Tunneling ionization of atoms and ions in an elliptical electromagnetic wave and a static magnetic field

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We consider the tunneling ionization of an electron, bound by a zero-range potential and a constant magnetic field, under the influence of a monochromatic laser beam with elliptical polarization. The exact solution of the Schrödinger equation and the Green's function for an electron moving in an arbitrary electromagnetic wave and crossed constant electric and magnetic fields are obtained. The exact expressions are found for the level shift and width of the electron in a zero-range force field, a constant magnetic field, and a monochromatic electromagnetic field. In the case of ionization of neutral atoms and positive ions, we also take into consideration the Coulomb interaction of the emerging electron with the atomic or ionic core. The first-order contributions from the Stark and Zeeman effects to the ionization rate are taken into account as well. The paper generalizes the results earlier obtained by V. M. Rylyuk and J. Ortner [Phys. Rev. A **67**, 013414 (2003)].

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I. INTRODUCTION

Over the past three decades, significant progress has been made in the development of lasers capable of producing strong pulses. In this regard, of unabated interest are the phenomena related to interaction of high-intensity laser radiation with atoms and ions [1,2]. Ionization is one of these effects which characterizes the interaction of light with matter. The basic concepts of the theory of ionization were developed by Keldysh [3], who demonstrated that the tunneling effect and the multiphoton ionization are two limiting cases of the common process of nonlinear photoionization. The authors of Refs. [4–7] developed a consistent quantum-mechanical theory of ionization of a level bound by short-range forces based on the replacement of the exact final-state wave function with the Volkov wave function. The theory [4–7] is valid for the case of low frequencies $\omega \ll |E_0|/\hbar$ ($|E_0|$ being the ionization potential of the level) and not-too-high electric fields $F \ll F_0$, where F is the magnitude of the perturbing field and F_0 is that of the inneratomic field. The indicated authors employed the standard saddle-point method with a quadratic expansion to calculate the ionization probability in the tunnel regime, which is equivalent to the quasiclassical approximation. This approach was further developed by Faisal [8] and Reiss [9] (the latter author called it the strong-field approximation). In the paper [10] a more general saddle-point method was developed which makes it possible to extend the validity region for the adiabatic approximation. The case of strong electric fields at high frequencies may be described within the framework of the high-frequency Floquet theory (HFFT) [11]. Since the Volkov wave function does not behave correctly near the origin of the binding potential, the use of this function as the final-state wave function imposes an upper limit on the electric-field magnitudes for which the adiabatic approximation remains valid. The Coulomb interaction between the emerging electron and the atomic core was included in consideration in Refs. [12–15]. It was shown in there that the Coulomb interaction leads to an increase of ionization rates. The questions considered above were also expounded in reviews [16,17].

On the other hand, it is very useful to have exact methods for calculating the decay rates. Such is the method of zero-range potential [18], which was developed in parallel with the adiabatic approach. The zero-range potential is an approximate model for a negative ion. The effect of the zero-range potential can be accounted for via the boundary condition on the wave function at the origin of the force field. In the rest of space the electron moves freely and its wave function is well known. In the absence of electric and magnetic fields the boundary condition yields a continuous energy spectrum for a free electron motion and a single energy level located below the continuum and describing the only bound state possible in the system. In the presence of an electric field the lower boundary of the continuum moves to minus infinity. As a result, the bound-state level is located inside the continuum. The level energy moves away from the real axis in the complex plane, its imaginary part describing the decay probability of the bound state [18].

The zero-range potential method was further used to find the bound-state energy of a charged particle in a zero-range field and a constant magnetic field [19]. It was also used to calculate the level shift and width for an electron in a zero-range field and crossed static electric and magnetic fields [20]. As the magnetic field is increased, the bound-state energy moves to the right along the real axis. As a result, the decay of the bound state becomes less probable. The ionization in crossed electric and magnetic fields was also considered within the adiabatic approach [21,22]. There the decrease of the ionization rate as the magnetic field increases is explained by the elongation of the electron underbarrier trajectory. The latter is screwlike in the presence of a magnetic field and, thus, longer than that without the magnetic field. The generalization of the short-range potential method to nonstationary problems was pursued in Refs. [23,24]. Its authors solved a full nonstationary problem on the decay of a weakly bound level in a monochromatic field with an arbitrary elliptical polarization.

Up to that moment, one theoretical problem still remained unsolved: the decay of a weakly bound level in a monochro-

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matic wave and a constant magnetic field. In Ref. [25], the decay of a weakly bound level in the circularly polarized wave and a constant magnetic field was considered. The authors of Ref. [25] gave the exact solution of the Schrödinger equation for an electron moving in the net electromagnetic field produced by an electromagnetic plane wave and a constant magnetic field. However, they neglected the Coulomb interaction between the emerging electron and the atomic core. It should be noted that the ionization induced by an electromagnetic wave is also of great practical importance. Strong magnetic fields of some mega-Gauss may be observed in laser-produced plasmas [26]. Weaker magnetic fields may have a great effect in semiconductors [27].

If an electromagnetic wave is present instead of a static electric field, then the effect of the magnetic field can no longer be reduced to a decrease of the decay probability. In quasiclassical terms, it is the polarization of the monochromatic wave that determines whether the underbarrier trajectory is elongated or shortened by the uniform magnetic field. Therefore, in contrast to the case of crossed static electric and magnetic fields, the magnetic field may either decrease or increase the ionization rate in the presence of a nonstationary electric field.

In this paper, we consider the tunneling ionization of atoms and ions under the influence of an elliptical electromagnetic wave and a static magnetic field. For this purpose, the exact solution and the Green's function of the Schrödinger equation for an electron moving in these fields are represented in a convenient form. On their basis, we solve the ionization problem with the short-range potential method and, in addition, take into account the Coulomb interaction between the emerging electron and the atomic or ion core. Note that the Coulomb correction in the case of crossed static electric and magnetic fields was considered in Ref. [28]. Thus, this work generalizes the results of Ref. [25] and also well-known results obtained for the ionization in crossed static electric and magnetic fields and for the ionization induced by a monochromatic electromagnetic field alone.

II. EXPRESSION FOR THE COMPLEX QUASIENERGY

In the integral form, the Schrödinger equation for the wave function $\hat{\Psi}_{\epsilon,s}(\boldsymbol{r},t) = \exp(-i\epsilon t/\hbar)\ \hat{\Phi}_{\epsilon,s}(\boldsymbol{r},t)$ of an electron in the field of potential $\mathrm{U}(\boldsymbol{r})$ reads

$$\hat{\Phi}_{\epsilon,s}(\boldsymbol{r},t) = \int_{-\infty}^{t} dt' e^{-i\epsilon(t-t')/\hbar} \int d\boldsymbol{r}'\ \hat{\mathrm{G}}(\boldsymbol{r},t;\boldsymbol{r}',t',s)\mathrm{U}(\boldsymbol{r}') \times \hat{\Phi}_{\epsilon,s}(\boldsymbol{r}',t'), \tag{1}$$

where $\hat{\mathrm{G}}(\boldsymbol{r},t;\boldsymbol{r}',t',s)$ is the retarded Green's function of an electron with the spin $\boldsymbol{s}$ moving under the influence of the short-range potential, the constant magnetic field $\boldsymbol{H}$, and the field of a monochromatic electromagnetic wave. The exact solution of Eq. (1) was obtained in Ref. [25].

