Critical exponents of steady-state phase transitions in fermionic lattice models

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We discuss reservoir-induced phase transitions of lattice fermions in the nonequilibrium steady state of an open system with local reservoirs. These systems may become critical in the sense of a diverging correlation length on changing the reservoir coupling. We here show that the transition to a critical state is associated with a vanishing gap in the damping spectrum. It is shown that, although in linear systems there can be a transition to a critical state, there is no reservoir-induced quantum phase transition between distinct phases with a nonvanishing damping gap. We derive the static and dynamical critical exponents corresponding to the transition to a critical state and show that their possible values, defining universality classes of reservoir-induced phase transitions, are determined by the coupling range of the independent local reservoirs. If a reservoir couples to N neighboring lattice sites, the critical exponent can assume all fractions from 1 to $1/(N-1)$.

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I. INTRODUCTION

Experiments with cold atoms allow unmatched control over quantum systems, giving access to a number of interesting many-body effects [1]. A prominent example is quantum phase transitions [2] in the ground state of many-body Hamiltonians. On the other hand, the tools of quantum optics allow us to control the interaction with the environment and systems can be prepared in the nonequilibrium steady state (NESS) of an open dynamics instead [3–6]. In general, the steady state of an open system differs markedly from the ground state of the corresponding Hamiltonian or even from thermal states and thus driving a system into that state uniquely differs from the canonical treatment of decoherence as a perturbing effect on ground states [7]. As the steady state or, more generally, the stationary space is an attractor of the nonunitary evolution, it is robust against further decoherence. Such a scheme, where the stationary state is a projector, i.e., a dark state of the nonunitary evolution, has recently been proposed, e.g., by Diehl *et al.* to prepare exotic states such as the Kitaev edge modes [8] or BCS-type states [9]. The time evolution within a higher-dimensional dark space can correspond to interesting effective many-body Hamiltonians. Examples include the recently observed dissipative Tonks-Girardeau gas of atoms [10] or corresponding proposals for photons [11].

As there is a growing theoretical and experimental interest in engineered open systems, see, e.g., the recent review [5], we want to discuss in the present paper the analog to a quantum phase transition in Hamiltonian systems in the NESS of an open system, described by a Lindblad Liouville operator and induced by changing reservoir couplings. In Hamiltonian systems a quantum phase transition results from the competition of two noncommuting parts of a microscopic Hamiltonian $H = H_1 + gH_2$ with different symmetries on changing their relative strength g [2]. A transition between two distinct quantum phases occurs at a critical value $g = g_c$ and can be identified by a nonanalytic behavior of an order parameter. Furthermore, at the critical point the excitation gap Δ_H, i.e., the energy difference between the lowest excited state of H and the ground state closes. At the same time, certain correlations become infinitely long ranged, indicated by a diverging correlation length ξ. Criticality induced by reservoirs in open many-body systems has recently been discussed by Eisert and Prosen [12] for free systems, where criticality was defined in terms of a diverging correlation length. Here we will reexamine free fermionic lattice models as generic models to study noise-induced quantum phase transitions. As analog to the excitation gap Δ_H in unitary systems we will consider the damping gap Δ of the Liouvillian as indicator of a critical point for reservoir-induced phase transitions. We will show that if the manifold of reservoir parameters is of dimension d the values for which the damping gap closes and the system becomes critical is generically of dimension $d-2$ for free systems. As a consequence, the gapped phases are always connected in parameter space and there are only isolated critical points or manifolds for reservoir-induced quantum phase transitions of linear systems.

An important aspect of quantum phase transitions in unitary systems is that close to criticality, microscopic details of the interaction become irrelevant leading to universal behavior and allowing to classify phase transitions according to universality classes. These classes are defined by common critical exponents λ and dynamical critical exponents z, characterizing the divergence of the correlation length and the closure of the excitation gap when approaching the critical point:

$$\Delta_H \sim |g-g_c|^{z\lambda},$$
$$\xi^{-1} \sim |g-g_c|^{\lambda}.$$

We here argue that similar universal exponents can be derived for reservoir-induced transitions where the possible critical exponents depend solely on the coupling range of the independent local reservoirs.

After introducing linear fermionic lattice models coupled to local reservoirs and summarizing the general methods to derive the NESS of these systems in Sec. II, we discuss a specific realization of such a model with reservoirs coupling to two neighboring lattice sites in cold atom systems in Sec. III. Using this example as illustration we then formulate the key questions addressed in the remainder of this paper: Under what conditions do reservoir-induced phase transitions occur and to what universality classes can they be attributed to? We will then show in Sec. IV for general reservoir couplings that noise-induced criticality defined by a diverging correlation length occurs at points in parameter space where the damping gap, i.e., the spectral gap of relaxation rates of the Liouvillian dynamics, closes. In contrast to Hamiltonian systems, where the parameters that drive a phase transition are real, in Liouvillian systems these parameters are complex. We then show that for linear models criticality occurs only at isolated singularities or parameter manifolds of dimension $d-2$, where d is the dimension of the reservoir parameters. An immediate consequence of this is that there is only one connected phase with a nonvanishing damping gap. We derive the possible critical exponents, defining universality classes of noise-induced phase transitions, and show that they are simple fractions determined by the number of sites that couple to the same reservoir. Finally, we establish a simple relation between the critical and dynamical critical exponents.

