Trirefringence in nonlinear metamaterials

Vitorio A. De Lorenci[1,*] and Jonas P. Pereira[1,2,3,†]

[1]Instituto de Ciências Exatas, Universidade Federal de Itajubá, Itajubá, MG 37500-903, Brazil

[2]Université de Nice Sophia Antipolis, Nice, CEDEX 2, Grand Chateau Parc Valrose

[3]Dipartamento di Fisica and ICRA, Sapienza Università di Roma, Piazzale Aldo Moro 5, I-00185 Rome, Italy

(Received 28 October 2011; published 2 July 2012)

We study the propagation of electromagnetic waves in the limit of geometrical optics for a class of nearly transparent nonlinear uniaxial metamaterials for which their permittivity tensors present a negative principal component. Their permeabilities are assumed positive and dependent on the electric field. We show that light waves experience triple refraction—trirefringence. In addition to the ordinary wave, two extraordinary waves propagate in such media.

DOI: 10.1103/PhysRevA.86.013801 PACS number(s): 42.15.−i, 41.20.Jb, 42.65.−k, 78.67.−n

I. INTRODUCTION

Electromagnetic waves in a material medium propagate according to Maxwell's equations complemented by certain relations linking strengths and induced fields—the constitutive relations. Depending on the dielectric properties of the medium and on the presence of applied external fields, a variety of optical effects can be found. One that has wide interest is the birefringence [1–3], occurring when electromagnetic waves propagate in media exhibiting two distinct refractive indexes [1] in a same wave-vector direction. This effect occurs naturally in some well-known crystalline solids, such as quartz and sapphire, for instance. In nonlinear media, where the dielectric coefficients are field dependent, birefringence could also be induced by the presence of external fields, leading to the known Kerr, Cotton-Mouton, and magnetoelectric effects [4,5]. Birefringence has been widely used in the technology of optical devices [3] and as a powerful experimental technique for investigating properties of physical systems, including biological [6] and astrophysical ones [7,8] among others. The phenomenon of triple refraction has been much less investigated. By trirefringence in a given wave-vector direction, we mean the existence of three distinct refractive indexes in that direction. In the realm of linear electrodynamics, trirefringence does not occur [9]. Nevertheless, nothing was studied when the dielectric tensors are field dependent (nonlinear electrodynamics).

Multirefringent properties have been measured in tailored photonic crystals [10]. Such materials are constructed to manipulate light propagation and, hence, can lead to a variety of applications [11]. However, in these media, structural details (lattice constants, defects, etc.) are imperative, and hence, they cannot be described in terms of effective dielectric tensors [12]. Developments in photonic band-gap materials and the so-called metamaterials have enabled the discovery of several new phenomena [12]. For instance, it was experimentally shown [13] that nearly transparent isotropic metamaterials allow light propagation when both effective dielectric coefficients (permittivity and permeability) are negative. In fact, this phenomenon and other unusual properties displayed by isotropic media with negative dielectric coefficients were proposed theoretically long ago [14]. It is worth emphasizing that these media must be dispersive, and the negative coefficients are obtained for convenient frequency ranges. Some other unusual properties exhibited by specific metamaterials are negative refractive index [15], trapping of light [16,17], perfect lens devices [18], the electromagnetic cloaking effect [19,20], and the occurrence of asymmetry for the propagation of light in opposite wave-vector directions [21]. Wave propagation in indefinite metamaterials (where not all the principal components of the dielectric tensors have the same sign) has been also considered [12], showing that effects already proposed in the context of isotropic metamaterials can be obtained and possibly can be improved. Indefinite metamaterials can also be used for investigating certain aspects of general relativity [22].

In this paper, we show that nonlinear metamaterials described in terms of effective dielectric tensors, may display trirefringence. Analytical expressions describing this effect are formally obtained from Maxwell's electromagnetism, and a simple theoretical model is numerically examined. A possible experimental realization of the media expected to display this effect is also addressed. The vectorial three-dimensional formalism [23] is used. The units are set such that $c = 1$.

In the next section, electromagnetic wave propagation in nonlinear materials is examined. The eigenvalue problem is stated and formally is solved for a class of nonlinear materials presenting nonisotropic permittivity tensors (for a given frequency) and isotropic permeability dependent on the modulus of the resultant electric field. In Sec. III, trirefringence phenomenon is theoretically described and is proposed to occur in nearly transparent nonlinear uniaxial metamaterials. The analysis is performed in terms of phase and group velocities. Final remarks and conclusions are presented in Sec. IV.