For calculating the decay rate we use the quasistationary-quasienergy-state formalism [23,24] (see also Refs. [29–33]), which is a generalization of the usual quasienergy-state formalism [34]. In this approach, the position and width of the level are determined in a unified manner as the real and imaginary parts of the complex quasienergy. In the short-range potential model [18,24], the boundary condition at $r \to 0$ is

$$\begin{aligned} \hat{\Phi}_{\epsilon,s}(\boldsymbol{r},t) &\simeq \frac{1}{4\pi}\left(\frac{1}{r} - \frac{1}{a} + 2i\frac{\boldsymbol{r}\boldsymbol{A}_f}{ra}\right)\hat{f}_{\epsilon,s}(t) + O(r), \\ \hat{f}_{\epsilon,s}(t) &= \hat{f}_{\epsilon,s}\left(t + \frac{2\pi}{\omega}\right), \end{aligned} \tag{2}$$

where $a = \hbar/\sqrt{-2mE_0}$ is the so-called scattering length, E_0 is the energy of an electron bound by the short-range potential alone, and $\boldsymbol{A}_f$ is the sum of the vector potentials of an electromagnetic wave and a constant magnetic field. Equation (2) is equivalent to the relation

$$\mathrm{U}(\boldsymbol{r})\hat{\Phi}_{\epsilon,s}(\boldsymbol{r},t) = -4\pi\delta(\boldsymbol{r})\hat{f}_{\epsilon,s}(t). \tag{3}$$

Using Eqs. (2) and (3) and the expression for the Green's function obtained in Ref. [25], we get from Eq. (1) the equation for the unknown complex quasienergy $\epsilon = E - i\Gamma$ and the function $\hat{f}_{\epsilon,s}(t)$:

$$\begin{aligned} &(\sqrt{\epsilon} - \sqrt{|E_0|})\hat{f}_{\epsilon,s}(t) \\ &= -\sqrt{\frac{\hbar}{4\pi i}}\int_0^\infty \frac{dt'}{t'^{3/2}}\exp\left(-\frac{i}{\hbar}\epsilon t'\right) \\ &\times \left\{\frac{\omega_H t'}{2\sin(\omega_H t'/2)}\hat{f}_{\epsilon,s}(t-t') \right. \\ &\left. \times \exp\left(\frac{i}{\hbar}\mathrm{S}(t,t') \pm i\frac{\omega_H}{2}t'\right) - \hat{f}_{\epsilon,s}(t)\right\}, \end{aligned} \tag{4}$$

where $\mathrm{S}(t,t')$ is the classical action for an electron (see Ref. [25]). Equation (4) is the main one in the short-range potential method. It generalizes Eq. (4) in Ref. [24] for the case where a static magnetic field is present.

Following Ref. [24], we expand the function $\hat{f}_{\epsilon,s}(t)$ into a Fourier series (with t measured in the units of ω)

$$\hat{f}_{\epsilon,s}(t) = \sum_{k=-\infty}^{\infty} \hat{f}_{\epsilon,s,k}\ \exp(2ikt) \tag{5}$$

and obtain the following homogeneous system of equations for the unknown coefficients $\hat{f}_{\epsilon,s,k}$:

$$\left[\left(\beta^2 + \frac{2k\hbar\omega}{|E_0|}\right)^{1/2} - 1\right]\hat{f}_{\epsilon,s,k} = \sum_{n=-\infty}^{\infty}\hat{\mathrm{M}}_{s,kn}(\epsilon)\hat{f}_{\epsilon,s,n}, \tag{6}$$

where

$$\begin{aligned} \hat{\mathrm{M}}_{s,kn}(\epsilon) = &-\frac{1}{\sqrt{4\pi i\lambda}}\int_0^\infty \frac{dt}{t^{3/2}}\exp(-i\lambda\beta^2 t) \\ &\times \left\{\frac{\omega_0 t}{2\sin(\omega_0 t/2)}\exp(-2ikt)\hat{\mathrm{I}}_{s,kn}(t) - \delta_{kn}\right\} \end{aligned} \tag{7}$$

and

$$\hat{\mathrm{I}}_{s,kn}(t) = \frac{1}{\pi}\int_0^\pi \exp\left(i\mathrm{S}(\tau,t) \pm i\frac{\omega_0}{2}t + 2i(k-n)\tau\right)d\tau, \tag{8}$$

where $\omega_0 = \omega_H/\omega$ is the ratio of the cyclotron frequency to the laser frequency, $\lambda = |E_0|/\hbar\omega$ is the multiphoton parameter determining the minimal number of photons necessary for ionization, and $\beta = \sqrt{\epsilon/|E_0|}$ describes the alteration of the electron energy by the electric and magnetic fields.

Equations (6)–(8) are closed equations for the determination of the complex quasienergy ϵ of an electron moving under the influence of three fields: (i) that of the short-range potential, (ii) the constant magnetic field, and (iii) the field of an arbitrary electromagnetic wave. They are exact for arbitrary magnitudes of the electric and magnetic fields. Setting in Eqs. (6)–(8) $H = 0$, we reproduce the general solution [Eq. (8) in Ref. [24] and Eq. (9)] for the complex quasienergy of an electron moving in a short range-potential field and the field of an elliptically polarized electromagnetic wave. On the other hand, setting in Eqs. (6)–(8) $\omega = 0$, we get the closed equation [20] [Eq. (4)] for the energy ϵ of an electron moving in a short-range potential field and being perturbed by crossed static electric and magnetic fields.

The closed equation for ϵ follows from Eq. (6),

$$\det\left|\left|\left[\left(\beta_\pm^2 + \frac{2k\hbar\omega}{|E_0|}\right)^{1/2} - 1\right]\delta_{k,n} - \hat{\mathrm{M}}_{s,kn}(\epsilon)\right|\right| = 0, \tag{9}$$

where the two signs $\pm$ correspond to the two electron spin directions: parallel and antiparallel to the constant magnetic field $\boldsymbol{H}$. To derive basic analytical results for $H \neq 0$ and $\omega \neq 0$, we will consider some simplifications of the general case. In what follows we analyze the low-frequency limit $\lambda \gg 1$ and then Eq. (9) is simplified as follows:

$$\beta_\pm \simeq 1 + \hat{\mathrm{M}}_{s,00}(\epsilon) + O(1/\lambda) \tag{10}$$

or

$$\beta \simeq 1 - \frac{1}{\sqrt{4\pi i\lambda}}\frac{1}{\pi}\int_0^\pi d\tau \int_0^\infty \frac{dt}{t^{3/2}} \exp(-i\lambda\beta^2 t) \times \left\{\frac{\omega_0 t}{2\sin(\omega_0 t/2)} \exp\left[i\lambda \mathrm{S}_0(\tau,t)\right] - 1\right\}, \tag{11}$$

where the spin term $\pm\omega_0/2$, giving a contribution of the order $\sim 1/\lambda$, was neglected. Equation (11) is an integral equation for the determination of the desired complex quasienergy $\epsilon = E - i\Gamma$. We note that the approximation (11) corresponds to setting $\hat{f}_{\epsilon,s,n} = \text{const}$ and averaging the right-hand side in Eq. (4) with respect to t over the period π. Equation (11) generalizes Eq. (16 b) in Ref. [24] for the case where a static magnetic field is present.

Thus far, all the expressions have been valid for a monochromatic wave with an arbitrary direction of propagation. In the next section, we restrict our consideration to the case of an elliptically polarized wave propagating along the magnetic field.

III. ANALYTICAL RESULTS FOR THE CASE OF AN ELLIPTICALLY POLARIZED WAVE PROPAGATING ALONG THE MAGNETIC FIELD

We now consider a monochromatic wave, of frequency ω, which propagates along the constant magnetic field $\boldsymbol{H}$ (i.e., along the z axis) and has an elliptical polarization. The components of its vector potential are

$$A_x(t) = -\frac{F}{\omega}\sin\omega t, \quad A_y(t) = g\frac{F}{\omega}\cos\omega t, \quad A_z(t) = 0, \tag{12}$$

where F is the amplitude of the electromagnetic field.