II. FREE LATTICE FERMIONS WITH LINEAR RESERVOIR COUPLING

As a generic system we study a free and translationally invariant fermionic chain described by the system Hamiltonian $\mathcal{H}$. In addition to the unitary evolution, there is a nonunitary interaction with reservoirs which is assumed to be Markovian. This allows for an effective description of the dynamical equation for the density operator ρ via Lindblad operators L_μ

$$\frac{d}{dt}\rho = \mathcal{L}(\rho) = -i[\mathcal{H},\rho] + \frac{1}{2}\sum_\mu (2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu,\rho\}). \quad (1)$$

In the following we will use Majorana operators,

$$w_{2j-1} = \hat{c}_j^\dagger + \hat{c}_j, \qquad w_{2j} = i(\hat{c}_j - \hat{c}_j^\dagger), \quad (2)$$

instead of the more familiar fermionic creation and annihilation operators $\hat{c}_j^\dagger$ and $\hat{c}_j$, as the Majorana operators allow for an easy representation of the steady state of the considered linear systems. They are analogs of the bosonic position and momentum operators, are Hermitian, and fulfill a simple anticommutation relation $\{w_j,w_k\} = 2\delta_{j,k}$. The signature of a free system is the bilinearity of the Hamiltonian and the linearity of Lindblad generators when expressed in terms of Majorana operators

$$\mathcal{H}_S = \sum_{j,k=1}^{2L} (H)_{j,k} w_j w_k, \quad (3)$$

$$L_\mu = \sum_{j=1}^{L} (l)_{\mu,j,1} w_{2j-1} + (l)_{\mu,j,2} w_{2j}. \quad (4)$$

In the following, we consider local reservoirs only, i.e., we restrict the reservoir coupling to a finite number of N adjacent sites. To explicitly take into account translation invariance we reformulate the Hamiltonian and the Lindblad generators

$$\mathcal{H}_S = \sum_j \tau_j \left(\sum_{m,n} (h)_{m,n} w_m w_n \right), \quad (5)$$

$$L_j = \tau_j \left(\sum_{m=0}^{N-1} \nu_m e^{i g_m} w_{2m-1} + \tilde{\nu}_m e^{i\tilde{g}_m} w_{2m} \right), \quad j \in \mathbb{Z}, \quad (6)$$

where we have introduced the operator τ_j, which shifts a local operator by j lattice sites. One recognizes that the Lindblad generators contain $2N$ independent complex parameters $s_m = \nu_m e^{i g_m}$ and $q_m = \tilde{\nu}_m e^{i\tilde{g}_m}$. Only $2N-1$ are relevant, however, in the context of reservoir-induced phase transitions, as one coefficient can be pulled out and determines only the overall time scale of the damping. Thus, we may set without loss of generality $s_0 = 1$.

The dynamical equation (1) contains only quadratic terms in fermionic operators, thus the NESS ρ_0 is Gaussian [13] and is fully described by the correlation matrix

$$(\Gamma)_{j,k} = \frac{i}{2}\mathrm{Tr}\{\rho_0(w_j w_k - w_k w_j)\}. \quad (7)$$

Higher-order correlation functions can be calculated using Wick's theorem [14]. Γ_{jk} is antisymmetric, has purely imaginary entries, and is directly related to the Grassman representation of the steady state $w(\rho_0,\theta) = \frac{1}{2^L}\exp(-\frac{i}{2}\theta_j \Gamma_{jk}\theta_k)$. The dynamics generated by Eq. (1) take the simple form of a linear matrix differential equation for the correlation matrix Γ [12]

$$\frac{d}{dt}\Gamma = X^T\Gamma + \Gamma X - Y. \quad (8)$$

The matrices X and Y represent the Hamiltonian and the Lindblad generators $X = -4iH - (R + R^*)$ and $Y = 2i(R - R^*)$ with the reservoir matrix $R = \sum_\mu l_\mu \otimes l_\mu^*$.

We are interested only in the steady state of the system given by the solution of the Lyapunov Sylvester equation

$$X^T\Gamma_0 + \Gamma_0 X = Y. \quad (9)$$

For translationally invariant and infinitely large systems, the correlation matrix of the steady state is circulant and can be represented by its Fourier transform, called the symbol function $\gamma(\phi)$, which is a 2×2 matrix corresponding to the two different types of Majorana species with even or odd indices,

$$\gamma(\phi) = \begin{bmatrix} \gamma_{11}(\phi) & \gamma_{12}(\phi) \\ \gamma_{21}(\phi) & \gamma_{22}(\phi) \end{bmatrix}, \quad (10)$$

where

$$\langle w_{2j-1} w_{2(j+d)-1}\rangle = \langle w_1 w_{1+2d}\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi\; e^{i\phi d}\gamma_{11}(\phi) \quad (11)$$

and analogously for other types of Majorana operators.

In terms of the 2×2 symbol functions the Lyapunov Sylvester equation reads

$$x(-\phi)^T\gamma(\phi) + \gamma(\phi)x(\phi) = y(\phi). \quad (12)$$

The matrices $x(\phi)$ and $y(\phi)$ are calculated in correspondence to the X and Y matrices of the finite-size model.

