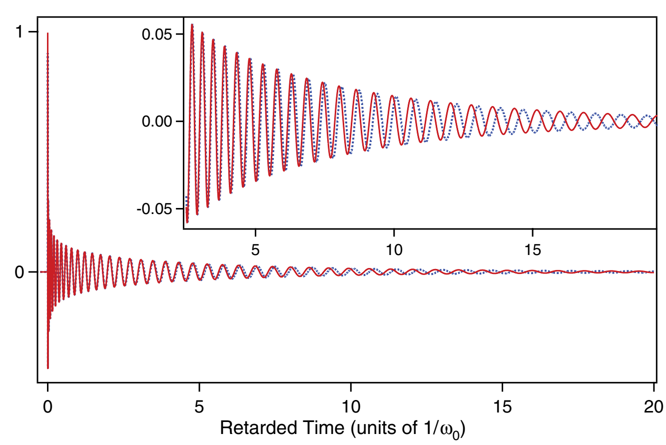

FIG. 2. (Color online) Sommerfeld precursor originated by the incident field $\cos(\omega_c t)u_H(t)$. The solid (dashed) line is the exact numerical solution (the approximate analytic solution). Parameters (units of ω_0) are $\omega_c = 1$, $\omega_p = 1.11$, $\gamma = 0.0707$, and $\xi = 831$. The inset is an enlargement of the tail of the precursor.

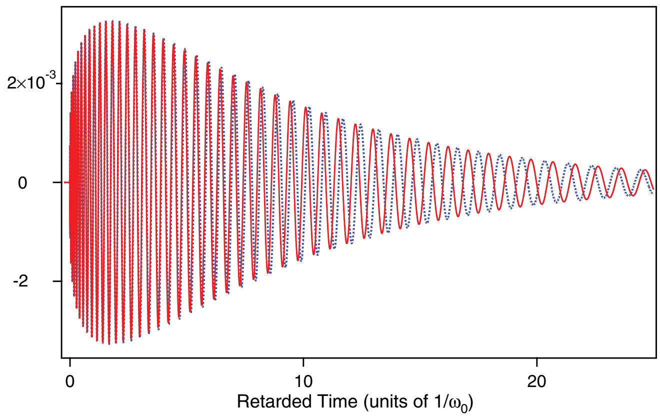

FIG. 3. (Color online) Sommerfeld precursor originated by the canonical incident field $\sin(\omega_c t)u_H(t)$. The solid (dashed) line is the exact numerical solution (the approximate analytic solution). Parameters are as in Fig. 2.

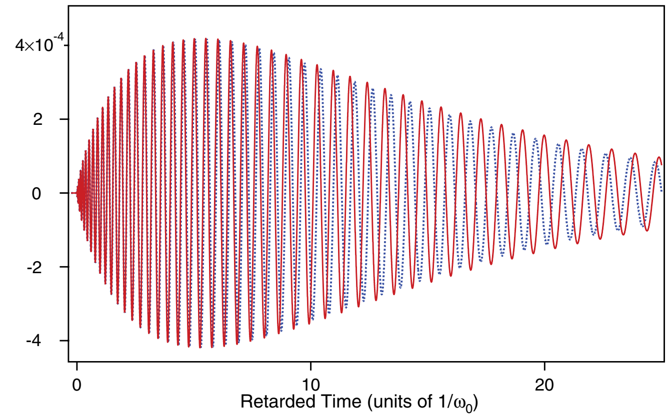

FIG. 4. (Color online) Sommerfeld precursor originated by the incident field $e(0,t) = \tanh(rt)\,\sin(\omega_c t)\,u_H(t)$. The solid (dashed) line is the exact numerical solution (the approximate analytic solution) obtained for $r = \omega_0$. Other parameters are as in Fig. 2.

with $\varepsilon \approx 0.34(\omega_c^2/\gamma\xi)$. The result given in Eq. (21) differs from that originally obtained by Sommerfeld [2] by the presence of the damping term $\exp(-2\gamma t)$. Though the formation of the Sommerfeld precursor is mainly governed by the medium dispersion, the presence of this term (associated with the absorption) is obviously necessary to avoid having $e_S(z,t)$ diverge with time. The precursor attains its maximum at $t \approx t_1 = 1/(8\gamma)$ ($\omega_0 t_1 = 1.77$), and its amplitude $a_S = a_1 \approx 0.26\,\omega_c\gamma^{-1/4}\xi^{-3/4}$ is *proportional to* ω_c. For $\omega_c = \omega_0$, $a_S \approx 1.8 \times 10^{-5}$ with $\varepsilon \approx 5.8 \times 10^{-6}$ in the Brillouin conditions, whereas $a_S \approx 3.25 \times 10^{-3}$ with $\varepsilon \approx 5.8 \times 10^{-3}$ in the simple asymptotic limit. In the latter case, Fig. 3 shows that Eq. (21) actually fits very well the exact result for $t \leqslant t_S$, again with a slight drift of the instantaneous frequency of the oscillations for $t > t_S$. In order to check the proportionality of the precursor to ω_c, we have compared the exact forms of $(\omega_0/\omega_c)\,e_S(z,t)$ obtained when ω_c lies at the boundaries ω_- or ω_+ of the opacity region to that obtained when $\omega_c = \omega_0$. As expected, we have found that the three results are nearly indistinguishable, except for an amplitude 1.3% larger for $\omega_c = \omega_+$ (below the corresponding value of ϵ, namely, $\varepsilon = 0.034$). For this value of ω_c, the amplitude of the precursor is $a_S \approx 0.082(\gamma/\xi)^{1/4}$, which is 1.40×10^{-3} in the Brillouin conditions and 7.9×10^{-3} in the simple asymptotic limit.

E. Rise-time effects

A gradual turning on of the incident field is expected to reduce the amplitude of the Sommerfeld precursor. To study this so-called rise-time effect, Ciarkowski [14,17] has considered the incident field $e(0,t) = \tanh(rt)\sin(\omega_c t)u_H(t)$ whose envelope has a $10-90\%$ rise time $T_r \approx 1.37/r$. In this case $p = 2$ with $d_2 = 2r\omega_c$, $q = 4$ with $d_4 = -4\omega_c r(2r^2 + \omega_c^2)$, and the asymptotic form of the precursor reads

$$e_S(z,t) \approx 2\omega_c r \left(\frac{t}{\xi}\right) J_2(2\sqrt{\xi t})\exp(-2\gamma t)u_H(t), \qquad (22)$$

with $\varepsilon \approx 1.21(2r^2 + \omega_c^2)/\gamma\xi$. The precursor attains its maximum at $t \approx t_2 = 3/(8\gamma)$ ($\omega_0 t_2 \approx 5.3$) with an amplitude $a_S = a_2 \approx 0.26\,r\omega_c\gamma^{-3/4}\xi^{-5/4}$. Compared to the precursor obtained with the canonical incident field [Eq. (21)], the maximum is shifted to larger time ($t_2 = 3t_1$), and its amplitude is reduced by a factor $\rho \approx \sqrt{\gamma\xi}/r$. Figure 4, obtained in the simple asymptotic limit, shows that Eq. (22) fits quite satisfactorily the exact precursor, although its maximum now lies at a time slightly larger than t_S. To check that the precursor is mainly determined by the lowest-order initial discontinuity of the incident field regardless of its subsequent evolution, we have compared the precursor obtained when the envelope $\tanh(rt)u_H(t)$ is replaced by $(1 - e^{-rt})u_H(t)$, having the same initial discontinuity. Though $q = 3$ (instead of 4) and $T_r \approx 2.20/r$ (instead of $1.37/r$), we have found that the precursor is actually very close to the previous one.