Far from the resonance frequency $\omega = \omega_H$, the action S_0 in Eq. (11) can be represented in the form

$$\mathrm{S}_0(\tau,t) = -\frac{1}{2\gamma^2}\frac{1+g^2+2g\omega_0}{1-\omega_0^2}t + \frac{1}{2\gamma^2}\frac{1-g^2}{1-\omega_0^2}\sin(t)\cos(2\tau - t) + \frac{1}{\gamma^2}\frac{\omega_0}{(1-\omega_0^2)^2}\sin^2\left(\frac{t}{2}\right)\cot\left(\frac{\omega_0 t}{2}\right) \times\left[(1+g^2)\left(1+\omega_0^2\right) + 4g\omega_0 - (1-g^2)\left(1-\omega_0^2\right)\cos(2\tau - t)\right] - \frac{1}{\gamma^2}\frac{\omega_0}{(1-\omega_0^2)^2}(1+g\omega_0)(g+\omega_0)\sin(t), \tag{13}$$

where $\gamma = \sqrt{2|E_0|}\omega/F$ is the adiabatic Keldysh parameter. Here and above we use the following atomic units: $e = m = c = \hbar = 1$.

Transforming the integral over t in Eq. (11) into one in the complex plane z and applying the standard saddle-point method, we obtain a system of equations for the width and the shift of the bound electron level,

$$\mathrm{Im}\beta = -\frac{\omega_H}{4|E_0|\sqrt{2z_0}\sinh(\omega_0 z_0)}\left(\frac{1}{|F''(z_0)|}\right)^{1/2} \times \mathrm{Re}\{\exp[-2\lambda F(z_0)]\},$$
$$\mathrm{Re}\beta = 1 - \frac{1}{2\sqrt{2\pi\lambda}}\mathrm{Re}\left\{\int_C \frac{dz}{z^{3/2}}\left[\frac{\omega_0 z}{\sinh(\omega_0 z)} \times \exp[-2\lambda F(z)] - \exp(-2\lambda\beta^2 z)\right]\right\}, \tag{14}$$

where $F(z) = \frac{i}{2}[-\mathrm{S}_0(\tau = t/2,t) + \beta^2 t]_{t=-2iz}$ and the contour C in Eq. (14) goes from the point $z = 0$ to the saddle point z_0 determined by the condition $F'(z_0) = 0$, i.e., by the equation

$$\sinh^2 z_0 - \frac{(g+\omega_0)^2}{\left(1-\omega_0^2\right)^2}\left[\cosh z_0 - \omega_0 \sinh z_0 \coth(\omega_0 z_0)\right]^2 = \gamma^2. \tag{15}$$

It is seen from Figs. 1 and 2 that for an elliptically polarized wave the left-hand side of Eq. (15) monotonously increases as a function of z for all ω_0. Therefore, there exists a unique saddle point z_0 for each γ (for a rigorous proof of this fact, see the Appendix).

For further simplifications, consider the case of weak electric and magnetic fields. Since their influence on the electron energy is small, we can put $\beta \simeq 1$ in the right-hand

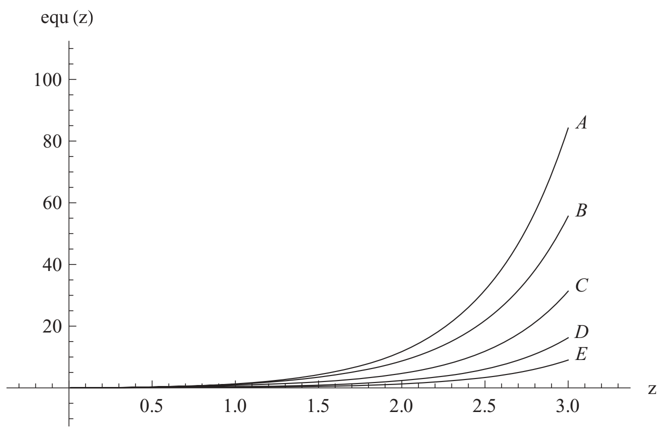

FIG. 1. The left-hand side equ(z) of Eq. (15) as a function of z at $g = +0.5$; $\omega_{0A} = 0.1$, $\omega_{0B} = 0.9$, $\omega_{0C} = 2$, $\omega_{0D} = 5$, $\omega_{0E} = 10$.

side of Eq. (14) to obtain

$$\Gamma = \frac{\omega_H}{2\sqrt{2z_0}\sinh(\omega_0 z_0)}\left(\frac{1}{|F''(z_0)|}\right)^{1/2}\exp[-2\lambda F(z_0)], \quad (16)$$

$$E = |E_0|\left\{-1-\frac{1}{4}\left(\frac{F}{F_0}\right)^2\left[1-\frac{7}{48\lambda^2}(g^2-3)-\frac{7}{12}\frac{g}{\lambda}\frac{H}{H_0}+\frac{13}{2}\left(\frac{F}{F_0}\right)^2\right]+\frac{1}{12}\left(\frac{H}{H_0}\right)^2-\frac{5}{144}\left(\frac{H}{H_0}\right)^4+\frac{1}{2}\left(\frac{HF}{H_0F_0}\right)^2\left[1+\frac{1}{2\lambda^2}\left(\frac{221}{96}-\frac{541}{576}g^2\right)\right]\right\}, \quad (17)$$

where $H_0 = F_0^{2/3} = 2|E_0|$ is the magnetic field with the Landau energy having the order of the level binding energy ($H_0 = 2.35 \times 10^5$ T for the H atom) and the function $F(z_0)$ is

$$F(z_0) = \left(1+\frac{1+g^2+2g\omega_0}{2\gamma^2(1-\omega_0^2)}\right)z_0 - \frac{1-3\omega_0^2-g(g+2\omega_0)(1+\omega_0^2)}{4\gamma^2(1-\omega_0^2)^2}\sinh(2z_0) - \frac{\omega_0(g+\omega_0)^2}{\gamma^2(1-\omega_0^2)^2}\sinh^2 z_0\,\coth(\omega_0 z_0). \quad (18)$$

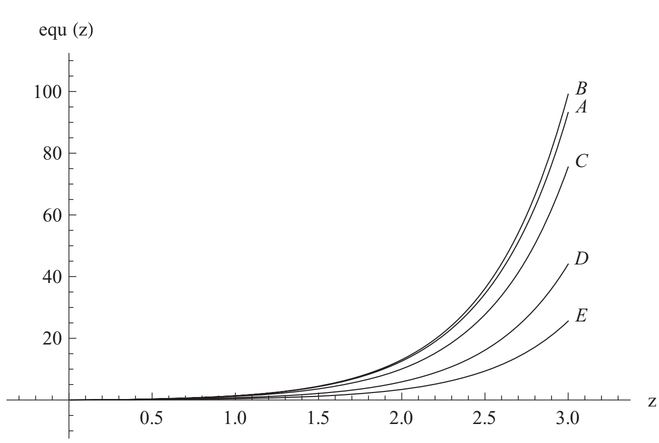

FIG. 2. The left-hand side equ(z) of Eq. (15) as a function of z at $g = -0.5$; $\omega_{0A} = 0.1$, $\omega_{0B} = 0.9$, $\omega_{0C} = 2$, $\omega_{0D} = 5$, $\omega_{0E} = 10$.

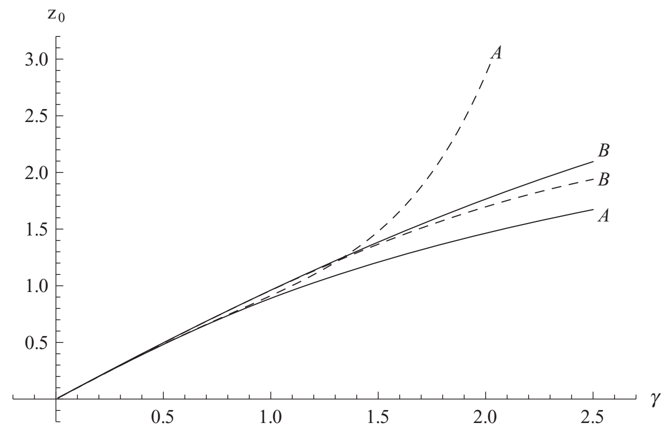

FIG. 3. The saddle point z_0 as a function of γ at $\omega_0 = 0.5$; $g_A = -1$ and $g_B = +1$. (Solid line) exact saddle-point equation (15); (dashed line) asymptote (19).