III. A QUANTUM-OPTICAL EXAMPLE: LATTICE FERMIONS WITH TWO-SITE RESERVOIR COUPLING

In order to illustrate the physical context of the present paper let us consider fermionic atoms with four internal states $|g\rangle,|r\rangle,|e_1\rangle,|e_2\rangle$ in state-selective, optical lattice traps as shown in Fig. 1. The lattice for atoms in states $|g\rangle,|e_1\rangle,|e_2\rangle$ is given by an optical standing wave and has lattice constant

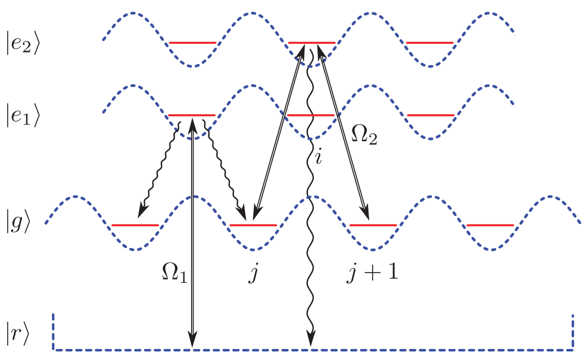

FIG. 1. (Color online) Fermions with four internal states are trapped in state-dependent optical lattices in the above configuration to realize the reservoir coupling of two sites in the $|g\rangle$ lattice. The shallow potential for atoms in internal state $|r\rangle$ acts as the reservoir, which is coupled in a controlled way to $|g\rangle$. The effective pumping is driven by a laser on the $|e_1\rangle,|r\rangle$ transition, whereas the decay is driven on the $|e_2\rangle,|g\rangle$ transition.

$\frac{\lambda}{4}$. The ground-state lattice is shifted by half a lattice constant compared to the other two. Atoms in internal state $|r\rangle$ feel a very shallow potential and are delocalized compared to the tightly confined atoms in other internal states. The transition $|r\rangle \leftrightarrow |e_1\rangle$ is driven by a laser field with Rabi frequency Ω_1, whereas $|g\rangle \leftrightarrow |e_2\rangle$ is coupled to a laser with Rabi frequency Ω_2. Spontaneous decay occurs from the two excited levels into the metastable ground states $|r\rangle,|g\rangle$. The optical lattices are deep so we can assume that only adjacent Wannier wave functions $\phi_i^\mu(x)$, $\mu \in e_1,e_2$, and ϕ_i^g or ϕ_{i+1}^g are optically coupled by the laser fields. Using this setup nontrivial pump and loss processes into and from the metastable states $|g\rangle$ are realized. Atoms in the shallow $|r\rangle$ potential act as a reservoir for the optical transitions.

First, atoms from the reservoir $|r\rangle$ are pumped via $|e_1\rangle$ into a superposition of atoms in neighboring lattice sites with fixed relative phase. To see this, we note that the spontaneous emission from $|e_1\rangle$ to $|g\rangle$ can be described by the interaction Hamiltonian

$$\mathcal{H}_{\text{int}} = \int dx \sum_{\vec{k}} [g_{\vec{k}} \hat{a}_{\vec{k}} \hat{\Psi}_g^\dagger(x) \hat{\Psi}_{e_1}(x) e^{ik_x x} + \text{H.a.}], \quad (13)$$

where $\hat{a}_{\vec{k}}$ is the annihilation operator of the electromagnetic mode with wave vector $\vec{k}$ and $g_{\vec{k}}$ is the corresponding coupling matrix element. Using the decomposition of the fermionic fields in internal states $|g\rangle$ and $|e_1\rangle$ into the Wannier basis $\hat{\Psi}_g(x) = \sum_j \phi_j^g(x)\hat{c}_j$ and $\hat{\Psi}_{e_1}(x) = \sum_j \phi_j^{e_1}(x)\hat{e}_{1,j}^\dagger$ yields

$$\mathcal{H}_{\text{int}} = \sum_j \sum_{\vec{k}} \left[g_{\vec{k}} \hat{a}_{\vec{k}} \hat{e}_j \left(\eta_{j1}^{(k)} \hat{c}_j^\dagger + \eta_{j2}^{(k)} \hat{c}_{j+1}^\dagger \right) + \text{H.a.} \right], \quad (14)$$

where $\eta_{j1}^{(k)} = \int dx\, \phi_j^{e_1}(x)\phi_j^g(x) e^{ik_x x}$ and $\eta_{j2}^{(k)} = \int dx\, \phi_j^{e_1}(x)\phi_{j+1}^g(x) e^{ik_x x}$ denote the Frank-Condon factors corresponding to the transitions $j \to j, j+1$. Due to the exponentially decreasing Frank-Condon overlaps, all transitions with $j' \neq j, j+1$ can safely be neglected. As the Wannier functions in a deep optical lattice are well localized, the products $\phi_j^{e_1}(x)\phi_j^g(x)$ and $\phi_j^{e_1}(x)\phi_{j+1}^g(x)$ are well-localized functions at positions $x_j \pm a/4$ with $a \sim \lambda/2$ being the lattice constant. Thus, (14) can be rewritten as

$$\mathcal{H}_{\text{int}} = \sum_j \sum_{\vec{k}} \left[g_{\vec{k}} \hat{a}_{\vec{k}} \hat{e}_j \eta_{j1}^{(k)} (\hat{c}_j^\dagger + \nu \hat{c}_{j+1}^\dagger e^{ik_x a/2}) + \text{H.a.} \right], \quad (15)$$

where the (real) parameter ν can be tuned by shifting the position of the $|e_1\rangle$ lattice, relative to that of $|g\rangle$. Coupling of the many motional states in $|r\rangle$ with a laser to $|e_1\rangle$ leads, after elimination of the vacuum modes, to an optical pumping that can be described by independent Lindblad generators

$$L_j^{\text{pump}} = \chi(c_j^\dagger + \nu c_{j+1}^\dagger) \quad (16)$$

with $\chi \sim \frac{\Omega_1^2}{\gamma}$, which describe the coupling to two adjacent lattice sites. Note that the relative phase term $e^{ik_x a/2}$ in Eq. (15) vanishes after averaging over the vacuum modes up to the first order in $k_0 a/2$.