Other things being equal, the reduction of the amplitude of the precursor is more and more important when the incident field is applied more and more smoothly, that is, when the order p of its initial discontinuity increases. It is easily deduced from Eq. (18) that for $p \geqslant 2$, $\rho = O[(\sqrt{\gamma\xi}/r)^{p-1}] \propto (T_r\sqrt{z})^{p-1}$. In light of this result, dramatic rise-time effects are expected when the incident field is ideally smooth, i.e., analytic with continuous derivatives at every point. Such fields have been considered [13,16,34], though they are not causal and, strictly speaking, not physically realizable (in the sense of the linear-system theory). We have made numerical simulations for $e(0,t) = \sin(\omega_c t)[1 + \mathrm{erf}(rt)]/2$, where $\mathrm{erf}(s)$ designates the error function. For z and $r = \omega_0$ as in Fig. 4, we get $\rho \approx 1.4 \times 10^3$ instead of $\rho \approx 7.7$ for $e(0,t) = \tanh(rt)\,\sin(\omega_c t)\,u_H(t)$.

IV. BRILLOUIN PRECURSOR IN THE STRICT ASYMPTOTIC LIMIT

A. Transfer function $H_B(z,\omega)$ and impulse response

In the limit considered now $\omega^2 \leqslant \omega_-^2 \ll \omega_0^2$, and $H_B(z,\omega)$ is conveniently developed under the form

$$H_B(z,\omega) = \exp\left(\sum_{n=1}^{\infty} \frac{(-i\omega)^n}{n!} k_n(z)\right). \qquad (23)$$

Here $k_n(z)$ are the so-called cumulants, generally introduced in probability theory [32], but also quite useful to study deterministic signals [35,36]. The cumulants $k_1(z)$, $k_2(z)$, and $k_3(z)$ have remarkable properties. $k_1(z)$ and $k_2^{1/2}(z)$ are the center-of-mass and the root-mean-square durations, respectively, of the impulse response $h_B(z,t)$, which is the inverse Fourier transform of $H_B(z,\omega)$, whereas $\kappa(z) = k_3(z)/k_2^{3/2}(z)$ is its normalized asymmetry or skewness [32]. From Eqs. (3) and (4), we easily get $k_1 = t_B$ (as expected), $k_2 = 4\gamma/(3b^3)$, $k_3 = -2/(\eta b^3)$, and $\kappa = -(1/4\eta)(3b/\gamma)^{3/2}$, where t_B, b, and η are defined by Eqs. (8)–(10). When $z \to \infty$ (strict asymptotic limit), $\kappa \propto b^{3/2} \propto z^{-1/2} \to 0$, and the expansion of Eq. (23) may be limited to the term $n = 2$. Taking a new origin of time at $t = t_B$, the transfer function then reads as

$$H_B(z,\omega) \approx \exp\left(-\frac{\omega^2}{4\beta^2}\right), \tag{24}$$

where $\beta = \sqrt{3b^3/8\gamma} \propto 1/\sqrt{z}$ is very small compared to ω_0. This Gaussian form is that of the normal distribution derived by means of the central limit theorem in probability theory. This theorem can also be used to obtain an approximate evaluation of the convolution of n deterministic functions [30]. It can be applied to our case by splitting the medium into n cascaded sections, with $h_B(z,t)$ being the convolution of the impulses responses of each section. By calculating the inverse Fourier transform of $H_B(z,\omega)$, we get

$$h_B(z,t) = \frac{\beta}{\sqrt{\pi}} \exp(-\beta^2 t'^2), \tag{25}$$

where $t' = t - t_B$. The impulse response has a duration (amplitude) proportional (inversely proportional) to $\sqrt{z}$, with an area constantly equal to 1 (in agreement with the general analysis). We remark that the approximation leading to Eqs. (24) and (25), valid in the strict asymptotic limit, amounts to neglecting the effects of the group-delay dispersion, the formation of the Brillouin precursor being then governed by the frequency dependence of the medium attenuation (dominant-attenuation limit).

The Gaussian forms of Eqs. (24) and (25) are not specific to the Lorentz medium but have some generality [37]. They hold for the Debye medium [38], for some random media [39], and, more generally, whenever the transfer function of the medium can be expanded in cumulants and the propagation distance is such that $|\kappa| \ll 1$. Stoudt *et al.* [38] showed in particular that the results of their experiments on water (Debye medium) at decimetric wavelengths can be numerically reproduced by neglecting the group-delay dispersion, that is the approximation made to obtain the analytical result of the Eq. (24). See also [40–43]. Using a purely temporal approach, Karlsson and Rikte [12] remarked early on that the impulse response of the Debye medium is very close to a normalized Gaussian. This property is obviously a consequence of the previous analysis. The complex refractive index now reads as $\widetilde{n}(\omega) = [1 + (n_0^2 - 1)/(1 + i\omega\tau)]^{1/2}$, where n_0 is the refractive index for $\omega \to 0$ and τ is the relaxation time for the orientation of the polar molecules [41]. Including $\widetilde{n}(\omega)$ in Eq. (3) and following the procedure used for the Lorentz medium, we easily get $\beta = [2(n_0^2 - 1)\tau z/cn_0]^{-1/2}$, and taking into account that $n_0^2 \gg 1$, $\kappa \approx 2.25\sqrt{c\tau/n_0 z}$. Note that β and κ depend on z as $1/\sqrt{z}$ (as in the Lorentz medium). The normalized Gaussian of Eq. (25) will thus also be obtained for sufficient propagation distances. Using the parameters of water [41], namely, $n_0 = \sqrt{79}$ and $\tau = 8.5 \times 10^{-12}$ s, we find that the skewness of 5.2%, obtained in a Lorentz medium for a propagation distance more than four orders of magnitude larger than the optical wavelengths considered, is now attained for a propagation distance $z \approx 0.55$ m, comparable to the wavelengths involved in the experiments reported in [38]. Despite strongly different scales, Brillouin precursors in the Lorentz medium in the strict asymptotic limit and in the Debye medium pertain to the same physics, namely, that of the dominant-attenuation limit, and will be described by the same laws. On the other hand, the Debye medium is fully opaque at high frequency, and Sommerfeld precursors cannot be generated in this medium.

B. Precursor generated by an incident field of nonzero area

The Brillouin precursor generated by an arbitrary incident field $e(0,t)$ is obtained by convoluting the latter with $h_B(z,t)$ or by multiplying its Fourier transform $E(0,\omega)$ by $H_B(z,\omega)$ and determining the inverse Fourier transform of the product. We consider first the case where $e(0,t)$ is rapidly varying compared to $h_B(z,t)$. This requires in particular that $\omega_c \gg \beta$. Compared to $E(0,\omega)$, $H_B(z,\omega)$ then appears as a narrow peak centered on $\omega = 0$, and provided that $E(0,0) \neq 0$, $E_B(z,\omega) \approx E(0,0)\, H_B(z,\omega)$. Remembering that $E(0,0)$ is the algebraic area $\mathcal{A}$ of the incident field (see Sec. II), we finally get

$$e_B(z,t) \approx \mathcal{A} h_B(z,t) = \frac{\mathcal{A}\beta}{\sqrt{\pi}} \exp(-\beta^2 t'^2). \tag{26}$$

For the canonical incident field $\sin(\omega_c t)u_H(t)$, $E(0,0) = 1/\omega_c$, and the precursor has an amplitude $a_B = \beta/(\omega_c\sqrt{\pi})$ *inversely proportional* to ω_c (no matter its value provided that $\omega_c \gg \beta$) and to $\sqrt{z}$. Note that the law $a_B \propto 1/\sqrt{z}$, sometimes considered as general, is only valid in the strict asymptotic limit considered here (for which $|\kappa| \ll 1$). Figure 5 shows that the