The expressions (16) and (17) are the quasiclassical ionization rate and the level shift for an electron in a short-range potential under the influence of an elliptically polarized wave and a static magnetic field. They generalize the results [5,20,35] for the decay probability and the level shift in the case of an elliptical wave and crossed fields. In the case of a circular electromagnetic wave ($g = \pm 1$), the ionization rate (16), with the function $F(z_0)$ from Eq. (18), and the level shift (17) coincide with the corresponding results in Ref. [25].

The real part of the quasienergy contains the contributions from the Stark effect, the Zeeman effect (both are written down here up to the fourth order), the cross terms starting with the biquadratic order, and the terms depending on the frequency of the electromagnetic wave. One sees from Eq. (17) that the term in the quadratic brackets, which is proportional to $g(H/H_0)$, increases the energy in the case of a right-polarized wave and decreases it in the case of a left-polarized wave.

In the adiabatic limit, where the inequality $\gamma \ll 1$ holds, the saddle point z_0 in Eq. (15) can be written in the following

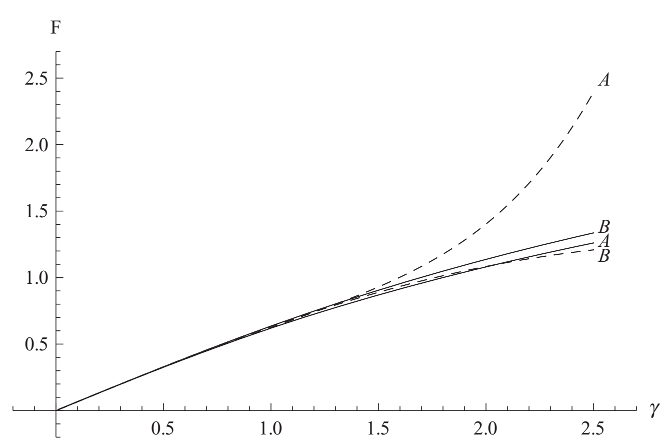

FIG. 4. The exact function $F(z_0)$ (18) (solid line) and the function $f_0(\gamma,g,\omega_0)$ (22) (dashed line) vs. γ at $\omega_0 = 0.9$; $g_A = -0.3$ and $g_B = +0.3$.

analytical form:

$$z_0 \simeq \gamma\left\{1-\frac{\gamma^2}{6}\left(1-\frac{g^2}{3}\right)+\frac{3}{40}\left[1-\frac{58}{81}g^2\left(1-\frac{35}{174}g^2\right)\right]\gamma^4 + g\frac{\gamma^2}{9}\left[1-\frac{29}{30}\gamma^2\left(1-\frac{35}{87}g^2\right)\right]\omega_0 + \frac{\gamma^2}{18}\left[1-\frac{29}{30}\gamma^2\left(1-\frac{31}{29}g^2\right)\right]\omega_0^2\right\}. \quad (19)$$

From Eq. (16) we then obtain

$$\Gamma_0 = |E_0|\frac{F}{2F_0}P_0(\gamma,g,H)\exp\left\{-\frac{2}{3}\frac{F_0}{F}f_0(\gamma,g,\omega_0)\right\}, \quad (20)$$

where

$$P_0(\gamma,g,H) \simeq 1-\frac{\gamma_H^2}{6}\left[1-\frac{8}{45}\left(1+\frac{3}{8}g^2\right)\gamma^2\right] - \frac{7}{135}g\gamma_H\left(1-\frac{10}{21}g^2\right)\gamma^3 + \frac{\gamma^4}{30}\left[1-\frac{7}{9}g^2\left(1-\frac{5}{21}g^2\right)\right] \quad (21)$$

and

$$f_0(\gamma,g,\omega_0) \simeq 1-\frac{1}{10}\left(1-\frac{g^2}{3}\right)\gamma^2 + \frac{3}{8}\left[\frac{3}{35}-\frac{2}{27}g^2\left(\frac{29}{35}-\frac{g^2}{6}\right)\right]\gamma^4 + g\frac{\gamma^2}{3}\left[\frac{1}{5}-\frac{1}{2}\left(\frac{29}{105}-\frac{g^2}{9}\right)\gamma^2\right]\omega_0 + \frac{\gamma^2}{30}\left[1-\frac{1}{42}(29-31g^2)\gamma^2\right]\omega_0^2. \quad (22)$$

In Eq. (21), $\gamma_H = \sqrt{2|E_0|}\omega_H/F$ is the magnetic Keldysh parameter, the ratio of the cyclotron frequency to the inverse tunnel time. In the limit $H \to 0$, Eqs. (20)–(22) coincide with the corresponding results in Ref. [36]. Note that in the case of a linearly polarized electromagnetic wave ($g=0$), the term in the function $f_0(\gamma,g,\omega_0)$ (22) which is proportional to ω_0 (the magnetic field H) vanishes. It follows from Eqs. (20)–(22) that in the case of a low-frequency monochromatic wave, the magnetic field causes the reduction or enhancement of the ionization probability. This effect can be explained by the distortion of the underbarrier trajectory due to the screwlike electron motion.

Some results of numerical calculations for the saddle point and the function $F(z_0)$ are shown in Figs. 3 and 4. Note that z_0 has a simple physical interpretation: $t_0 = -iz_0/\omega$ is the time of the underbarrier motion of the electron. One sees from Figs. 3 and 4 that the asymptotes of expressions (19)–(22) have a remarkable property: They can be extended to the region where $\gamma \sim 1$.

In order to take into account the influence of the Stark and Zeeman effects on the ionization rate, we take into account in the right-hand side of Eq. (14) the level shift for an electron under the influence of an elliptically polarized wave and a static magnetic field [see Eq. (17)]. Then, in the first order,we obtain

$$\Gamma = |E_0|\frac{F}{2F_0}P(\gamma,g,F,H)\exp\left\{-\frac{2}{3}\frac{F_0}{F}f(\gamma,g,F,H,\omega_0)\right\}, \quad (23)$$

where

$$P(\gamma,g,F,H) \simeq 1+\delta(F,H) - \frac{\gamma_H^2}{6}\left[1-\frac{8}{45}\left(1+\frac{3}{8}g^2\right)[1-\delta(F,H)]\gamma^2\right] - \frac{7}{135}g\gamma_H\left(1-\frac{10}{21}g^2\right)[1-\delta(F,H)]\gamma^3 + \frac{\gamma^4}{30}\left[1-\frac{7}{9}g^2\left(1-\frac{5}{21}g^2\right)\right][1-\delta(F,H)] \quad (24)$$

and

$$f(\gamma,g,F,H,\omega_0) \simeq 1-\frac{3}{2}\delta(F,H)-\frac{\gamma^2}{10}\left(1-\frac{g^2}{3}\right)\left[1-\frac{5}{2}\delta(F,H)\right]+\frac{3}{8}\left\{\frac{3}{35}-\frac{2}{27}g^2\left(\frac{29}{35}-\frac{g^2}{6}\right) - \frac{3}{10}\left[1-\frac{58}{81}g^2\left(1-\frac{35}{174}g^2\right)\right]\delta(F,H)\right\}\gamma^4+\frac{\gamma^2}{3}\left\{g\left[\frac{1}{5}-\frac{1}{2}\left(\frac{29}{105}-\frac{g^2}{9}\right)\gamma^2\right] - \frac{g}{2}\left[1-\frac{29}{30}\left(1-\frac{35}{87}g^2\right)\gamma^2\right]\delta(F,H)\right\}\omega_0+\frac{\gamma^2}{6}\left\{\frac{1}{5}-\frac{1}{2}\delta(F,H)-\frac{1}{210}(29-31g^2)\left[1-\frac{7}{2}\delta(F,H)\right]\gamma^2\right\}\omega_0^2, \quad (25)$$

where $\delta(F,H) = [F^2/F_0^2 - H^2/(3H_0^2)]/4$ takes into account the Stark and Zeeman effects.