We now show that optical pumping from $|g\rangle$ via $|e_2\rangle$ leads to a loss of fermions in all superpositions of neighboring sites except for one dark mode. The corresponding Hamiltonian describing the coherent part of the interaction reads

$$\mathcal{H}_{\text{int},2} = \sum_j \Omega_2 \hat{f}_j (\tilde{\eta}_{j1} \hat{c}_j + \tilde{\eta}_{j2} \hat{c}_{j+1}) + \text{H.a.}, \quad (17)$$

where $\tilde{\eta}_{j1}^{(k)} = \int dx\, \phi_j^{e_1}(x)\phi_j^g(x) e^{iq_x x}$ and $\tilde{\eta}_{j2}^{(k)} = \int dx\, \phi_j^{e_1}(x)\phi_{j+1}^g(x) e^{iq_x x}$, where $q_x = \vec{q}\cdot\vec{e}_x$ and $\vec{q}$ is the wave vector of the laser corresponding to Ω_2. Note that since the wave vector of Ω_2 is well defined, $\tilde{\eta}_{j1}$ and $\tilde{\eta}_{j2}$ differ in both amplitude and phase. Considering a fast subsequent decay from $|e_2\rangle$ finally gives rise to an optical pumping out of state $|g\rangle$ described by independent Lindblad generators

$$L_j^{\text{decay}} = \gamma(c_j + \nu e^{ig} c_{j+1}). \quad (18)$$

Without loss of generality, we can set $\gamma = 1$, which fixes the overall time scale of the process. The free parameters of the Liouvillian then are the amplitudes χ and ν and the phase g.

Solving the Lyapunov-Sylvester equation (9), we find for the symbol function of the correlation matrix

$$\gamma(\phi) = \frac{1}{d(\phi)} \begin{bmatrix} n_{11}(\phi) & n_{12}(\phi) \\ -n_{12}(\phi) & n_{11}(\phi) \end{bmatrix}, \quad (19)$$

where

$$n_{11}(\phi) = 4i\nu\chi^2(1 + \nu^2 + 2\nu\cos\phi)\sin g \sin\phi, \quad (20)$$

$$n_{12}(\phi) = (1 + \nu^2 + 2\nu\cos g\cos\phi)^2 - 4\nu^2\sin^2 g\sin^2\phi - \chi^4(1 + \nu^2 + 2\nu\cos\phi)^2 \quad (21)$$

and

$$d(\phi) = [(1 + \nu^2)(1 + \chi^2) + 2\nu(\chi^2 + \cos g)\cos\phi]^2 - 4\nu^2\sin^2 g\sin^2\phi. \quad (22)$$

The Fourier-transform according to Eq. (11) yields the correlations of Majorana fermions. From the symmetry of $\gamma(\phi)$ it is immediately clear that, as expected, only normal correlations of fermionic creation and annihilation operators are nonzero. Following Ref. [12] we can define criticality by a diverging correlation length

$$\xi^{-1} = -\lim_{|i-j|\to\infty} \frac{\ln |\langle \hat{c}_i^\dagger \hat{c}_j \rangle_{\text{SS}}|}{|i-j|}. \quad (23)$$

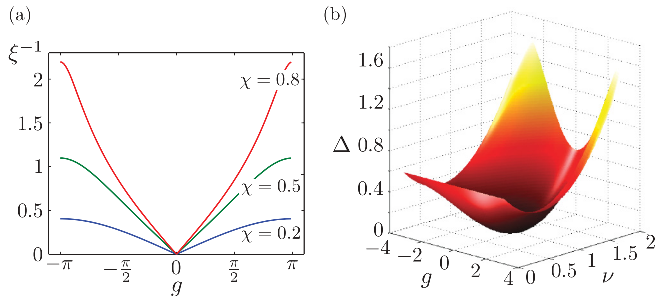

FIG. 2. (Color online) For the quantum-optical example the inverse correlation length shown in (a) shows a linear behavior around a critical point at $g_c = 0, \nu_c = 1$. Near the critical point the damping gap Δ, shown in (b), approaches zero with exponent 2. Both length and time scales diverge in the vicinity of the critical point.