II. WAVE PROPAGATION

The electrodynamics of a continuum medium at rest in the absence of sources is governed by the Maxwell field equations,

$$\vec{\nabla}\cdot\vec{D} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \tag{1}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} \tag{2}$$

*delorenci@unifei.edu.br

†jonaspedro.pereira@gmail.com

taken together with the constitutive relations between the fundamental fields $\vec{E}$ and $\vec{B}$ and the induced ones $\vec{D}$ and $\vec{H}$, written here as

$$D_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j, \quad H_i = \sum_{j=1}^{3} \mu_{ij} B_j. \tag{3}$$

The dielectric coefficients ε_{ij} and μ_{ij} are the components of the permittivity and the inverse permeability tensors, respectively, and they encompass all information about the electromagnetic properties of the medium. Furthermore, for any vector $\vec{\alpha}$, we denote its ith component by α_i $(i = 1,2,3)$.

The propagation of monochromatic electromagnetic waves is here examined within the limit of geometrical optics [1] using the method of field disturbances [24]. This method can be summarized as follows [23]. Let Σ, defined by $\phi(t,\vec{x}) = 0$, be a smooth (differentiable of class $\mathcal{C}^n$, $n > 2$) hypersurface. The function ϕ is understood to be a real-valued smooth function of the coordinates $(t,\vec{x})$ and regular in a neighborhood U of Σ. The spacetime is divided by Σ into two disjoint regions U^-, for which $\phi(t,\vec{x}) < 0$, and U^+, corresponding to $\phi(t,\vec{x}) > 0$. The discontinuity of an arbitrary function $f(t,\vec{x})$ (supposed to be a smooth function in the interior of $U^{\pm}$) on Σ is a smooth function in U and is given by [24]

$$[f(t,\vec{x})]_\Sigma \doteq \lim_{\{P^\pm\}\to P} [f(P^+) - f(P^-)], \tag{4}$$

with P^+, P^-, and P belonging to U^+, U^-, and Σ, respectively. The electromagnetic fields are supposed to be smooth functions in the interior of U^+ and U^- and continuous across Σ (ϕ is now taken as the eikonal [1] of the wave). However, they have a discontinuity in their first derivatives such that [24]

$$[\partial_t E_i]_\Sigma = \omega e_i, \quad [\partial_t B_i]_\Sigma = \omega b_i, \tag{5}$$

$$[\partial_i E_j]_\Sigma = -q_i e_j, \quad [\partial_i B_j]_\Sigma = -q_i b_j, \tag{6}$$

where e_i and b_i are related to the derivatives of the electric and magnetic fields on Σ and correspond to the components of the polarization of the propagating waves [25]. The quantities ω and q_i are the angular frequency and the ith component of the wave-vector. (Incidentally, we note that the negative signs appearing in Eq. (6) are missing in the corresponding equations in Ref. [23].)

For the cases of interest in this paper, the permittivity of the media under study is described by real diagonal tensors (losses have been neglected) whose components ε_{ij} are dependent only upon the constant frequency of the wave. We set the magnetic permeability of these media to be real functions of the modulus of the electric field such that

$$\mu_{ij}(|\vec{E}|) = \frac{\delta_{ij}}{\mu(|\vec{E}|)}, \tag{7}$$

where $\delta_{ij} = \mathrm{diag}(1,1,1)$. Thus, applying the boundary conditions stated by Eqs. (5) and (6) to the field equations (1) and (2), we obtain the eigenvalue equation [5,26],

$$\sum_{j=1}^{3} Z_{ij} e_j = 0, \tag{8}$$

where the Fresnel matrix Z_{ij} is given by

$$Z_{ij} = \varepsilon_{ij} - \frac{\mu'}{\omega\mu^2}(\vec{q}\times\vec{B})_i E_j - \frac{1}{\mu\omega^2}(q^2\delta_{ij} - q_i q_j), \tag{9}$$

with

$$\mu' \doteq \frac{1}{|\vec{E}|}\frac{\partial\mu}{\partial|\vec{E}|}, \tag{10}$$

and $q^2 \doteq \vec{q}\cdot\vec{q}$.