The following two subsections deal with the cases of small and large cyclotron frequencies as compared to the frequency of the elliptical electromagnetic wave.

A. The limit of small magnetic fields, $\omega_0 \ll 1$

When the applied constant magnetic field is small, i.e., $\omega_0 \ll 1$ ($\gamma_H \ll \gamma$), Eq. (15) for the saddle point z_0 reduces to

$$\sinh^2 z_0 - g(g+2\omega_0)\left[\cosh z_0 - \frac{\sinh z_0}{z_0}\right]^2 + \frac{2}{3}gz_K\sinh z_K\sqrt{\sinh^2 z_K-\gamma^2}\,\omega_0^2 - (1+2g^2)\left[\cosh z_K - \frac{\sinh z_K}{z_K}\right]^2\omega_0^2 = \gamma^2, \quad (26)$$

whence

$$z_0 = z_K + a\omega_0 + b\omega_0^2,$$

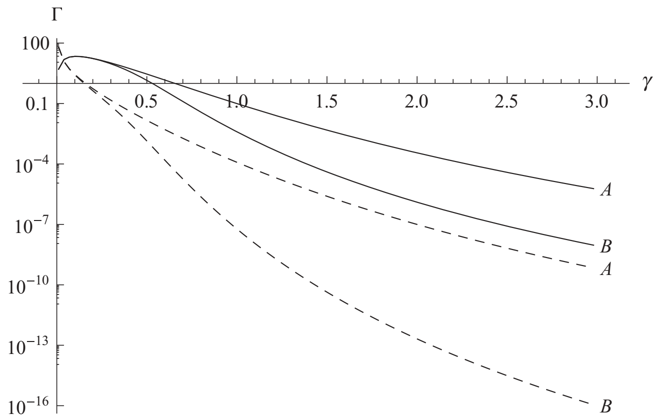

FIG. 5. The ionization rates Γ for a linearly polarized electromagnetic wave vs. γ at $\lambda = 7$. The weakly bound level (dashed line) and the $1s$ state of the H atom (solid line); $\omega_{0A} = 0.5$ and $\omega_{0B} = 5$.

where

$$a = g\frac{\sinh(2z_0)/z_0 - \cosh^2 z_0 - \sinh^2 z_0/z_0^2}{\gamma^2/z_0 - (1+g^2)\sinh^2 z_0/z_0 - (1-g^2)\sinh(2z_0)/2}$$

and z_K can be determined from the saddle-point equation (see, e.g., Eq. (5) in Ref. [36]) for zero magnetic field,

$$\sinh^2 z_K - g^2\left[\cosh z_K - \frac{\sinh z_K}{z_K}\right]^2 = \gamma^2. \tag{27}$$

We then find the ionization probability to be

$$\Gamma = P(z_0)\,\exp\left[-2\lambda F(z_0)\right], \tag{28}$$

where

$$F(z_0) = f(z_K) + F_1\omega_0 + F_2\omega_0^2$$

and

$$f(z_K) = \left(1 + \frac{1+g^2}{2\gamma^2}\right)z_K - \frac{g^2}{\gamma^2 z_K}\sinh^2 z_K - \frac{1-g^2}{4\gamma^2}\sinh 2z_K \tag{29}$$

is the so-called Keldysh function and

$$\begin{aligned} F_1 &= \frac{3g}{2\gamma^3}\left(z_K + \frac{1}{2}\sinh 2z_K - 2\frac{\sinh^2 z_K}{z_K}\right), \\ F_2 &= \frac{3}{2\gamma^3}\left[\frac{1+g^2}{2}z_K + \frac{1+3g^2}{4}\sinh 2z_K \right.\\ &\left. -\left(\frac{1+2g^2}{z_K} + \frac{g^2}{3}z_K\right)\sinh^2 z_K\right] + \frac{3a}{g\gamma}\left(\frac{\sinh^2 z_K}{\gamma^2} - 1\right) \\ &- \frac{3a^2}{2\gamma^3}\left\{\left(\frac{1-g^2}{2} - \frac{g^2}{z_K^2}\right)\sinh 2z_K \right.\\ &\left. + g^2\left[1 + \left(2 + \frac{1}{z_K^2}\right)\frac{\sinh^2 z_K}{z_K}\right]\right\}. \end{aligned} \tag{30}$$

The preexponential function $P(z_0)$ and the constant b will not be given here. For $\gamma \ll 1$ the saddle point is

$$z_0 \simeq \gamma\left\{1 - \frac{1}{6}\left(1 - \frac{g^2}{3}\right)\gamma^2 + \frac{\gamma_H}{9}\left(g\gamma + \frac{\gamma_H}{2}\right)\right\} \tag{31}$$

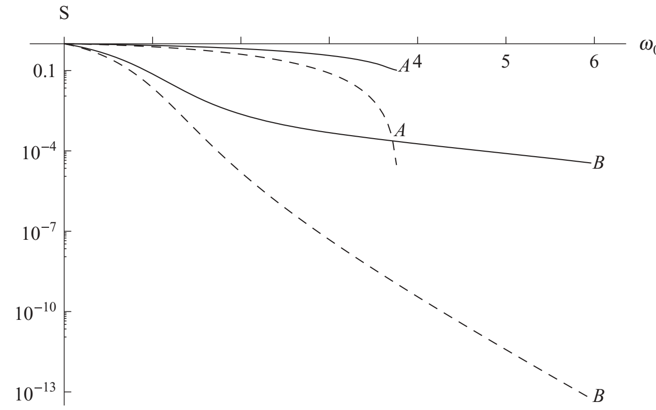

FIG. 6. The function S vs. ω_0 at $\lambda = 7$. The weakly bound level (dashed line) and the $1s$ state of the H atom (solid line); $g_A = +1$, $\gamma_A = 0.6$ and $g_B = +0.8$, $\gamma_B = 2.1$.

and the level width and the level shift become

$$\begin{aligned} \Gamma &= |E_0|\frac{F}{2F_0}\left[1 + \frac{\gamma^3}{30}\left(\gamma - \frac{14}{9}g\gamma_H\right)\right] \\ &\quad\times\exp\left\{-\frac{2}{3}\frac{F_0}{F}\left[1 - \frac{1}{10}\left(1 - \frac{g^2}{3}\right)\gamma^2 \right.\right.\\ &\quad\left.\left. + \frac{\gamma_H}{15}\left(g\gamma + \frac{\gamma_H}{2}\right)\right]\right\}, \\ E &= |E_0|\left\{-1 - \frac{1}{4}\left(\frac{F}{F_0}\right)^2\left[1 + \frac{7}{48}\tilde{\omega}^2(3 - g^2 - 2g\omega_0)\right]\right\}, \end{aligned} \tag{32}$$

where $\tilde{\omega} = \omega/|E_0|$. Equations (32) generalize the results of [see Eq. (8) in Ref. [35]], where the motion of an electron in a circularly polarized electromagnetic wave was considered, to the case where the wave is elliptically polarized and an additional constant magnetic field pointing along the direction of the wave propagation is present.