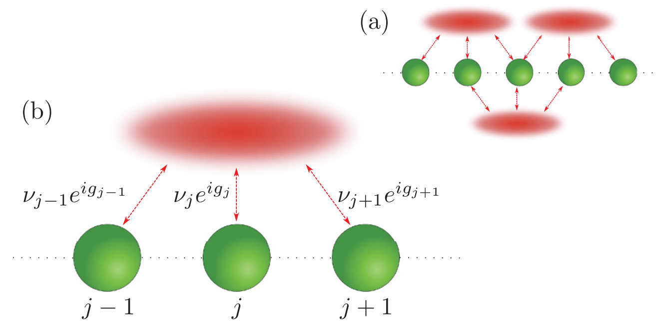

FIG. 3. (Color online) (a) Schematic representation of a linear chain translationally invariant coupled to independent reservoirs. (b) The single reservoir is characterized by $2N$ real parameters as indicated for a three-site reservoir coupling.

The correlations $\langle \hat{c}_i^\dagger \hat{c}_j \rangle$ become infinitely long ranged for $\nu = \nu_c = 1$ and $g = g_c = 0$ and any nonvanishing value of χ. In Fig. 2(a) the dependence of ξ on the phase $g - g_c$ is shown for $\nu = 1$. One recognizes a linear dependence and the same holds for the dependence on $\nu - \nu_c$. Hence, the critical exponent for this example is $\lambda = 1$. The same critical exponent has been found by Eisert and Prosen in Ref. [12].

The NESS with correlation matrix Γ_0 is an attractor of the dynamics, i.e., small deviations $\delta\Gamma$ from it will decay to zero after some time. An important quantity is the smallest decay rate for such deviations as it defines the time scale of decay back to the stationary state. Since close to the critical point correlations become infinitely long ranged, one expects that the time scale for reestablishing long-range order after a small perturbation tends to infinity. This corresponds to a closure of the damping gap Δ, i.e., the smallest nonvanishing eigenvalue of the real part of the Liouvillian. The damping gap Δ is here the direct counterpart to the excitation gap Δ_H in unitary systems. Staying in the manifold of Gaussian states, the dynamical equation for $\delta\Gamma$ reads

$$\frac{d}{dt}\delta\Gamma = X^T\,\delta\Gamma + \delta\Gamma\, X. \tag{24}$$

Thus, one has to consider the eigenvalues of X, i.e., $R + R^*$. One again finds that these eigenvalues vanish for arbitrary values of χ for $\nu = \nu_c = 1$ and $g = g_c = 0$. Figure 2(b) shows the damping gap for the present example. One recognizes a closure at the isolated point $s_c = \nu_c e^{ig_c} = 1$ with a quadratic dependence in the immediate vicinity. Thus, the dynamical critical exponent is $z = 2$.

In this particular example, we have seen that a critical point is associated with both a vanishing of the damping gap Δ and a diverging correlation length. One recognizes, furthermore, that although the parameter space of the reservoir coupling is two dimensional (disregarding the irrelevant parameter χ), there is only a single point where the system is critical. The parameter region with nonvanishing Δ is a connected manifold and, thus, there is only one distinct gapped phase. Continuous changes of the parameter $s = \nu e^{ig}$ in the complex plane circumventing the critical point $s_c = 1$ will smoothly connect the entire gapped region.

The present example gives rise to a number of questions: Is a diverging correlation length always connected to a vanishing damping gap? What are the possible universality classes, characterized by the critical exponents λ and the dynamical critical exponents z? In the following we will discuss these issues for linear fermion models with general local reservoir couplings.

IV. FREE LATTICE FERMIONS WITH GENERAL LOCAL RESERVOIRS

Let us now consider fermionic lattice models with reservoirs that couple simultaneously to N adjacent lattice sites. A schematic representation of our model is shown in Fig. 3. To simplify the calculations and the final expressions we restrict ourselves to Lindblad generators which only contain a single type of Majorana fermions. A generalization is, however, straightforward and all conclusions hold. In particular, we consider only Majorana operators of the first kind (w_{2j-1}) in the generators as they are invariant under the exchange of creation and annihilation operators. The dissipative dynamics is decoupled from the even Majorana modes, leading to degeneracies in the NESS. This degeneracy can be lifted by a free, translation-invariant Hamiltonian and the addition of this unitary term does not change the properties of the uneven Majorana modes. In this case, the steady-state equation (12) has a trivial solution in terms of the symbol function $r(\phi)$, which represents the reservoir coupling

$$r(\phi) = \sum_{m,n} \nu_m \nu_n e^{-i\phi(m-n)} e^{i(g_m - g_n)}, \tag{25}$$

$$\gamma(\phi) = \frac{r(\phi) - r(-\phi)}{r(\phi) + r(-\phi)} \mathbb{1}_{2\times 2}. \tag{26}$$

Apart from the fact that the Hamiltonian guarantees its uniqueness, the NESS is independent of the Hamiltonian details and, therefore, possible critical features of the ground state of $\mathcal{H}$ are not recovered in the steady state. Moreover, (26) does not, in general, correspond to a pure state. The particle-hole symmetry of the Lindblad generators leads to a mean occupation of $\frac{1}{2}$ in the NESS. The eigenvalues of the circulant correlation matrix Γ are given by the entries of the symbol function $\gamma(\phi)$ which is positive. They are bounded between $[-1,1]$, which is in contrast to pure states for which it can be shown that all eigenvalues must be ± 1.