Nontrivial solutions for the eigenvalue problem in Eq. (8) can be found if and only if $\det|Z_{ij}| = 0$, which is known as the generalized Fresnel equation. This equation also gives the dispersion relations of the media under study. Using the covariant formulas for the traces of linear operators [27], the Fresnel equation can be cast as

$$\det|Z_{ij}| = -\tfrac{1}{6}(Z_1)^3 + \tfrac{1}{2}Z_1 Z_2 - \tfrac{1}{3}Z_3 = 0, \tag{11}$$

where we defined the traces,

$$Z_1 \doteq \sum_{i=1}^{3} Z_{ii}, \tag{12}$$

$$Z_2 \doteq \sum_{i,j=1}^{3} Z_{ij} Z_{ji}, \tag{13}$$

$$Z_3 \doteq \sum_{i,j,l=1}^{3} Z_{ij} Z_{jl} Z_{li}. \tag{14}$$

As a requirement of the geometrical optics limit, the wave fields are considered to be negligible when compared with the external fields. Thus, we assume from now on that the fields are approximated by their external counterparts $\vec{E}_{\rm ext}$ and $\vec{B}_{\rm ext}$. We set $\vec{E} \approx \vec{E}_{\rm ext} \doteq E\hat{x}$ and $\vec{B} \approx \vec{B}_{\rm ext} \doteq B\hat{y}$, which could be arbitrary functions of space and time coordinates. Let us examine the particular case of uniaxial media [1,2] with permittivity

$$\varepsilon_{ij} = \mathrm{diag}(\varepsilon_\parallel,\varepsilon_\perp,\varepsilon_\perp). \tag{15}$$

Using Eqs. (9) and (12)–(14), straightforward calculations show that Eq. (11) results in the following algebraic fourth-degree equation for the phase velocity $v = \omega/q$ of the propagating waves,

$$av^4 + bv^3 + cv^2 + dv + e = 0, \tag{16}$$

with

$$a = 6\varepsilon_\perp^2\varepsilon_\parallel, \tag{17}$$

$$b = \frac{6\mu'}{\mu^2}\varepsilon_\perp^2 EB\hat{q}_z, \tag{18}$$

$$c = -\frac{6}{\mu}\varepsilon_\perp\big[2\varepsilon_\parallel\hat{q}_x^2 + (\varepsilon_\perp + \varepsilon_\parallel)\big(\hat{q}_y^2 + \hat{q}_z^2\big)\big], \tag{19}$$

$$d = -\frac{6\mu'}{\mu^3}\varepsilon_\perp EB\hat{q}_z, \tag{20}$$

$$e = \frac{6}{\mu^2}[\varepsilon_\parallel\hat{q}_x^2 + \varepsilon_\perp(\hat{q}_y^2 + \hat{q}_z^2)]. \tag{21}$$

We defined $\hat{q}_\alpha \doteq (\hat{q}\cdot\hat{\alpha})$, where $\hat{q} \doteq \vec{q}/q$ for any unit vector $\hat{\alpha}$. Then, $\hat{q} = \hat{q}_x\hat{x} + \hat{q}_y\hat{y} + \hat{q}_z\hat{z}$ and $\hat{q}^2 \doteq \hat{q}\cdot\hat{q} = 1$. Notice that, when the propagation occurs in the xy plane (spanned by the external electric and magnetic fields), the coefficients b and d are null, and the generalized Fresnel equation reduces to a quadratic equation in v^2, therefore, allowing only birefringence. The same behavior occurs if $\mu' = 0$. In fact, in this situation, we recover linear electrodynamics.

Solving Eq. (16), we obtain

$$v_o = \pm\frac{1}{\sqrt{\mu\varepsilon_\perp}}, \tag{22}$$

$$v_e^{\pm} = -\sigma\hat{q}_z \pm \sqrt{(\sigma\hat{q}_z)^2 + \frac{1}{\mu\varepsilon_\parallel}\left(\frac{\varepsilon_\parallel}{\varepsilon_\perp}\hat{q}_x^2 + \hat{q}_y^2 + \hat{q}_z^2\right)}, \tag{23}$$

where

$$\sigma \doteq \frac{\mu' EB}{2\mu^2\varepsilon_\parallel}. \tag{24}$$

The solution v_o does not depend on direction of the wave propagation and is called the ordinary wave, whereas, $v_e^{\pm}$ depend on direction of the wave propagation and are called extraordinary waves [1]. By definition, the velocities of the waves are given by $\vec{v} = v\hat{q}$, where v is given by Eqs. (22) and (23). In order to achieve more simplicity, in the following analysis, we assume the external fields to be constant and set the wave-vector in a given direction of the xz plane, i.e., $\hat{q}_y = 0$, $\hat{q}_x = \sin\theta$, and $\hat{q}_z = \cos\theta$. In this notation, θ indicates the angle between the $\hat{q}$ and the z directions.