We see from the first of Eqs. (32) that if the electromagnetic wave is right polarized, then the magnetic field causes the ionization rate to decrease, and, if left polarized, then the ionization rate increases as the magnetic field increases.

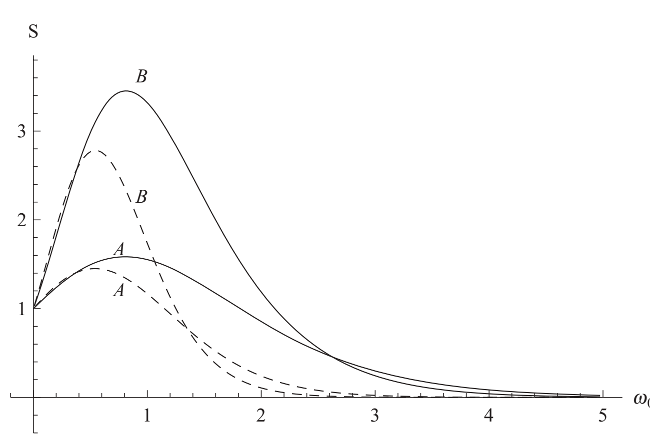

FIG. 7. The function S for vs. ω_0 at $\lambda = 7$ and $g = -0.8$. The weakly bound level (dashed line) and the $1s$ state of the H atom (solid line); $\gamma_A = 2.1$ and $\gamma_B = 5.1$.

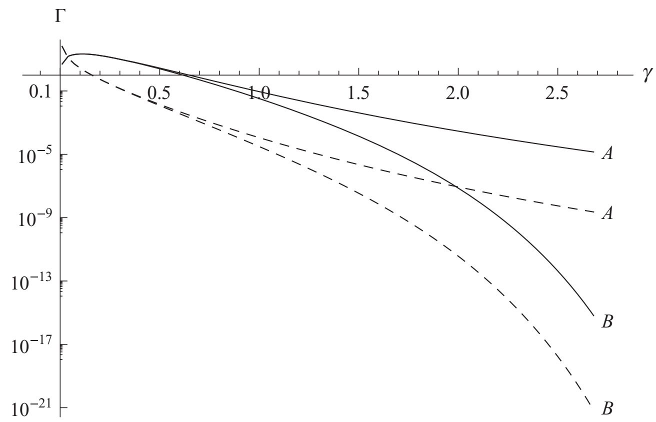

FIG. 8. The ionization rates Γ vs. γ at $\lambda = 7$ and $\omega_0 = 1.1$. The weakly bound level (dashed line) and the $1s$ state of the H atom (solid line); $g_A = -1$ and $g_B = +1$.

The motion of the electron in a magnetic field and a right-polarized electromagnetic wave propagating along the magnetic field is screwlike. As a result, the underbarrier motion of the electron becomes longer and the ionization rates decreases. However, if the electromagnetic wave is left polarized, then the magnetic field and the electromagnetic wave field rotate the electron into opposite directions. As a result, the underbarrier screwlike electron trajectory becomes shorter and the ionization probability increases.

B. The limit of low frequencies, $\omega_0 \gg 1$

In the limit where the frequency of the electromagnetic wave is small, i.e., $\overline{\omega}_0 = 1/\omega_0 = \omega/\omega_H \ll 1$, we recover the case of static crossed electric and magnetic fields, with a small frequency-dependent correction due to the electromagnetic wave. The equation for the saddle point takes the form

$$(1 - x_0 \coth x_0)^2 \{1 + 2g\,\overline{\omega}_0 + (2 + g^2)\overline{\omega}_0^2\} = -\gamma_H^2 + x_0^2 \left\{1 - \left[1 + \frac{x_0}{3}\left(\frac{x_0}{\sinh^2 x_0} - 4\coth x_0\right)\right]\overline{\omega}_0^2\right\}. \tag{33}$$

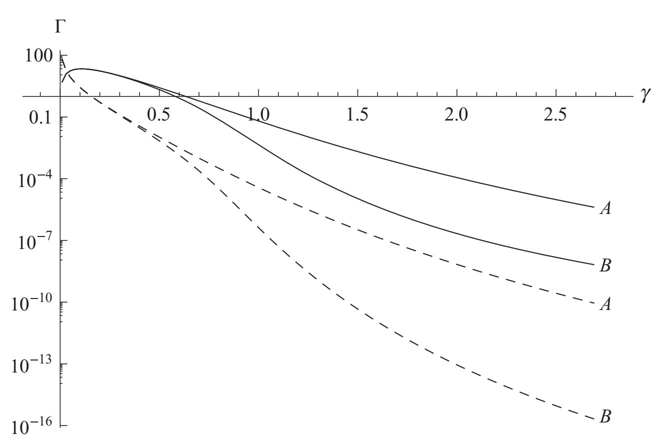

FIG. 9. The ionization rates Γ vs. γ at $\lambda = 7$ and $\omega_0 = 2.5$. The weakly bound level (dashed line), the $1s$ state of the H atom (solid line); $g_A = -0.8$ and $g_B = +0.8$.

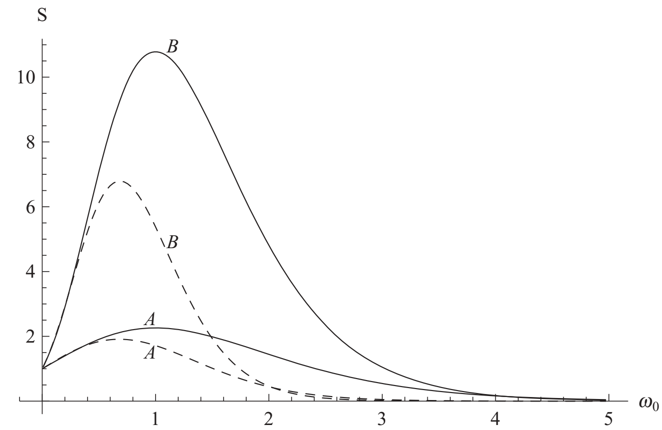

FIG. 10. The function S vs. ω_0 at $\lambda = 7$ and $g = -1$. The weakly bound level (dashed line) and the $1s$ state of the H atom (solid line); $\gamma_A = 2.1$ and $\gamma_B = 5.1$.

The result for the ionization probability is

$$\Gamma = P_s(x_0)\,\exp\left[-2\lambda_H F_s(x_0)\right], \tag{34}$$

where $\lambda_H = |E_0|/\omega_H$ and

$$\begin{aligned} F_s(x_0) = {} & \left(1 + \frac{1}{\gamma_H^2}\right) x_0 - \frac{x_0^2}{\gamma_H^2}\coth x_0 \\ & + \frac{2g x_0}{\gamma_H^2}\left(1 + \frac{x_0^2}{3} - x_0 \coth x_0\right)\overline{\omega}_0 \\ & + \frac{x_0}{\gamma_H^2}\left[2 + g^2 + x_0^2\left(1 + \frac{g^2}{3}\right)\right. \\ & \left. - x_0\left(2 + g^2 + \frac{x_0^2}{3}\right)\coth x_0\right]\overline{\omega}_0^2. \end{aligned} \tag{35}$$

For $\gamma_H \ll 1$ the saddle point is

$$x_0 \simeq \gamma_H \left\{1 + \frac{\gamma_H^2}{18}\left[1 + 2g\overline{\omega}_0 - (3 - g^2)\,\overline{\omega}_0^2\right]\right\} \tag{36}$$