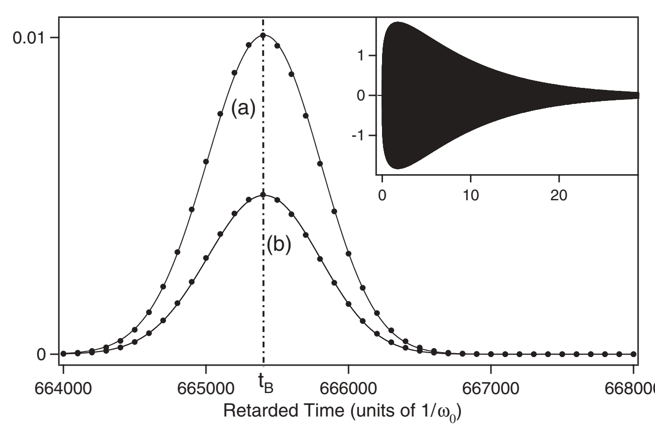

FIG. 5. Brillouin precursor obtained under the Brillouin conditions, namely, for $\omega_c = 0.1$, $\omega_p = 1.11$, $\gamma = 0.0707$, and $\xi = 8.31 \times 10^5$ (units of ω_0). For these parameters, $\omega_0 t_B \approx 6.654 \times 10^5$ and $\beta \approx 1.78 \times 10^{-3}\omega_0 = 1.78 \times 10^{-2}\omega_c$. Solid lines (dots) are the exact numerical solutions (the analytic solutions). Curve (a) is the precursor obtained with the canonical incident field $\sin(\omega_c t)u_H(t)$. The precursor of curve (b) is originated by the incident field $e(0,t) = \sin(\omega_c t)[1 + \mathrm{erf}(rt)]/2$ for $r = \omega_c/2\sqrt{2}$. The inset shows the Sommerfeld precursor magnified by a factor 10^6 obtained with the conditions of curve (a). It fully vanishes under the conditions of curve (b).

precursor obtained in *all* the Brillouin conditions [curve (a)] is perfectly fitted by the Gaussian form of Eq. (26). We incidentally note that, for the carrier frequency retained by Brillouin ($\omega_c = \omega_0/10$), the medium is fully opaque at this frequency [$\alpha(\omega_c)z \approx 800$], in contradiction to his artist's view showing a main field (at ω_c) larger than the precursors. On the other hand, the condition $\omega_c \gg \beta$ is well satisfied. The inset in Fig. 5 shows the Sommerfeld precursor obtained under the same conditions. As already mentioned, it is perfectly fitted by the analytical expression of Eq. (21). Note, however, that its amplitude is about four orders of magnitude smaller than that of the Brillouin precursor. Equation (26) also holds when the envelope of the incident field rises in a finite time provided that both the rate r and the frequency ω_c are large compared to β. Curve (b) of Fig. 5 shows the Brillouin precursor generated by the incident field $e(0,t) = \sin(\omega_c t)[1 + \mathrm{erf}(rt)]/2$. We have then $E(0,0) = (1/\omega_c)\exp(-\omega_c^2/4r^2)$, and the area of the incident pulse, equal to $1/\omega_c$ for $r \to \infty$, falls to $1/2\omega_c$ for $r = \omega_c/2\sqrt{\ln(2)}$ ($r \approx 0.60\omega_c$). As expected, the Brillouin precursor is identical to the previous one with the amplitude reduced by half, and the corresponding Sommerfeld precursor completely vanishes.

C. Precursor originated by an incident field of zero area

Even if $\omega_c, r \gg \beta$, Eq. (26) obviously fails when $\mathcal{A} = E(0,0) = 0$. This occurs in particular in the extreme case where the incident field is instantaneously turned on, with $e(0,t) = \cos(\omega_c t)u_H(t)$. It is then necessary to consider the next term in the expansion of $E(0,\omega)$ in powers of $i\omega$. We get in this case $E(0,\omega) \approx i\omega/\omega_c^2$ and $E_B(z,\omega) \approx i\omega H_B(z,\omega)/\omega_c^2$. Using the correspondence $i\omega \leftrightarrow d/dt$ between frequency and time descriptions [30] and denoting by a dot the time derivative, we finally get

$$e_B(z,t) \approx \frac{1}{\omega_c^2}\dot{h}_B(z,t) = -\frac{2\beta^2}{\omega_c^2\sqrt{\pi}}\,\beta t'\exp(-\beta^2 t'^2). \tag{27}$$

As shown Fig. 6 [curve (a)], the analytical expression of Eq. (27) perfectly fits the exact numerical results obtained by FFT. The precursor is a Gaussian derivative with a peak amplitude $a_B = [2/(\pi e)]^{1/2}(\beta/\omega_c)^2$, which is smaller than that attained with the canonical incident field by a factor $\omega_c\sqrt{e}/(\beta\sqrt{2})$ ($\approx$65 in *all* the Brillouin conditions) and decreases much more rapidly with the propagation distance (as $1/z$ instead of as $1/\sqrt{z}$). We, however, remark that the precursor so obtained is not robust. Indeed it suffices that the incident field suffers a short rise time to retrieve a precursor mainly governed by the area law of Eq. (26). To illustrate this point, we have again considered an incident field of the form $(1 - e^{-rt})\cos(\omega_c t)u_H(t)$ that tends to $\cos(\omega_c t)u_H(t)$ for $r \to \infty$. For $r \gg \omega_c$ (very short rise time), $E(0,\omega) \approx -1/r + i\omega/\omega_c^2$. The incident field has gained a (negative) area $\mathcal{A} = -1/r$. The precursor is then the sum of two contributions given by Eq. (26) with $\mathcal{A} = -1/r$ and by Eq. (27). Curve (b) of Fig. 6 shows the result obtained when the two contributions have the same amplitude, that is, when $r/\omega_c = \omega_c\sqrt{e}/(\beta\sqrt{2}) \approx 65$. When r decreases by remaining large compared to ω_c, the Gaussian part of the precursor rapidly prevails on the Gaussian-derivative part, and as shown [curve (c)], the precursor becomes nearly Gaussian (downwards) for r as large as $20\omega_c$.

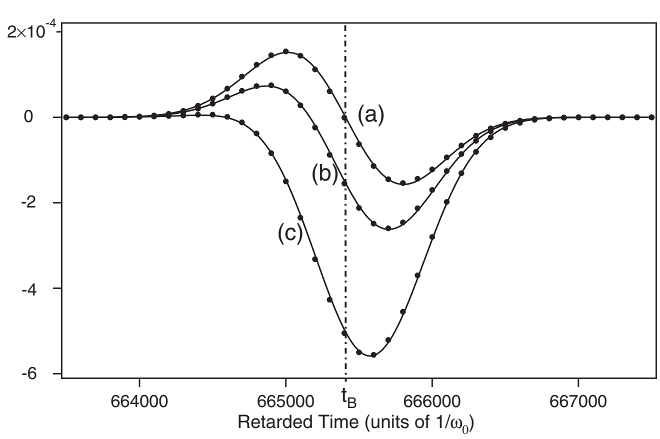

FIG. 6. Brillouin precursor obtained with the incident fields $(1 - e^{-rt})\cos(\omega_c t)u_H(t)$ for (a) $r \to \infty$, (b) $r = 65\omega_c$, and (c) $r = 20\omega_c$ (solid lines). Other parameters are as in Fig. 5. The dots correspond to the analytical solutions given by Eq. (27) or by the combination of this equation with Eq. (26).