III. TRIREFRINGENCE

Two distinct solutions for $\vec{v}_e^{\pm}$ in a same given direction $\hat{q}$ can be obtained from Eq. (23) if

$$-1 < \frac{1}{\mu\varepsilon_\parallel\sigma^2}\left(\frac{\varepsilon_\parallel}{\varepsilon_\perp}\tan^2\theta + 1\right) < 0. \tag{25}$$

Thus, taking into account the ordinary wave, Eq. (25) defines a region inside which trirefringence occurs in any chosen direction $\hat{q}$. Let us examine this effect closer.

In order to guarantee the existence of an ordinary wave, we must set $\mu\varepsilon_\perp > 0$, otherwise, v_o is not real. This is true when both coefficients μ and $\varepsilon_\perp$ present the same sign, which can be positive for usual media or negative for left-handed materials (in this case, a negative refractive index occurs [14,15,28]). Let us set these coefficients to be positive. Now, in order to satisfy Eq. (25), we set $\varepsilon_\parallel = -\epsilon_\parallel < 0$. Hence, trirefringence occurs in directions determined by

$$\frac{\varepsilon_\perp}{\epsilon_\parallel} > \tan^2\theta > \frac{\varepsilon_\perp}{\epsilon_\parallel}(1 - \epsilon_\parallel\mu\sigma^2). \tag{26}$$

The above-discussed phenomenon is displayed in Fig. 1 where the normal surfaces [1,2] associated with the ordinary and extraordinary waves are depicted for some specific values of the quantities appearing in Eq. (26). For any given direction encompassed by the angles between the two dashed straight lines, defined by Eq. (26), there are three distinct solutions: the dashed and dot-dashed curves representing the extraordinary waves and the circular solid curve representing the ordinary wave. For angles between the dashed and the solid straight lines, only birefringence occurs, and finally, only one refraction occurs for directions encompassed by the angles between the two solid straight lines. From Fig. 1, it is also worth noticing that, in the sectors where more than one refractive index occur, the medium under consideration behaves as a positive or negative medium [1,2], depending on subsectors and extraordinary waves.

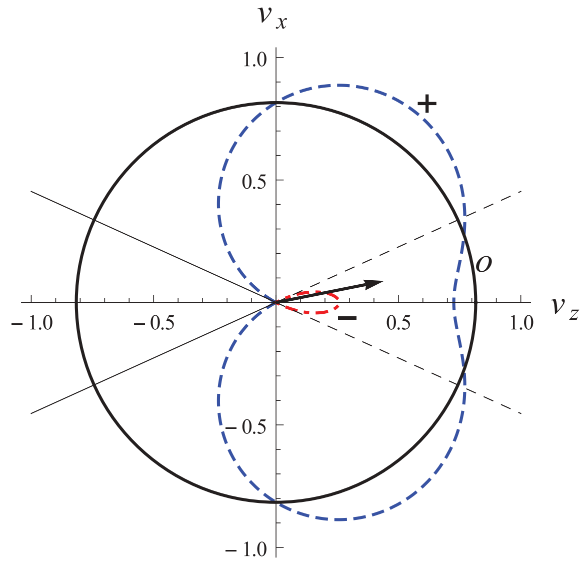

FIG. 1. (Color online) Normal surfaces [1,2] of a nonlinear medium with dielectric coefficients given by $\varepsilon_{ij} = \mathrm{diag}(-\epsilon_\parallel, \varepsilon_\perp, \varepsilon_\perp)$ and $\mu(E)$. We set a model where the numerical values were taken such that $\epsilon_\parallel = 6.69$, $\varepsilon_\perp = 1.88$, $\mu = 0.8$, and $\mu' EB = 4.21$. The ordinary wave is represented by the circular solid line, and the extraordinary waves are represented by the dashed and dot-dashed curves. The dashed straight lines encompass an angular region in which trirefringence occurs. In this case, there exist two extraordinary waves denoted by + and −. The symbol o refers to the ordinary wave.