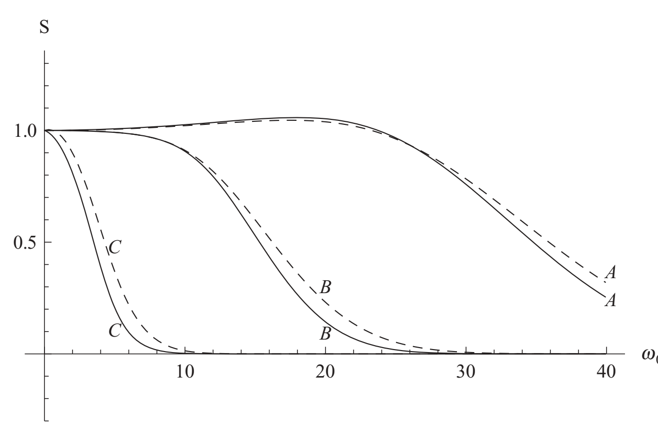

FIG. 11. The function S for the $1s$ state of the H atom vs. ω_0 at $\lambda = 7$: $g = +0.3$ (solid line) and $g = -0.3$ (dashed line); $\gamma_A = 0.1$, $\gamma_B = 0.2$, $\gamma_C = 0.6$.

and the level width and the level shift are

$$\Gamma = |E_0| \frac{F}{2F_0} \left[1 - \frac{\gamma_H}{6} \left(\gamma_H + \frac{14}{45} g\gamma^3 \right) \right]$$

$$\times \exp\left(-\frac{2}{3}\frac{F_0}{F} \left\{ 1 + \frac{\gamma_H^2}{30} \left[1 + 2g\overline{\omega}_0 - (3 - g^2)\overline{\omega}_0^2 \right] \right\} \right),$$

$$E = |E_0| \left[-1 - \frac{1}{4}\left(\frac{F}{F_0}\right)^2 + \frac{1}{12}\left(\frac{H}{H_0}\right)^2 - \frac{13}{8}\left(\frac{F}{F_0}\right)^4 - \frac{5}{144}\left(\frac{H}{H_0}\right)^4 + \frac{1}{2}\left(\frac{HF}{H_0F_0}\right)^2 \left(1 + g\frac{7}{12}\overline{\omega}_0\right) \right]. \quad (37)$$

Equations (37) generalize the expression for the complex energy obtained in Ref. [20] for an electron moving in crossed static electric and magnetic fields. Our result (37) additionally takes into account the frequency-dependent corrections due to an elliptically polarized electromagnetic wave.

IV. THE COULOMB CORRECTION

In the case of ionization of neutral atoms and positive ions, the Coulomb interaction of the emerging electron with an atomic or ionic core has to be taken into consideration, especially if $F \ll F_0$. For this purpose, we can employ the quasiclassical perturbation theory to calculate the correction to the classical action, $\delta S = Z\int dt/r(t)$. But since this integral diverges at $r \to 0$, we use the procedure of sewing with the asymptote of the wave function of a free atom, $\chi_k(r) \simeq \exp\{-[(kr) - \eta ln(kr) + O(1)]\}$ (see Ref. [12] for details). This approach gives the Coulomb factor $Q(z_0,H)$,

$$Q(z_0,H) = 2\lambda z_0 \exp\{J(z_0)\},$$

$$J(z_0) = \int_0^1 \left[\frac{\gamma z_0}{|\boldsymbol{r}([1-s]z_0)|} - \frac{1}{s} \right] ds \quad (38)$$

in the expression for the probability of the tunneling ionization Γ,

$$\Gamma = \frac{\omega_H C_{\kappa l}^2 Q^{2\eta}(z_0,H)}{2\sqrt{2z_0}\sinh(\omega_0 z_0)} \left(\frac{1}{|F''(z_0)|} \right)^{1/2} \exp[-2\lambda F(z_0)], \quad (39)$$

where $\eta = Z/(|E|/|I_H|)^{-1/2}$ is the Zommerfeld parameter (Z is the charge of the atomic core, and $|I_H| = 13.6$ eV). The parameter η is usually close to unity (for the H atom $\eta = 1$). Moreover, $C_{\kappa l}$ is the asymptotic coefficient in the atomic wave function at infinity. In particular, $C_{\kappa l} = 2$ for the $1s$ state of the H atom [5].

Equations (38) and (39) generalize the corresponding expressions in Refs. [12,28] for the case where an arbitrary electromagnetic wave and a static magnetic field are present simultaneously. Here we note that the condition of applicability of the expression (38) for the Coulomb factor $Q(z_0,H)$ is $\gamma \ll 2E/\sqrt{ZF}$. In Eq. (38),

$$|\boldsymbol{r}(z)| = \sqrt{r_x^2(z) + r_y^2(z)},$$

where (r_x, r_y) is the sub-barrier trajectory of the electron in the atom. To find this trajectory, we use the "imaginary-time" method, originally proposed in Refs. [5,37] in order to solve the problem of ionization of nonrelativistic bound systems in the field of an intensive light wave. Being a generalization of the quasiclassical WKB approximation to the case of time-dependent fields, this method describes the tunneling transition of an electron from a bound state to the continuum by using the classical equations of motion but with an imaginary time. Integrating the equation of motion subject to the initial condition

$$\boldsymbol{r}(z_0) = \boldsymbol{0}$$

and taking into account that at $t = 0$ the electron overcomes the barrier (leaves the atom) and changes the imaginary values of the coordinate and momentum into real ones,

$$\text{Im}\,\boldsymbol{r}(0) = \text{Im}\,\dot{\boldsymbol{r}}(0) = \boldsymbol{0},$$

we obtain the following result for the extremal trajectory:

$$r_x(z) = \frac{1}{1-\omega_0^2} \left\{ (1 + g\omega_0)(\cosh z - \cosh z_0) - (g + \omega_0)\frac{\sinh z_0}{\sinh(\omega_0 z_0)} [\cosh(\omega_0 z) - \cosh(\omega_0 z_0)] \right\},$$

$$r_y(z) = i\,\frac{g + \omega_0}{1 - \omega_0^2} \left\{ \sinh z - \sinh z_0 \frac{\sinh(\omega_0 z)}{\sinh(\omega_0 z_0)} \right\}. \quad (40)$$

In the general case the integral in Eq. (38) can be calculated only numerically. In the adiabatic limit $\gamma \ll 1$ the function $Q(z_0,H)$ has the asymptote

$$Q(z_0,H) \simeq \frac{2F_0}{F} \left\{ 1 - \frac{g^2}{3} \left[\frac{\gamma^2}{6} - \frac{1}{15}\left(1 - \frac{25}{72}g^2\right)\gamma^4 \right] + \frac{\gamma_H}{9}\left[\gamma_H + g\frac{\gamma}{2} \right] \right\}, \quad (41)$$

which at $H = 0$ coincides with the corresponding result in Ref. [1].

At $\gamma \ll 1$, the ionization rate (39) can be represented in the form

$$\Gamma = |E_0| C_{\kappa l}^2 \left(\frac{2F_0}{F} \right)^{2\eta - 1} P_\eta(\gamma, g, F, H) \exp\left\{ -\frac{2}{3}\frac{F_0}{F} f_\eta(\gamma, g, F, H, \omega_0) \right\}, \quad (42)$$

where

$$f_\eta(\gamma,g,F,H,\omega_0) \simeq 1 - \frac{3}{2}\delta(F,H) - \frac{\gamma^2}{10}\left\{1 - \frac{5}{2}\delta(F,H) - \frac{g^2}{3}\left[1 + 10\eta\frac{F}{F_0} - \frac{5}{2}\delta(F,H)\right]\right\}$$
$$+ \frac{3}{8}\left\{\frac{3}{35} - \frac{2}{27}g^2\left(\frac{29}{35} - \frac{g^2}{6}\right) - \frac{3}{10}\left[1 - \frac{58}{81}g^2\left(1 - \frac{35}{174}g^2\right)\right]\delta(F,H) - \frac{8}{15}\eta\frac{F}{F_0}g^2\left(1 - \frac{5}{18}g^2\right)\right\}\gamma^4$$
$$+ \frac{\gamma^2}{3}\left\{g\left[\frac{1}{5} - \frac{1}{2}\left(\frac{29}{105} - \frac{g^2}{9}\right)\gamma^2\right] + \frac{1}{2}\eta g\frac{F}{F_0} - \frac{g}{2}\left[1 - \frac{29}{30}\left(1 - \frac{35}{87}g^2\right)\gamma^2\right]\delta(F,H)\right\}\omega_0$$
$$+ \frac{\gamma^2}{6}\left\{\frac{1}{5} - \eta\frac{F}{F_0} - \frac{1}{2}\delta(F,H) - \frac{29}{210}\left(1 - \frac{31}{29}g^2\right)\left[1 - \frac{7}{2}\delta(F,H)\right]\gamma^2\right\}\omega_0^2. \quad (43)$$