By inverse Fourier transform we calculate real-space correlations in the steady state

$$\langle w_1 w_{1+2d} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \; e^{i\phi d} \gamma_{11}(\phi). \tag{27}$$

The integration cannot be carried out in general but we can understand the characteristic properties for large spatial distances d by using arguments of complex calculus. To this end we rewrite the symbol $\gamma(\phi)$ as a function in the complex plane using the substitution $e^{i\phi} = z$

$$\gamma_{11}(z) = i \frac{\sum_{j,l} \nu_j \nu_l e^{i(g_j - g_l)}(z^{j-l} - z^{l-j})}{\sum_{j,l} \nu_j \nu_l e^{i(g_j - g_l)}(z^{j-l} + z^{l-j})}, \tag{28}$$

$$\langle w_1 w_{1+2d} \rangle = \sum_{a \in S_1} \mathrm{Res}_a [z^{d-1} \gamma_{11}(z)], \tag{29}$$

where Res_a denotes the residues inside the unit circle S_1, ($|z| \leqslant 1$). The residue is nonzero only in singular points of $\gamma(z)$. Because numerator $n(z)$ and denominator $d(z)$ of (29) are holomorphic, only zeros of the denominator inside the unit circle contribute to the correlations. The symbol function has at most simple poles and it has been pointed out in Ref. [12] that the zero closest to the unit circle is relevant for the large d behavior. As the denominator $d(z)$ of Eq. (26) is just the symbol function of the matrix X, zeros of $d(z)$ on the unit circle correspond to both a diverging correlation length *and* a closure of the damping gap Δ, making both definitions of noise-induced criticality for linear models identical.

A. Conditions for criticality

The critical points, i.e., the singularities of the symbol function, are the roots of $d(z)$ on the unit circle. For general reservoir couplings to multiple sites, explicit expressions for the roots of $d(z)$ are hard to obtain and we need a different criterion to find the critical parameters. On the unit circle the denominator $d(z)$ is strictly non-negative because it can be rewritten as $d(z,\{g_j\},\{\nu_j\}) = |\sum_j \nu_j e^{i g_j} z^j|^2 + |\sum_j \nu_j e^{i g_j} z^{-j}|^2$. A configuration $\{s_j\} = \{\nu_j e^{i g_j}\}$ of the complex parameters of the Liouvillian leads to a critical behavior if a z_0 on the unit circle exists, such that the individual sums inside the absolute value vanish for z_0 and its complex conjugate. This gives a pair of implicit equations

$$1 + \sum_{j=1}^{N-1} s_j \, z_0^j = 1 + \sum_{j=1}^{N-1} s_j \, z_0^{j*} = 0, \tag{30}$$

where we have used that without loss of generality s_0 can be set equal to unity. Apparently reservoir couplings to a single site ($N = 1$) cannot induce criticality; however, couplings to $N > 1$ sites may. For a given z_0, $2(N-2)$ of the $2(N-1)$ real parameters ν_j, g_j can be chosen arbitrarily. As there are only a finite number of roots z_0 the nontrivial complex solutions $\{s_j = \nu_j e^{i g_j}\}$ to these equations are limited to a $d-2$-dimensional manifold in the $d = 2(N-2)$-dimensional parameter space. As a consequence, there can never be two extended, nonconnected regions in parameter space with a finite damping gap Δ. Thus, linear systems can become critical, but there are no reservoir-induced phase transitions between distinct gapped phases.

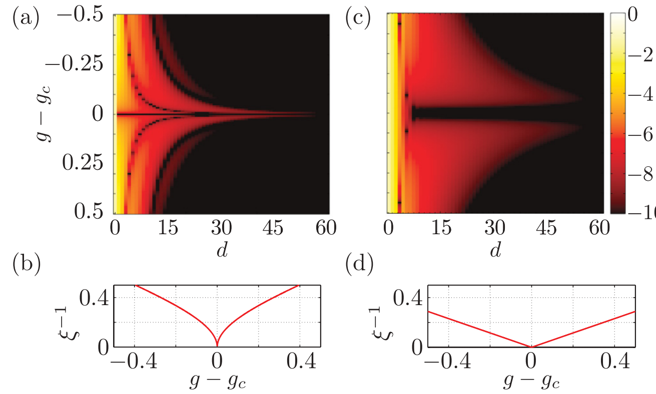

FIG. 4. (Color online) (a) and (c) show two-site correlations $|w_1 w_{1+2d}|$ on a logarithmic color scale in dependence of the spatial distance d and a parameter change around critical phase g_c. The corresponding inverse correlation length ξ^{-1} is plotted in (b) and (d) and reveals the critical exponent. The left side is the $\lambda = \frac{1}{2}$ transition and the right side the $\lambda = 1$ transition for the chain with three-site reservoir coupling discussed in the text.

B. Correlation length and critical exponents

In the vicinity of the critical point the behavior of ξ is determined by the leading-order exponent λ of the singularity, which itself is determined by the properties of the denominator $d(z)$ in Eq. (28). If a zero of $d(z)$ approaches the unit circle from the inside, ξ diverges, which corresponds to a phase transition to criticality. Let z_0 be the closest singularity to the unit circle, then the correlation length is given by

$$\xi^{-1} = -\ln |z_0| \approx 1 - |z_0|. \tag{31}$$

In the following, we will analyze the dependence of ξ on the system parameters in the vicinity of such singularities and determine the corresponding critical exponents.