D. Case where the carrier frequency lies below the opacity region

The previous results are valid for the Lorentz medium in the strict asymptotic limit (also as in the Debye medium) when $\omega_c \gg \beta$, that is, when ω_c lies in the opacity region. Fortunately enough, the simplicity of the Gaussian impulse response enables us to obtain exact expressions of the transmitted field for arbitrary values of the ratio ω_c/β. This occurs in the Lorentz medium when ω_c resides below the opacity region, and *direct* observations of the field transmitted under such conditions have been performed by Stoudt *et al.* in a Debye medium [38]. The transmitted field $e(z,t)$ is calculated directly in the time domain by convoluting $h_B(z,t)$ given Eq. (25) with the incident field. For the canonical incident field, the convolution product can be written as

$$e(z,t) = \frac{\beta}{\sqrt{\pi}}\int_{-\infty}^{t'} e^{-\beta^2\theta^2}\sin[\omega_c(t'-\theta)]d\theta. \tag{28}$$

After some simple transformations, we finally get

$$e(z,t) = \frac{1}{2}\,e^{-\omega_c^2/4\beta^2}\mathrm{Im}\left\{\left[1+\mathrm{erf}\left(\beta t'+\frac{i\omega}{2\beta}\right)\right]e^{i\omega_c t'}\right\}, \tag{29}$$

where $e^{-\omega_c^2/4\beta^2} \approx e^{-\alpha(\omega_c)z}$ and, as previously, $t' = t - t_B$. For $t' \to \infty$, $e(z,t)$ tends to $e^{-\omega_c^2/4\beta^2}\sin(\omega_c t')$, which is nothing more than the steady state or main field, which is not negligible when ω_c and β are comparable. If we take t_B ($1/\beta$) as the time origin (time unit), the transmitted field only depends on the ratio ω_c/β, *regardless of the particular system considered*. When $\omega_c \gg \beta$, it tends to $\beta\exp(-\beta^2t'^2)/(\omega_c\sqrt{\pi})$, in agreement with Eq. (26), with the main field then being negligible. When $\omega_c \geqslant 4\beta$, Eq. (29) is well approximated by the expression

$$e(z,t) \approx \frac{1+\mathrm{erf}(\beta t')}{2}\,\sin(\omega_c t')\,e^{-\alpha(\omega_c)z} + \frac{\beta'}{\omega_c\sqrt{\pi}}\,e^{-\beta'^2t'^2}, \tag{30}$$

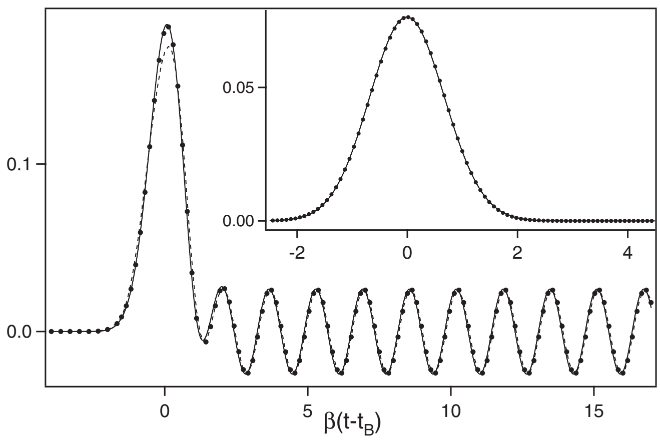

FIG. 7. Brillouin precursor and main field obtained for $\omega_c \approx 3.84\beta$ as a function of $\beta(t-t_B)$. The solid line, the dots, and the dashed line are, respectively, the exact numerical solution, the analytical solution given by Eq. (29), and its approximate form given by Eq. (30). The inset shows the Brillouin precursor obtained for $\omega_c \approx 7.67\beta$. The two analytical solutions are indistinguishable in this case, and the amplitude of the main field is negligible.

where $\beta' = \beta(1+2\beta^2/\omega_c^2) \to \beta$ for $\omega_c \gg \beta$. The first (second) term of Eq. (30) obviously corresponds to the main field (the Brillouin precursor). Figure 7 shows the transmitted field as a function of $\beta t' = \beta(t-t_B)$ for $\omega_c \approx 3.84\beta$ and $\omega_c \approx 7.67\beta$ (inset). In the study on water (Debye medium) at decimetric wavelengths [38], these values are obtained with $\omega_c = 2\pi \times 10^9\ \mathrm{s}^{-1}$ for $z = 0.75$ m and $z = 3$ m, respectively. As expected, Eq. (29) perfectly fits the exact numerical result in both cases. Equation (30) provides a good approximation for $\omega_c \approx 3.84\beta$ and an excellent approximation for $\omega_c \approx 7.67\beta$. In the latter case, the Brillouin precursor prevails over the main field, whose relative amplitude is negligible. The signals shown Fig. 7 are in good agreement with those directly observed in the experiments reported in [38].

V. EXTENDED EXPRESSION OF THE BRILLOUIN PRECURSOR

We come back in this section to the Brillouin precursor in the Lorentz medium. Numerical simulations show that the solutions obtained in the strict asymptotic or dominant-attenuation limit continue to provide good (not too bad) approximations of the exact solutions when the propagation distance is 10 times (100 times) shorter than that considered by Brillouin [29], though the skewness κ then rises up to 16% (52%). For shorter distances, it is obviously necessary to take into account the effects of the group-delay dispersion neglected in the strict asymptotic approximation.

A. Transfer function $H_B(z,\omega)$ and impulse response

Taking into account the term in ω^3 in Eq. (23), the transfer function then reads

$$H_B(z,\omega) \approx \exp\left[-i\omega t_B - \frac{i}{3\eta b^3}(\omega^3 - 2i\eta\gamma\omega^2)\right], \quad (31)$$

where t_B, b, and η are defined by Eqs. (8)–(10), with $2\gamma/(3b^3) = 1/4\beta^2$. Remarking that $(\omega^3 - 2i\eta\gamma\omega^2)$ is the beginning of $(\omega - 2i\eta\gamma/3)^3$ and taking a new origin of time at $t_B + 4\eta\gamma^2/9b^3$, we get

$$H_B(z,\omega) \approx \exp\left[-\frac{i}{3\eta b^3}\left(\omega - \frac{2}{3}i\eta\gamma\right)^3 - \frac{\eta^2}{3}\left(\frac{2\gamma}{3b}\right)^3\right]. \quad (32)$$

By means of an inverse Fourier transform, we finally find

$$h_B(z,t) \approx B\ \mathrm{Ai}(-\eta^{1/3}bt'')\exp(-2\eta\gamma t''/3). \quad (33)$$

Here $B = \eta^{1/3}b\ \exp[-(\eta^2/3)(2\gamma/3b)^3]$, $t'' = t - t_B - 4\eta\gamma^2/9b^3$, and $\mathrm{Ai}(s)$ designates the Airy function. The range of validity of Eq. (33) can be roughly estimated by means of a strategy similar to that used for the Sommerfeld precursor. By taking into account the cumulants k_4 (correction of the attenuation) and k_5 (correction of the dispersion), the transfer function associated with the Brillouin precursor approximately reads $H_B(z,\omega)(1 - a_4\omega^4 - ia_5\omega^5)$, where $a_4 = -k_4/24 > 0$ and $a_5 = k_5/120 > 0$. $H_B(z,\omega)$ will be a good approximation if $a_4\omega^4$ and $a_5\omega^5$ are small compared to 1 (say $\leqslant 1/\sqrt{10}$). For the sake of simplicity, we take for the ratios ω_p/ω_0 and γ/ω_0 the values retained by Brillouin, representative of a dense Lorentz medium with moderate damping. We get then $\eta \approx 1.018 \approx 1$. In addition, in a cavalier manner, we assimilate ω to the instantaneous frequency derived from the asymptotic form $\mathrm{Ai}(-s) \approx \pi^{-1/2}s^{-1/4}\sin(2s^{3/2}/3 + \pi/4)$, which provides a good approximation of $\mathrm{Ai}(-s)$ when $s > 1$. We get thus $\omega \approx \sqrt{b^3t''}$. With all these hypotheses, we finally find that the corrections due to cumulants k_4 and k_5 will be small if $\omega_0 t'' \leqslant 2(\omega_0/b)^{3/2}$ and $\omega_0 t'' \leqslant (\omega_0/b)^{9/5}$, respectively. Despite the roughness of the procedure leading to these conditions, it will be shown below that they are realistic and even too severe.