In geometrical optics, the directions of light rays are given by the directions of the group velocities,

$$\vec{u} = \frac{\partial\omega}{\partial\vec{q}} = v\hat{q} + q\frac{\partial v}{\partial\vec{q}}, \tag{27}$$

which are considered as the physical velocities of propagation of the rays [1]. As we see from Eq. (23), the extraordinary waves depend on the wave-vector. Thus, the directions of the extraordinary light rays do not, in general, coincide with the directions of the extraordinary phase velocities as explicitly shown by Eq. (27). Taking $v = \{v_o, v_e^+, v_e^-\}$ into the definition of the group velocity, we obtain that the ordinary group velocity is identified with the ordinary phase velocity $\vec{u}_o = v_o\hat{q}$. Nevertheless, the extraordinary group velocities are given by

$$\vec{u}_e = u_x\hat{x} + u_z\hat{z}, \tag{28}$$

with

$$u_x = \frac{v_e^3\sin\theta + \sigma v_e^2\sin 2\theta + \frac{v_e}{\mu\varepsilon_\parallel}\left(\frac{\varepsilon_\parallel}{\varepsilon_\perp} - 1\right)\sin\theta\cos^2\theta}{v_e^2 + (1-\eta)\sigma v_e\cos\theta + \frac{\eta}{2\mu\varepsilon_\parallel}\cos^2\theta}, \tag{29}$$

$$u_z = \frac{v_e^3\cos\theta + \sigma v_e^2\cos 2\theta - \frac{v_e}{\mu\varepsilon_\parallel}\left(\frac{\varepsilon_\parallel}{\varepsilon_\perp} - 1\right)\cos\theta\sin^2\theta}{v_e^2 + (1-\eta)\sigma v_e\cos\theta + \frac{\eta}{2\mu\varepsilon_\parallel}\cos^2\theta}, \tag{30}$$

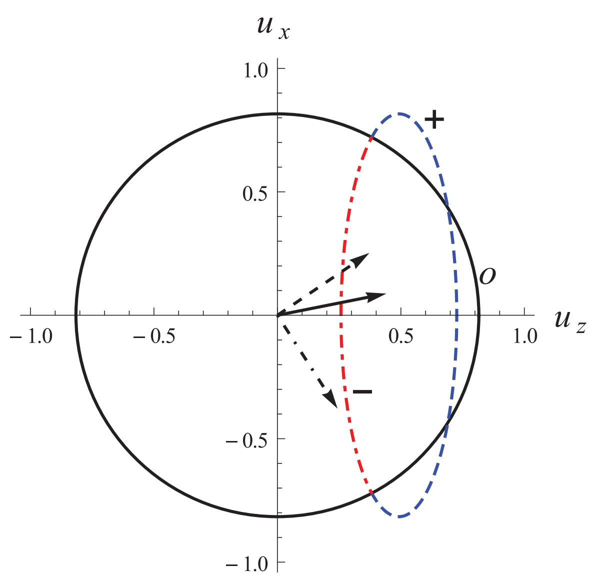

FIG. 2. (Color online) Ray surfaces [1,2] of the nonlinear medium considered in Fig. 1. The ordinary group velocity is represented by the circular solid line, and the extraordinary group velocities are represented by the dashed and dot-dashed curves.

and where we defined

$$\eta \doteq \frac{\omega}{\varepsilon_\parallel}\frac{\partial \varepsilon_\parallel}{\partial \omega}. \tag{31}$$

In these equations, $v_e \doteq v_e^{\pm}$ represents the extraordinary wave velocities in direction $\hat{q}$, which can be obtained from Fig. 1 by taking the distances from the associated points on its wave-vector surfaces to the origin of the coordinate system. For the particular model set in Fig. 1, the corresponding ray surfaces [1,2] are depicted in Fig. 2 as a function of the angle φ between $\vec{u}$ and the z axis where it was assumed for simplicity that $|\eta| \ll 1$. Notice that this assumption resides in the realm of nonlinear media. In the context of linear electrodynamics, for which $\mu' = 0$, Eqs. (28)–(30) show that $|\eta| \ll 1$ leads to superluminal group velocity solutions. Therefore, this assumption is not in disagreement with the so-called causality requirement [12]. For the ordinary waves, we have $\varphi = \theta$ as anticipated. For the extraordinary waves, the relationship between these angles can be obtained directly from Eqs. (29) and (30) as $\varphi = \arctan[u_x(\theta)/u_z(\theta)]$. From Fig. 2, we see that there is a sector inside which there exist two extraordinary rays in any chosen direction of observation. For the complementary sector, there are no extraordinary rays. We point out that any chosen direction for the rays inside the trirefringent sector is related to two different extraordinary wave-vectors in Fig. 1. Similarly, for any given wave-vector inside the two dashed straight lines in Fig. 1, there are two different extraordinary rays associated with it in Fig. 2. For instance, let us take the wave-vector in the direction $\theta = \pi/16$ as indicated by the arrow drawn in Fig. 1. As is clear, in this direction, there are three solutions for the phase velocity, indicating that trirefringence occurs. These solutions are represented by the points on the $+$, $-$, and o curves, found in this particular direction, and their moduli are $v_e^+ = 0.76$, $v_e^- = 0.20$, and $v_o = 0.82$, respectively. From Eqs. (28)–(30), the corresponding group velocities present magnitudes $u_e^+ = 0.83$, $u_e^- = 0.55$, and $u_o = 0.82$. Their associated directions are indicated by the arrows appearing in Fig. 2 where the dashed and dot-dashed arrows correspond to $\vec{u}_e^+$ and $\vec{u}_e^-$, respectively.