For two-site coupling the implicit equations can have a nontrivial solution and we find that the generator $L_{\mathrm{two}} = \nu_0 e^{i g_0} w_1 + \nu_1 e^{i g_1} w_3$ leads to a critical NESS for $g_0 - g_1 \in \pi\mathbb{Z}$ and $\nu_0 = \pm\nu_1$. Analyzing the behavior in the vicinity of the critical points we find a critical exponent of $\lambda = 1$ in agreement with the results of Ref. [12]. The question arises if this value is the only possible one in free fermionic lattice models. To answer this question let us consider a more general reservoir with coupling to three adjacent sites: $L_{\mathrm{three}} = L_{\mathrm{two}} + \nu_2 e^{i g_2} w_5$.

A possible solution of the implicit equations (30) is $\nu_0 = \nu_1 = \nu_2$ and $g_0 = -\frac{2\pi}{3}, g_1 = 0, g_2 = \frac{2\pi}{3}$. Under variation of, for example, g_1 the critical exponent is here again $\lambda = 1$. However, there is another solution $g_0 = g_1 = g_2$ and $\nu_0 = \frac{1}{2}\nu_1 = \nu_2$. In this case the critical exponent under variation of g_1 differs and is given by $\lambda = \frac{1}{2}$.

In Fig. 4 we have plotted the two-point correlation functions for sites separated by a distance d in dependence of the phase g that drives the transition for the case of a three-site reservoir coupling. The left part of the figure corresponds to the $\lambda = \frac{1}{2}$ case, whereas the right part corresponds to a transition with $\lambda = 1$. The analytic behavior of the inverse correlation length around the critical points, shown in the lower parts of the plots, clearly distinguishes the two cases.

One recognizes that the amplitude of the component with long-range correlations vanishes as one approaches the critical points. More precisely for the case of the smaller critical exponent, $\lambda = 1/2$, all nonlocal correlations vanish, while for the case of the larger critical exponent, $\lambda = 1$, some component with finite correlation length remains also at the critical point.

In the following we want to get some general insight into what are the possible critical exponents of a reservoir-driven phase transition. As we are interested in the behavior in the vicinity of the critical point, we need to expand the denominator $d(z)$ of the symbol function in terms of the relevant system parameter $\{g_j\},\{\nu_j\}$ around their critical values. Since we do not have an explicit expression for the roots z_0 of $d(z)$, we need to do this in an implicit way, i.e., expanding $d(z)$ both in terms of $z - z_0$ and in their explicit dependence on $\{g_j\},\{\nu_j\}$.

Due to the positivity of $d(z)$ on the unit circle, all first-order partial derivatives with respect to z as well as to the system parameter g_k and ν_k must be zero at the critical point. To find the leading-order expansion in $z - z_0$ we, thus, evaluate higher-order partial derivatives with respect to z using the implicit equations (30)

$$\frac{\partial^2 d}{\partial z^2} = z^{-2} \sum_{j,l} \nu_j \nu_l e^{i(g_j - g_l)}[(j-l)(j-l-1)z^{j-l} + (l-j)(l-j-1)z^{l-j}]. \tag{32}$$

If the second-order derivative is nonzero at $z = z_0$, we can stop at this level. On the other hand, the second-order derivative vanishes if a second pair of independent implicit equations is fulfilled, which can easily be read off from (32). This procedure can be continued and each term $\frac{\partial^{2m} d}{\partial z^{2m}}|_{z_0}$, which is zero, yields a new pair of implicit equations

$$\sum_j \nu_j j^m e^{i g_j} z_0^j = \sum_j \nu_j j^m e^{i g_j} z_0^{*j} = 0. \tag{33}$$

Here we have used that the first nonvanishing derivative must be an even one. Let us assume that all derivatives in z vanish up to order $2M - 1$. It then can be shown that mixed derivatives of the type $\partial_{(\nu_k, g_k)} \partial_z^m d|_{z_0}$ vanish for all $m < M$. Thus, what remains are the second-order partial derivatives with respect to ν_k or g_k. Second-order partial derivatives in the same parameter are always nonzero on the unit circle (except for trivial cases), as $\partial^2_{\nu_k} d(z)|_{z_0} = 2$, $\partial^2_{g_k} d(z)|_{z_0} = 2\nu_k^2$. Thus, we can write the power expansion of $d(z)$ in the following general way:

$$d(\tilde{z},\tilde{x}) \approx C_2 \tilde{x}^2 + \tilde{x}\left[C_{1,\tilde{M}} \tilde{z}^{\tilde{M}} + O(\tilde{z}^{\tilde{M}+1})\right] + C_{0,2\tilde{M}} \tilde{z}^{2\tilde{M}} + O(\tilde{z}^{2\tilde{M}+1}), \tag{34}$$

where C_2 and $C_{0,2\tilde{M}}$ are nonzero constants. Here $\tilde{x}$ is a linear combination of parameter variations from the critical values $\tilde{g}_k = g_k - g_{kc}$ and $\tilde{\nu}_k = \nu_k - \nu_{kc}$ and $\tilde{z} = z - z_0$. The lowest nonvanishing contributions determine the critical exponent and, therefore, $\tilde{M}$ is the minimal M of all parameters included in $\tilde{x}$. The zeros z_0 are algebraic functions of the system parameters and, therefore, we can write $\tilde{z} \approx \tilde{x}^\lambda + O(\tilde{x}^{\lambda+1})$. At least two terms in the expansion must be of the lowest order and we find $\lambda = \tilde{M}^{-1}$ along the line $\tilde{x}$ if the first $\tilde{M}$ implicit equations are fulfilled. We see that all possible critical exponents are the inverse of integer numbers. The smallest possible critical exponent is determined by the maximum $\tilde{M}$, which is just given by M.