B. Precursor generated by the canonical incident field

When $h_B(z,t)$ is slowly varying compared to $e(0,t)$, the Brillouin originated by the canonical incident field $\sin(\omega_c t)u_H(t)$ takes again the simple form $e_B(z,t) = \mathcal{A}\, h_B(z,t)$, that is,

$$e_B(z,t) \approx \frac{B}{\omega_c}\mathrm{Ai}(-\eta^{1/3}bt'')\exp(-2\eta\gamma t''/3). \quad (34)$$

It is assumed by writing Eq. (34) that the instantaneous frequency $\sqrt{b^3t''}$ is small compared to ω_c (say $\sqrt{b^3t''} \leqslant \omega_c/\sqrt{10}$) and that the conditions of the validity of $h_B(z,t)$ are met. All these restrictions are summarized by the inequality

$$\omega_0 t'' \leqslant \min[2(\omega_0/b)^{3/2},\ (\omega_0/b)^{9/5},\ \omega_0\omega_c^2/10b^3]. \quad (35)$$

Figure 8 shows the Brillouin precursor obtained in the simple asymptotic limit considered in the study of the Sommerfeld precursor (Fig. 3). The inequality of Eq. (35) then leads to $\omega_0 t \leqslant \min[750, 760, 840]$. Insofar as the amplitude of the precursor is negligible for $\omega_0 t = 750$, the analytical expression of Eq. (34) perfectly fits the exact numerical result.

Surprisingly enough, Eq. (34) remains a not too bad approximation of the exact result even when the opacity region is not broad in the sense given to this expression in the present paper. Figure 9 shows the precursor obtained at a distance ten times smaller than the previous one. Though the width of the opacity region is then of the order of ω_0 [see curve (e) of Fig. 1],

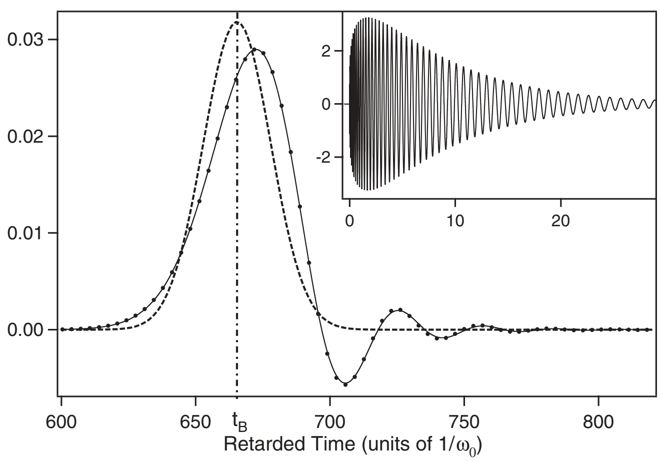

FIG. 8. Brillouin precursor obtained in the simple asymptotic limit with the canonical incident field $\sin(\omega_c t)u_H(t)$. Parameters (units of ω_0) are $\omega_c = 1$, $\omega_p = 1.11$, $\gamma = 0.0707$, and $\xi = 831$, leading to $\omega_0 t_B \approx 665.4$, $b \approx 8.44 \times 10^{-2}\omega_0$, and $\beta \approx 5.64 \times 10^{-2}\omega_0$. The solid line, the dots, and the dashed line are, respectively, the exact numerical solution, the analytical solution given by Eq. (34), and the Gaussian that would be obtained in the dominant-attenuation approximation. The conditions are those of Fig. 3. The corresponding Sommerfeld precursor magnified by a factor 1000 is given in the inset for reference.

the entirety of the first oscillation of the Brillouin precursor is very well fitted by Eq. (34). The corresponding Sommerfeld precursor (inset) is itself well reproduced by Eq. (21) up to its maximum.

C. Dominant-dispersion limit

The expression of the Brillouin precursor given by Eq. (34) obviously includes as a particular case the Gaussian obtained in the dominant-attenuation limit. In fact, retrieving the Gaussian precursor directly from Eq. (34) requires long and tedious calculations, and this probably explains why the Gaussian solution has been generally overlooked. Another particular form of Eq. (34), also of special importance, is that obtained when the damping is very small, so that the formation of the Brillouin precursor is mainly governed by the group-delay dispersion (dominant-dispersion limit). This requires in particular that $\gamma \ll b$. We then get $t'' \approx t - t_B$, $B \approx b$, and

$$e_B(z,t) \approx \frac{b}{\omega_c}\mathrm{Ai}[-b(t-t_B)]\exp\left[-\frac{2}{3}\gamma(t-t_B)\right]. \quad (36)$$

Except for the exponential damping term, this result was established by Brillouin himself by means of the stationary-phase method [4,44]. When the group-delay dispersion is fully dominant (when $\gamma/b < 1/100$), the precursor has a well-marked oscillatory behavior with a very weak damping, and its maximum practically coincides with the first maximum of $\mathrm{Ai}[-b(t-t_B)]$, attained for $t - t_B \approx 1.02/b$. The corresponding amplitude is $a_B \approx 0.536(b/\omega_c)$, which scales as $z^{-1/3}$ instead of as $z^{-1/2}$ in the strict or dominant-attenuation limit. Figure 10 shows an example of the Brillouin precursor obtained under such conditions ($\gamma/b \approx 3.9 \times 10^{-3}$). It is worth emphasizing that, since $b \propto z^{-1/3}$, the condition $\gamma/b \ll 1$ requires that the propagation distance is not too large. On the other hand, it should be large enough for the inequality of Eq. (35) to be satisfied for a time larger than or at least comparable to the half-maximum duration of the precursor. In fact, the most severe restriction originates from the condition $\omega_0(t - t_B) \leqslant (\omega_0/b)^{9/5}$ associated with the dispersion correction. When $\gamma \ll b$, we easily deduce from the asymptotic form of the Airy function that the half maximum of the precursor will be attained for $\omega_0(t - t_B) \approx 20(\omega_0/b)$. The precursor will thus be well reproduced by the expression $e_B(z,t) \approx (b/\omega_c)\mathrm{Ai}[-b(t-t_B)]$ beyond its half-maximum amplitude if $\gamma \ll b$ and if $(\omega_0/b)^{4/5} > 20$, that is, if $b/\omega_0 < 0.024$. The latter condition is approximately met in Fig. 10, where $b/\omega_0 = 0.026$. As expected, the maximum amplitude of the precursor is $a_B \approx 0.536(b/\omega_c) \approx 0.0165$, with $\exp[-2\gamma(t - t_B)/3] \approx 0.997$ at the corresponding time.

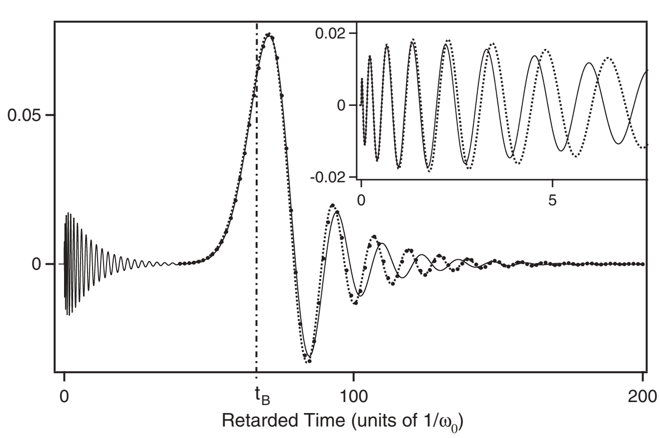

FIG. 9. Comparison of the Brillouin precursor obtained outside the asymptotic limit (solid line) with the analytical forms given by Eqs. (34) (dots) and (36) (dashed line). Parameters (units of ω_0) are $\omega_c = 1$, $\omega_p = 1.11$, $\gamma = 0.0707$, and $\xi = 83.1$, leading to $\omega_0 t_B \approx 66.54$, $b \approx 0.182\omega_0$, and $\beta \approx 0.178\omega_0$. The inset shows the corresponding Sommerfeld precursor (solid line) compared to the analytic form given by Eq. (21) (dashed line).