IV. FINAL REMARKS

If the principal permittivity components are also dependent upon the modulus of the electric field, then analogous calculations leading to the equations for the phase and group velocities and conditions for having trirefringence derived in this paper still hold by identifying

$$\varepsilon_\parallel \to \varepsilon_\parallel(E) + E\frac{\partial \varepsilon_\parallel(E)}{\partial E}, \tag{32}$$

and

$$\varepsilon_\perp \to \varepsilon_\perp(E). \tag{33}$$

Once we work in the regime of high frequencies (required by geometrical optics), losses must be assumed to be low [1]. Metamaterials with low losses are still a matter of current investigation [29] with some achievements already obtained in such a regime [30].

In what concerns the role played by the nonlinearities, Eqs. (23) and (24) show us that if they exist (i.e., $\mu' \neq 0$), irrespective of their strengths, then they will lead to the presence of two extraordinary waves (hence, allowing trirefringence). Given the physical quantities appearing in Eqs. (28)–(30), the thresholds for the strengths of the nonlinearities are such that causality is not violated. The important quantity related to the nonlinearities is σ, given by Eq. (24), which is experimentally controllable, thereby making the predictions here derived feasible.

Layered media [2,31] seem good candidates for experimental realizations of our assumptions. Consider a system constituted by a repetition of two thin low loss layers. One of them is composed of a nonmagnetic ($\mu_1 = 1$) and dispersive [$\epsilon_1 = \epsilon_1(\omega)$] medium. The other one is composed of a liquid medium whose permeability and permittivity are dependent upon the modulus of the resultant electric field as $\mu_2 = 1 - f(|\vec{E}|)$ and $\epsilon_2 = 1 - g(|\vec{E}|)$, where $f(|\vec{E}|)$ and $g(|\vec{E}|)$ are usually much smaller than unity [32]. Additionally, if convenient external fields and layer parameters are present, then it is always possible to guarantee that our constraints are fulfilled.

Refraction analyses were disregarded in our paper once they are just a straightforward extension of the analyses valid for birefringent crystals [2]. The only difference now is the existence of two extraordinary waves.

Summing up, working in the limit of geometrical optics, we studied the propagation of electromagnetic waves in nearly transparent nonlinear uniaxial metamaterials in the presence of external electric and magnetic fields. We assumed a constant nonisotropic permittivity tensor (for a given frequency) presenting a negative principal component and an isotropic permeability dependent on the modulus of the electric field. We solved the corresponding eigenvalue problem and obtained the general fourth-degree polynomial equation, whose solutions describe the propagation of waves. In this context, we showed that trirefringence is a phenomenon allowed to occur and could be described both in terms of wave and ray propagations. With the present technology in manipulating the dielectric coefficients in metamaterials [22,31], we hope that the effect here derived can be experimentally tested and if verified, could lead to applications.

ACKNOWLEDGMENTS

This work was partially supported by the Brazilian research agencies CNPq, FAPEMIG, and CAPES. J.P.P. thanks FAPEMIG and the Erasmus Mundus IRAP Ph.D. Program for support.