We now can relate the M to the number N of adjacent sites coupled by each local reservoir. It is clear that equations (33) are linearly independent for different m. On the other hand, only N equations can be independent for a finite reservoir coupling. This proves that M can be at most $N - 1$ or all orders vanish, in which case the symbol function must be zero everywhere and the system is not critical. We conclude that if the reservoir coupling is restricted to N sites, the critical exponent is out of a bounded set of fractional numbers

$$\lambda \in \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N-1}\right\}. \tag{35}$$

This is the main result of the present paper.

The corresponding Taylor expansion of the numerator $n(z)$ in a critical point is of higher order than the denominator. Therefore, the amplitude of the critical correlations vanishes as ξ diverges. This is seen in the graphs of Fig. 4. The remaining nonlocal correlations, visible, for example, in Fig. 4(b) are due to additional singularities inside the unit circle. These singularities cannot exist due to fundamental laws of algebra only in the case of the minimal critical exponent $\lambda = \frac{1}{N-1}$. In this case, the NESS is completely mixed in the critical point.

Another result that can be drawn from our analysis is the dimensionality of the critical parameter space. Critical points have to fulfill the set of equations (33) and, for a given critical exponent, the corresponding dimensionality is given by

$$\dim(P_\lambda) = 2\left(N - 1 - \frac{1}{\lambda}\right). \tag{36}$$

It is clear that the critical points are always a zero measure subset of parameter space, but they are not necessarily isolated points, with the exception of critical points with minimal exponent for the given configuration, which are always singular.

C. Spectral gap of relaxation rates and dynamical critical exponent

For unitary lattice models it is well established that the presence of a finite gap in the excitation spectrum leads to a finite correlation length, while the transition to criticality is associated with a vanishing gap [15]. In the following, we want to establish a corresponding relation for reservoir-driven phase transitions and discuss the dynamical critical exponents.

The relaxation rates of the system are determined by the homogeneous part of (8) and, therefore, the damping matrix X. More precisely, the damping matrix X describes the dynamics on the submanifold of Gaussian states, whereas the full system is spanned by the Liouville operator $\mathcal{L}$ in Eq. (1). The trace preservation of the Lindblad dynamics requires $\mathcal{L}$ to have at least one eigenvalue with a vanishing real part. If this zero eigenvalue is unique, the gap in the real spectrum sets the slowest relaxation rate for arbitrary initial states. The gap Δ of X thus gives an upper bound for the gap of $\mathcal{L}$, and the gap in the full damping spectrum must vanish as one approaches the critical points. The eigenvalues of X are purely real, when neglecting the Hamiltonian contributions and strictly negative. In the translation-invariant system the eigenvalues are given by the symbol function $-r(\phi) - r(-\phi)$, which we have identified before as relevant for the correlation length. In the vicinity of a critical point, the slowest relaxation rate, defining the spectral gap of relaxation, is determined by the roots z_j of $d(z,\{g_j\},\{\nu_j\})$ closest to the unit circle

$$\Delta = \min_{|z|=1} d(z,\{g_j\},\{\nu_j\}). \tag{37}$$

Therefore, the dynamical exponent is immediately related to the critical exponent λ. The exponent of the divergence, however, must be modified by the number of roots κ_c that merge at the same point on the unit circle when the Liouville parameters approach their critical values:

$$\Delta \sim |g - g_c|^{\kappa_c \lambda}. \tag{38}$$

The number of roots κ_c is, thus, identical to the dynamical critical exponent z. For the minimum critical exponent $\lambda = 1/(N-1)$, all $2(N-1)$ complex roots merge simultaneously on the same point and, thus,

$$z\lambda = 2, \qquad \text{for} \quad \lambda = \lambda_{\min}. \tag{39}$$

Moreover, for all examples we have considered we found that the damping gap closes as a quadratic function, which suggests that (39) is more general.

V. SUMMARY

To summarize, we have analyzed the nonequilibrium steady state of translation-invariant chains of free fermions coupled to local Markovian reservoirs described by linear Lindblad generators. Such couplings can be generated, e.g., in ultracold atomic lattice gases as we have shown for the example of the two-site coupling. A general expression for the correlation matrix of the NESS can be obtained using a symbol function ansatz. We showed that under certain conditions the NESS goes into a critical state on changing reservoir parameters. The critical state is characterized by a simultaneous divergence of a correlation length and a critical slow-down of relaxation, i.e., a closing of the gap in the damping spectrum. We showed that the dimension of the critical parameter space is at most $d-2$, where d is the dimension of the full space of reservoir parameters. As a consequence, all gapped phases are smoothly connected and, although there is a transition into a critical phase, there is no reservoir-induced quantum phase transition between distinct gapped phases for linear models. The transitions to a critical state can be classified by the leading-order exponent in the dependence of the inverse correlation length on the system parameter that drives the transition. We have shown that this critical exponent must be the inverse of an integer between 1 and $N-1$, where N is the number of sites coupled by a single reservoir. Furthermore, a general expression for the dynamical critical exponent, describing the closure of the damping gap, was derived.

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