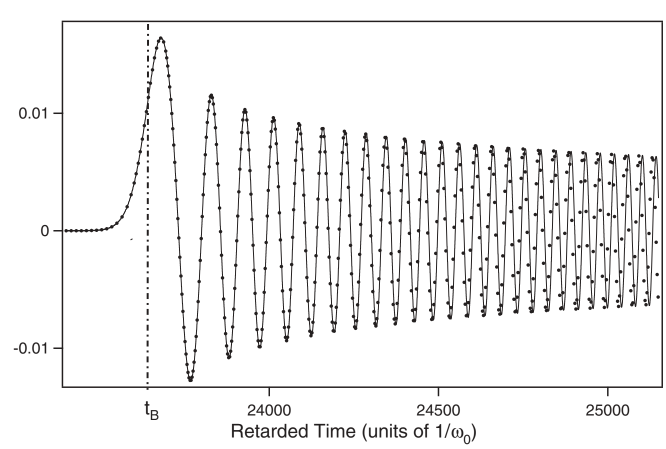

FIG. 10. Brillouin precursor in the dispersion-dominant limit. The solid line (dots) is the exact numerical solution (the analytical solution). Parameters (units of ω_0) are $\omega_c = 0.836$, $\omega_p = 1.11$, $\gamma = 10^{-4}$, and $\xi = 2.95 \times 10^4$, leading to $\omega_0 t_B \approx 2.3641 \times 10^4$, $b \approx 0.0257\omega_0$, and $\beta \approx 0.252\omega_0$. The carrier frequency ω_c is at the lower boundary of the opacity region $[\alpha(\omega_c)z \approx 20]$.

VI. PROPAGATION OF PULSES WITH A SQUARE OR GAUSSIAN ENVELOPE

Up to now, in the spirit of the pioneering work of Sommerfeld and Brillouin, we have considered incident fields of infinite duration. In actual or even numerical experiments, this duration is naturally finite. As a matter of fact the simulations made to corroborate our previous analytical calculations were made by using a square-wave modulation (eventually suitably filtered) and choosing a square duration long enough to avoid having the precursors generated by the rise and the fall of the square overlap. On the contrary, we consider in this section the case where the duration of the incident field is small compared to the time delay t_B separating the Brillouin precursor from the Sommerfeld precursor and does not exceed a few periods of the carrier. We will restrict the analysis to the Brillouin precursor. Indeed the Sommerfeld precursor, if it exists, is generally much smaller and will be often filtered out by rise-time effects, to which the Brillouin precursor is much less sensitive.

A. Square pulse

We consider first a square-modulated incident field $[u_H(t) - u_H(t-T)]\sin(\omega_c t)$. Of particular interest is the case where the square duration is an integer n of half periods of the carrier, that is, $T = nT_c/2 = n\pi/\omega_c$. The incident field can then be rewritten as $e(0,t) = u_H(t)\sin(\omega_c t) - (-1)^n u_H(t-T)\sin[\omega_c(t-T)]$, and the transmitted field reads as $e'(z,t) = e(z,t) - (-1)^n e(z,t-T)$, where $e(z,t)$ designates the transmitted field when only the incident field $u_H(t)\sin(\omega_c t)$ is on. This equation applies to the whole field and in particular to the Brillouin precursor to yield

$$e'_B(z,t) = e_B(z,t) - (-1)^n e_B(z,t-T), \tag{37}$$

where $e_B(z,t)$ is given by Eq. (26) or Eq. (34), depending on the system and the parameters considered. The two components of e'_B are of opposite (same) sign when n is even (odd) and are well separated when it is large enough, so that T significantly exceeds the duration of the elementary precursor. On the other hand, with $e_B(z,t)$ evolving slowly at the scale of T_c, the two components overlap and interfere if n is small. When $n = 2$ ($T = T_c$), as considered in [40,45], the two components interfere nearly destructively to give a precursor $e'_B(z,t) \approx T_c\dot{e}_B(z,t-T_c/2)$. The case where n is odd and, in particular, where $n = 1$ ($T = T_c/2$) is much more favorable. Indeed the two precursors then interfere constructively to yield a precursor $e'_B(z,t) \approx 2e_B(z,t-T_c/4)$ whose amplitude is twice that obtained with a step modulation. This result is not really a surprise since the pulse area is itself twice that of $u_H(t)\sin(\omega_c t)$. On the contrary the pulse area equals zero when n is even. The previous results are illustrated Fig. 11, which shows the Brillouin precursors obtained for $n = 1,2$ for a Lorentz medium when attenuation and dispersion comparably contribute to the formation of the Brillouin precursor (simple asymptotic limit).

When the detection of the Brillouin precursor is not time resolved, an important parameter is the integrated "energy" $W_B(z) = \int_{-\infty}^{+\infty} |e'_B(z,t)|^2 dt$ [18,21]. Thanks to the Parseval-Plancherel theorem [30], it can be written as

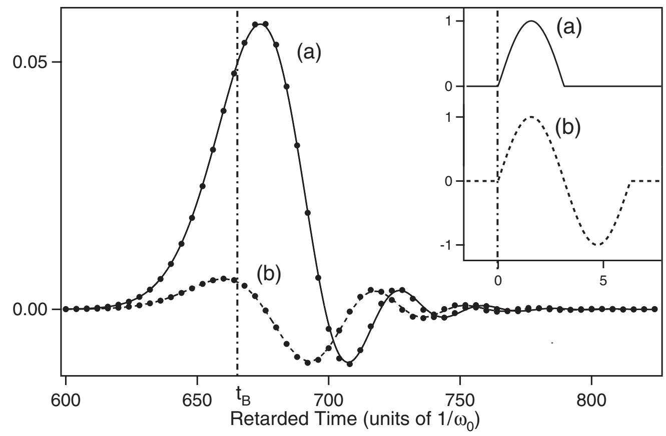

FIG. 11. Comparison of the Brillouin precursors $e'_B(z,t)$ generated by an incident square-modulated field of duration (a) $T = T_c/2$ and (b) $T = T_c$. The parameters are those of Fig. 8. The solid and dashed lines are the exact numerical solutions, indiscernible from the analytical solutions given by Eq. (37). The dots are the approximate solutions (a) $2e_B(z,t-T_c/4)$ and (b) $T_c\dot{e}_B(z,t-T_c/2)$. As expected the precursor amplitude for $T = T_c/2$ is twice that attained with a step-modulated field (see Fig. 8), whereas that attained for $T = T_c$ is much smaller. The inset shows the corresponding incident fields.

$$W_B(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |H_B(z,\omega)|^2 |E(0,\omega)|^2 d\omega. \tag{38}$$

In this expression all phases are eliminated, and $|H_B(z,\omega)|^2$ is reduced to $\exp(-4\gamma\omega^2/3b^3) = \exp(-\omega^2/2\beta^2)$ in both strict and simple asymptotic cases. For $T = T_c/2$, $|H_B(z,\omega)E(0,\omega)|^2 \approx (4/\omega_c^2)\exp(-\omega^2/2\beta^2)$, and we get an energy $W_B(z) = 2^{3/2}\pi^{-1/2}\beta/\omega_c^2$, which slowly decays with the propagation distance (as $1/\sqrt{z}$). On the other hand, for $T = T_c$, $|H_B(z,\omega)E(0,\omega)|^2 \approx (2\pi\omega/\omega_c^2)^2\exp(-\omega^2/2\beta^2)$ and $W_B(z) = (2\pi)^{3/2}\beta^3/\omega_c^4$. As expected, $W_B(z)$ then decays very rapidly with the propagation distance (as $z^{-3/2}$). As already mentioned, the previous expressions of the energy are valid regardless of the relative contributions of the absorption and the dispersion to the formation of the precursor. For the Debye medium and the Lorentz medium in the dominant-attenuation limit, it is also possible to derive from Eqs. (37) and (26) explicit expressions of the maximum amplitude $a'_B(z)$ of the precursor $e'_B(z,t)$. We find that this amplitude, equal to $2\beta/(\omega_c\sqrt{\pi}) \approx 1.1(\beta/\omega_c) \propto 1/\sqrt{z}$ when $T = T_c/2$, falls to $2\sqrt{2\pi/e}(\beta/\omega_c)^2 \approx 3.0(\beta/\omega_c)^2 \propto 1/z$ when $T = T_c$.