[1] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Vol. 8. (Pergamon, Oxford, 1984).
[2] M. Born and E. Wolf, *Principles of Optics* (Cambridge University Press, Cambridge, UK, 1999).
[3] R. Paschotta, *Encyclopedia of Laser Physics and Technology* (Wiley-VCH, Weinheim, 2008).
[4] T. Roth and G. L. J. A. Rikken, Phys. Rev. Lett. **85**, 4478 (2000); **88**, 063001 (2002).
[5] V. A. De Lorenci and G. P. Goulart, Phys. Rev. D **78**, 045015 (2008).
[6] L. Liu, J. R. Trimarchi, R. Oldenbourg, and D. L. Keefe, Bio. Reprod. **63**, 251 (2000); Y. Lim *et al.*, Biomed. Opt. Express **2**, 2392 (2011).
[7] G. D. Fleishman, Q. J. Fu, M. Wang, G.-L. Huang, and V. F. Melnikov, Phys. Rev. Lett. **88**, 251101 (2002); H. J. M. Cuesta, J. A. de Freitas Pacheco, and J. M. Salim, Int. J. Mod. Phys. A **21**, 43 (2006).
[8] L. Pagano, P. de Bernardis, G. De Troia, G. Gubitosi, S. Masi, A. Melchiorri, P. Natoli, F. Piacentini, and G. Polenta, Phys. Rev. D **80**, 043522 (2009).
[9] V. E. Wood and R. E. Mills, Opt. Acta **16**, 133 (1969).
[10] M. C. Netti, A. Harris, J. J. Baumberg, D. M. Whittaker, M. B. D. Charlton, M. E. Zoorob, and G. J. Parker, Phys. Rev. Lett. **86**, 1526 (2001).
[11] J. D. Joannopoulos, S. G. Johnson, J. N. Winn, and R. D. Meade, *Photonic Crystals: Molding the Flow of Light*, 2nd ed. (Princeton University Press, Princeton, 2008).
[12] D. R. Smith and D. Schurig, Phys. Rev. Lett. **90**, 077405 (2003).
[13] D. R. Smith, W. J. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, Phys. Rev. Lett. **84**, 4184 (2000).
[14] V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968).
[15] R. A. Shelby, D. R. Smith, and S. Schultz, Science **292**, 77 (2001).
[16] K. L. Tsakmakidis, A. D. Boardman, and O. Hess, Nature (London) **450**, 397 (2007)
[17] I. I. Smolyaninov *et al.*, Appl. Phys. Lett. **96**, 211121 (2010).
[18] J. B. Pendry, Phys. Rev. Lett. **85**, 3966 (2000).
[19] T. Ergin, N. Stenger, P. Brenner, J. B. Pendry, and M. Wegener, Science **328**, 337 (2010).
[20] I. I. Smolyaninov, V. N. Smolyaninova, A. V. Kildishev, and V. M. Shalaev, Phys. Rev. Lett. **102**, 213901 (2009).
[21] C. Menzel, C. Helgert, C. Rockstuhl, E.-B. Kley, A. Tünnermann, T. Pertsch, and F. Lederer, Phys. Rev. Lett. **104**, 253902 (2010).
[22] I. I. Smolyaninov and E. E. Narimanov, Phys. Rev. Lett. **105**, 067402 (2010).
[23] V. A. De Lorenci and D. D. Pereira, Phys. Rev. E **82**, 036605 (2010).
[24] J. Hadamard, *Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique* (Hermann, Paris, 1903); G. Boillat, J. Math. Phys. **11**, 941 (1970); A. Papapetrou, *Lectures on General Relativity* (Reidel, Dordrecht, 1974).
[25] V. A. De Lorenci and R. Klippert, Phys. Rev. D **65**, 064027 (2002); V. A. De Lorenci, Phys. Rev. E **65**, 026612 (2002).
[26] V. A. De Lorenci, R. Klippert, and D. H. Teodoro, Phys. Rev. D **70**, 124035 (2004).
[27] R. R. Silva, J. Math. Phys. **39**, 6206 (1998).
[28] D. R. Smith and N. Kroll, Phys. Rev. Lett. **85**, 2933 (2000).
[29] A. Boltasseva and H. A. Atwater, Science **331**, 290 (2011).
[30] P. Tassin, L. Zhang, T. Koschny, E. N. Economou, and C. M. Soukoulis, Phys. Rev. Lett. **102**, 053901 (2009).
[31] B. Wood, J. B. Pendry, and D. P. Tsai, Phys. Rev. B **74**, 115116 (2006).
[32] N. B. Baranova, Y. V. Bogdanov, and B. Y. Zeldovich, Usp. Fiz. Nauk. **123**, 349 (1977).