B. Gaussian pulse

The Gaussian pulses are probably the sole smooth pulses for which it is possible to obtain exact analytic expressions of the Brillouin precursor, both in the strict and simple asymptotic limits. Nonchirped incident fields of the forms $e^{-t^2/T^2}\cos(\omega_c t)$ and $e^{-t^2/T^2}\sin(\omega_c t)$ have been respectively considered by Oughstun and Balictsis [46] and by Ni and Alfano [47]. When the pulses are linearly chirped, it is convenient to consider them as the real and imaginary parts of $\widetilde{e}(0,t) = \exp(i\omega_c t - t^2/T^2 + i\chi^2 t^2)$, where χ^2 is the chirping parameter. The Fourier transforms of $\widetilde{e}(0,t)$ and of the corresponding transmitted field $\widetilde{e}_B(z,t)$ simply read $\widetilde{E}(0,\omega) = \widetilde{T}\sqrt{\pi}\exp[-(\omega-\omega_c)^2\widetilde{T}^2/4]$ and

$$\widetilde{E}_B(z,\omega) = \widetilde{\mathcal{A}} H_B(z,\omega)\exp(-\omega^2\widetilde{T}^2/4 + \omega\omega_c\widetilde{T}^2/2). \quad (39)$$

In these expressions $\widetilde{T} = T/\sqrt{1-i\chi^2T^2}$ and $\widetilde{\mathcal{A}} = \widetilde{T}\sqrt{\pi}\exp(-\omega_c^2\widetilde{T}^2/4)$ may be respectively seen as the (complex) duration and area of the pulse $\widetilde{e}(0,t)$. In the strict asymptotic limit [see Eq. (24)], we get

$$\widetilde{E}_B(z,\omega) = \widetilde{\mathcal{A}}\exp\left[-\frac{\omega^2}{4}\left(\frac{1}{\beta^2}+\widetilde{T}^2\right)+\omega\frac{\omega_c\widetilde{T}^2}{2}\right], \quad (40)$$

and $\widetilde{e}_B(z,t)$, the inverse Fourier transform of $\widetilde{E}_B(z,\omega)$, reads

$$\widetilde{e}_B(z,t) = \frac{\widetilde{\mathcal{A}}\beta}{\sqrt{\pi(1+\beta^2\widetilde{T}^2)}}\exp\left[-\frac{\beta^2(t'-i\omega_c\widetilde{T}^2/2)}{1+\beta^2\widetilde{T}^2}\right], \quad (41)$$

where $t' = t - t_B$. In the simple asymptotic limit (see Sec. V), Eqs. (31) and (39) yield

$$\widetilde{E}_B(z,\omega) = \widetilde{\mathcal{A}}\exp\left[-i\omega\left(t_B+\frac{i\omega_c\widetilde{T}^2}{2}\right)\right] \times\exp\left[-\omega^2\left(\frac{2\gamma}{3b^3}+\frac{\widetilde{T}^2}{4}\right)-i\omega^3\left(\frac{1}{3\eta b^3}\right)\right]. \quad (42)$$

This equation is easily transformed in an equation similar to Eq. (32). In this way, we find

$$\widetilde{e}_B(z,t) = \widetilde{\mathcal{A}}\,\widetilde{B}\,\mathrm{Ai}(-\eta^{1/3}b\,\widetilde{t})\exp\left(-\frac{2}{3}\eta\widetilde{\gamma}\,\widetilde{t}\right), \quad (43)$$

where $\widetilde{\gamma} = \gamma + 3b^3\widetilde{T}^2/8$, $\widetilde{B} = \eta^{1/3}b\,\exp[-(\eta^2/3)(2\widetilde{\gamma}/3b)^3]$, and $\widetilde{t} = t - t_B - 4\eta\widetilde{\gamma}^2/9b^3 - i\omega_c\widetilde{T}^2/2$. Finally, the precursors generated by the incident fields $e^{-t^2/T^2}\cos(\omega_c t + \chi^2t^2)$ and $e^{-t^2/T^2}\sin(\omega_c t+\chi^2t^2)$ respectively read as $e_{cos}(z,t) = \mathrm{Re}[\widetilde{e}_B(z,t)]$ and $e_{sin}(z,t) = \mathrm{Im}[\widetilde{e}_B(z,t)]$. Equation (41), Eq. (43), and the derived expressions of $e_{cos}(z,t)$ and $e_{sin}(z,t)$ hold whatever the duration of the incident pulse may be. However, as shown below, the amplitude of the Brillouin precursor will only be significant when this duration does not exceed a few periods of the carrier. In the Fourier transform $H_B(z,\omega)\widetilde{E}(0,\omega)$ of the transmitted field, $H_B(z,\omega)$ is then again much narrower than $\widetilde{E}(0,\omega)$, which may be approximated by its first order expansion in powers of ω. We get thus $\widetilde{E}_B(z,\omega) \approx \widetilde{\mathcal{A}}(1+\omega\omega_c\widetilde{T}^2/2)H_B(z,\omega)$, and finally,

$$\widetilde{e}_B(z,t) \approx \widetilde{\mathcal{A}}[h_B(z,t) - (i\omega_c\widetilde{T}^2/2)\dot{h}_B(z,t)]. \quad (44)$$

When there is no chirping, $\widetilde{T}$ and $\widetilde{\mathcal{A}}$ are real, with $\widetilde{T} = T$ and $\widetilde{\mathcal{A}} = \mathcal{A} = T\sqrt{\pi}\exp[-\omega_c^2T^2/4]$. Equation (44) then leads to

$$e_{cos}(z,t) \approx \mathcal{A}\,h_B(z,t) = T\sqrt{\pi}\exp\left[-\frac{\omega_c^2T^2}{4}\right]h_B(z,t), \quad (45)$$

$$e_{sin}(z,t) \approx -\frac{\mathcal{A}\omega_cT^2}{2}\,\dot{h}_B(z,t) = -\frac{\omega_cT^2}{2}\,\dot{e}_{cos}(z,t). \quad (46)$$

As illustrated in Fig. 12, obtained in the simple asymptotic limit, these approximate analytic solutions perfectly fit the exact numerical solution. It is easily deduced from Eq. (45) [Eq. (46)] that the amplitude of the precursor $e_{cos}(z,t)$ [$e_{sin}(z,t)$] is maximum for a pulse duration $T = T_m = \sqrt{2}/\omega_c$ [$\sqrt{6}/\omega_c$]. The energy of the precursors can be obtained by the method already used in

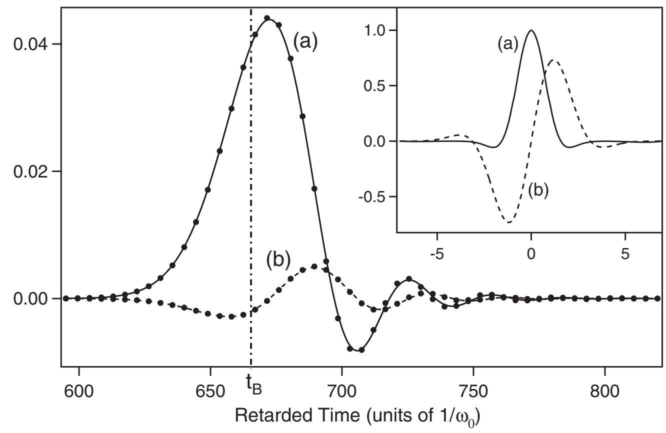

FIG. 12. Brillouin precursors generated by the incident fields of Gaussian envelope (a) $e^{-(t/T)^2}\cos(\omega_c t)$ with $T = \sqrt{2}/\omega_c$ and (b) $e^{-(t/T)^2}\sin(\omega_c t)$ with $T = \sqrt{6}/\omega_c$. The parameters are those of Fig. 8. In both cases, the pulse duration has been chosen in order to maximize the precursor amplitude (see text). The solid and dashed lines are the exact numerical solutions, whereas the dots are the analytical solutions obtained in the short-pulse approximation [Eqs. (45) and (46)], indiscernible from those obtained without approximation [Eq. (43)]. The inset shows the corresponding incident fields. Numerical calculations show that the Sommerfeld precursors generated by these fields have negligible amplitudes, 5.6×10^{-7} for (a) and 1.15×10^{-10} for (b).

the case of a square modulation. We get thus $W_B \approx (\pi/2)^{1/2}(\beta T^2 e^{-\omega_c^2T^2/2}) \propto 1/\sqrt{z}$ for $e(0,t) = e^{-t^2/T^2}\cos(\omega_c t)$ and $W_B \approx (\pi/32)^{1/2}(\beta^3\omega_c^2T^6e^{-\omega_c^2T^2/2}) \propto z^{-3/2}$ for $e(0,t) = e^{-t^2/T^2}\sin(\omega_c t)$. In fact the scaling laws in $z^{-1/2}$ or $z^{-3/2}$ are general and hold for every short incident pulse. In all cases, the transmitted pulse is indeed proportional to $h_B(z,t)$ when $E(0,0) = \mathcal{A} \neq 0$ or to $\dot{h}_B(z,t)$ when $\mathcal{A} = 0$, the proportionality coefficient depending only on the characteristics of the incident pulse and not on the propagation distance. For Gaussian incident pulses and, more generally, for smooth pulses, the amplitude and the energy of the Brillouin precursor rapidly decreases with the pulse duration. For example, the amplitude of the Brillouin precursor generated by the incident field $e^{-t^2/T^2}\cos(\omega_c t)$ is reduced by a factor exceeding 400 when T is taken as four times larger than its optimum value $\sqrt{2}/\omega_c$ [see Eq. (45)]. This reduction of amplitude can, however, be compensated by using chirped pulses. When the pulse duration remains small enough, Eq. (44) holds and the Brillouin precursor generated by the incident field $e^{-t^2/T^2}\cos(\omega_c t+\chi^2T^2)$ reads

$$e_B(z,t) \approx h_B(z,t)\,\mathrm{Re}(\widetilde{\mathcal{A}}) - \dot{h}_B(z,t)\,\mathrm{Re}(i\omega_c\widetilde{\mathcal{A}}\widetilde{T}^2/2). \quad (47)$$

Anticipating that the second term of this equation is small compared to the first one, we easily get the approximate expression

$$e_B(z,t) \approx \mathcal{A}\,h_B[z,t'' - \mathrm{Re}(i\omega_c\widetilde{\mathcal{A}}\widetilde{T}^2/2\mathcal{A})], \quad (48)$$

where $\mathcal{A} = \mathrm{Re}(\widetilde{\mathcal{A}})$ is the area of the incident pulse. This result differs from that obtained without chirping [see Eq. (45)] by an extra time delay $\mathrm{Re}(i\omega_c\widetilde{\mathcal{A}}\widetilde{T}^2/2\mathcal{A})$ and moreover by the pulse area $\mathcal{A}$, which may be considerably larger than that attained when the pulse is not chirped. Figure 13 shows the

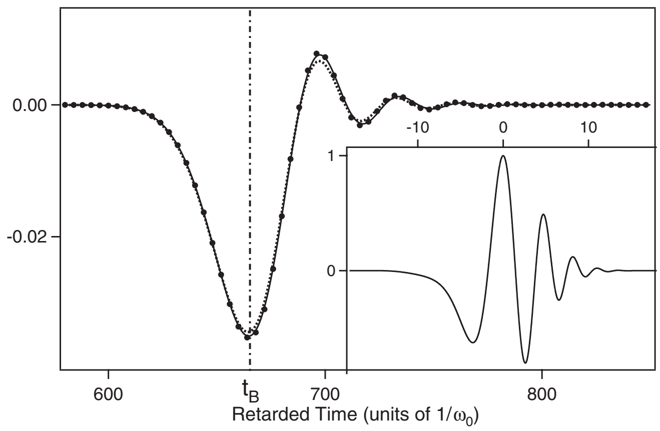

FIG. 13. Brillouin precursor generated by a chirped incident pulse $e^{-(t/T)^2}\cos(\omega_c t + \chi^2 t^2)$, with $T = 4\sqrt{2}/\omega_c$ and $\chi = \omega_c/4$. The other parameters are the same as those of Figs. 8 and 12. The solid line, the dots, and the dashed line are respectively the exact numerical solution, the analytic solution derived from Eq. (43), and the approximate analytic solution of Eq. (48), obtained in the short-pulse approximation. The inset shows the incident pulse. The corresponding Sommerfeld precursor has fully negligible amplitude (9×10^{-11}).

result obtained for a pulse duration $T = 4\sqrt{2}/\omega_c$. In order to maximize the precursor amplitude, we have chosen for the chirping the value $\chi = \omega_c/4$ for which the function $\mathcal{A}(\chi)$ reaches its first extremum (negative minimum). For these parameters, $\mathrm{Re}(i\mathcal{A}\widetilde{T}^2/2\mathcal{A})$ is also negative (time advancement). We remark that, despite the numerous approximations having led to Eq. (48), it provides a very good approximation of the exact result.

VII. CONCLUSION

We have analytically studied the propagation of light pulses in a dense Lorentz medium at distances z so large that the medium is opaque in a broad spectral region and the Sommerfeld and Brillouin precursors are far apart from each other. Assuming that the carrier frequency ω_c lies in the opacity region (below, inside, or beyond the anomalous dispersion region), we have shown that the Sommerfeld precursor has a shape independent of ω_c and that it is entirely determined by the order p and the importance d_p of the initial discontinuity of the incident field, regardless of its subsequent evolution. When the incident field is discontinuous ($p = 0$), its amplitude is independent of z and ω_c. For $p > 0$, this amplitude is proportional to $\omega_c z^{-(2p+1)/4}$ and rapidly decreases with the rise time of the incident field. These results, exact in the strict asymptotic limit where $z \to \infty$, provide excellent approximations for the propagation distance considered by Brillouin and remain good approximations even when z is 1000 times shorter.

In the strict asymptotic limit, the formation of the Brillouin precursor is uniquely determined by the frequency dependence of the medium attenuation. When ω_c lies in the opacity region, we have shown that the Brillouin precursor is a Gaussian of amplitude $a_B \propto 1/(\omega_c\sqrt{z})$ or a Gaussian derivative of amplitude $a_B \propto 1/(\omega_c^2 z)$, depending on whether or not the area of the incident field differs from zero. We have also determined the transmitted field when ω_c is outside the opacity region, evidencing the "pollution" of the Brillouin precursor by the field that is then transmitted at ω_c (Fig. 7).

In a simple asymptotic limit, both attenuation and group-delay dispersion contribute to the formation of the Brillouin precursor. We have established in this case an expression of the Brillouin precursor containing as particular cases the previous one (dominant-attenuation limit) and that obtained by Brillouin by means of the stationary-phase method (dominant-dispersion limit).

We have finally obtained exact analytical expressions of the Brillouin precursors originated by pulses of a square or Gaussian envelope. We have in particular determined the pulse parameters optimizing the precursor amplitude and demonstrated that the *energy* of the precursor decreases with the propagation distance as slowly as $z^{-1/2}$ when the area of the incident field differs from zero but as rapidly as $z^{-3/2}$ in the contrary case. We have also shown that, for a given duration, the precursor amplitude can be greatly enhanced by using frequency-chirped pulses.

Our explicit analytic expressions of the precursors contrast, by their simplicity, those currently derived by the uniform saddle-point methods. The complexity of the latter [13] is often such that it is difficult and sometimes impossible to retrieve from them our asymptotic forms. On the other hand, it should be kept in mind that our results only hold in the limit where the medium is opaque in a spectral region whose width is much larger than the resonance frequency. We, however, remark that they provide a not too bad reproduction of the Sommerfeld and Brillouin precursors even when this width is of the order of the resonance frequency (Fig. 9). We finally mention that the study of the precursors is greatly simplified when the complex index of the medium is such that $|\widetilde{n}(\omega) - 1| \ll 1 \forall\, \omega$ [16]. As in the study of the quasiresonant precursors [48], the equation giving the saddle points can then be reduced to a biquadratic form, and the saddle-point method is expected to provide simple solutions even when the Sommerfeld and Brillouin precursors partially overlap. This work is in progress.

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