For the 1s state of the H atom the preexponential factor $P_\eta(\gamma,g,F,H)$ ($\eta = 1$) is

$$P_1(\gamma,g,F,H) \simeq 1 + \delta(F,H) - \frac{\gamma_H}{18}(\gamma_H - 4g\gamma) + \frac{\gamma^2}{9}\left[g^2 - \frac{11}{30}\gamma_H^2\left(1 - \frac{11}{34}g^2\right)s(F,H)\right]$$
$$- \frac{41}{135}g\gamma_H\left(1 - \frac{50}{123}g^2\right)s(F,H)\gamma^3 + \frac{\gamma^4}{30}\left[1 - \frac{41}{9}g^2\left(1 - \frac{25}{123}g^2\right)\right]s(F,H), \quad (44)$$

where $s(F,H) = [1 - \delta(F,H)]$. In Eqs. (43) and (44) we took into account the first-order contributions from the Stark and Zeeman effects for an electron being acted on by the laser beam and the static magnetic field.

The case of the H atom in crossed fields is of special interest. In view of Eqs. (42)–(44), the ionization probability is

$$\Gamma = 8|I_H|\frac{F_H}{F}P(F,H)\,\exp\left\{-\frac{2F_H}{3F}\left[1 - \frac{3}{2}\delta(F,H) + \frac{\gamma_H^2}{6}\left(\frac{1}{5} - \frac{F}{F_H} - \frac{1}{2}\delta(F,H)\right)\right]\right\}, \quad (45)$$

where

$$P(F,H) \simeq 1 + \delta(F,H) - \frac{\gamma_H^2}{18} \quad (46)$$

and $F_H = 5.142 \times 10^9$ V/cm is the magnitude of the electric field at the first Bohr orbit. As can be seen from Eqs. (42)–(45), taking into account the Coulomb interaction increases the ionization rates of a neutral atom in comparison with those for a negative ion. This can be explained by the fact that the electron density at the "merge" of the atom increases. In the limit $H \to 0$ from Eqs. (45) and (46) we obtain

$$\Gamma = 8|I_H|\left[\frac{F_H}{F} + \frac{F}{4F_H}\right]\exp\left\{-\frac{2F_H}{3F} + \frac{F}{4F_H}\right\}. \quad (47)$$

Neglecting the Stark effect contribution in Eq. (47), we arrive at the well-known Landau-Lifshitz formula (see in Ref. [38]) for the ionization probability from the ground state of the H atom due to a constant electric field,

$$\Gamma = 8|I_H|\frac{F_H}{F}\exp\left\{-\frac{2F_H}{3F}\right\}. \quad (48)$$

The results of numerical calculations for the ionization rates are shown in Figs. 5–7.

It is seen from Figs. 5–7 that taking into account the Coulomb interaction increases the ionization rates of a neutral atom as compared to those of a negative ion.

Let us define the "stabilization factor" S (see in Ref. [28]): $S = \Gamma/\Gamma_0$, Γ_0 being here the ionization probability at $H = 0$.

It is seen from Fig. 8 that if the electromagnetic wave is right polarized, then the magnetic field causes the ionization rate to decrease, i.e., it stabilizes the bound level. This stabilization effect is greater for the short-range potential than for the Coulomb one, which can be explained by the fact that the Coulomb interaction tends to increase the ionization rates. On the other hand, Figs. 9 and 10 show that if the electromagnetic wave is left polarized, then the ionization rate can grow with the magnetic field. As can be seen from Figs. 9 and 10, the ionization probability grows at a greater rate if the Coulomb interaction is taken into account. Figures 11 and 12 show the so-called effect of the "plateau," which occurs for the electromagnetic wave with an arbitrary elliptical polarization at small γ only if the Coulomb interaction of the emerging electron with the atomic core is taken into consideration. Figures 11 and 12 show that the "plateau" shortens as the parameter γ increases. The cause of the appearance of the "plateau" is that the Coulomb interaction tends to prevent the suppression of the ionization of the bound level by the constant magnetic field.

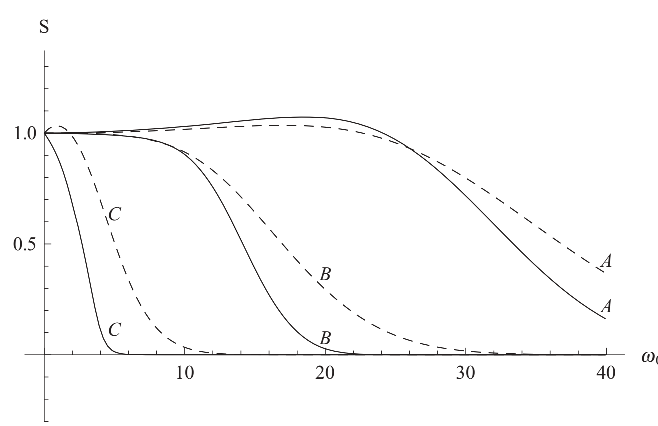

FIG. 12. The function S for the 1s state of the H atom vs. ω_0 at $\lambda = 7$: $g = +0.9$ (solid line) and $g = -0.9$ (dashed line); $\gamma_A = 0.1$, $\gamma_B = 0.2$, $\gamma_C = 0.6$.

V. CONCLUSIONS

We have considered the tunneling ionization of atoms and ions under the influence of an elliptical monochromatic electromagnetic wave and a static uniform magnetic field. In doing so, we found the exact wave function and Green's function for a charged particle in these fields. We then used the boundary condition in the short-range potential method to obtain a transcendental equation for the complex quasienergy. The real and imaginary parts of the latter gave the level shift and the decay probability.

The general solution was analyzed for the case where the wave propagates along the magnetic field. Simple analytical expressions were obtained for the tunnel regime and compared with the results known in the literature. The limits of weak magnetic fields and low-frequency waves were considered in detail. It was shown that in the presence of a static electric field the magnetic field stabilizes the bound level. In the presence of a nonstationary electromagnetic wave, the constant magnetic field can cause the level decay rate to either decrease or increase. This dynamic effect was shown to depend on the polarization direction of the wave: The level decay slows down in the case where the wave and the magnetic field rotate the electron in the same direction and speeds up in the opposite case.

In the case of ionization of neutral atoms and positive ions, we took into account the Coulomb interaction of the emerging electron with the atomic or ionic core and exploited the quasiclassical perturbation theory, complemented by the "imaginary-time" method. Within this approach, we calculated the ionization rate for the ground state of the H atom in an elliptical electromagnetic wave and a static magnetic field. Passing to the static limit, we obtained the formulas for the ionization probability of the H atom in crossed constant electric and magnetic fields and that in a constant electric field. It was shown that taking into account the Coulomb interaction results in the ionization rates of a neutral atom being greater than those of a negative ion. In addition, we took into account the contributions from the Stark and Zeeman effects to the ionization rate.

To support the above statements, we performed numerical calculations of the ionization rate and the "stabilization factor," S. These calculations show that if the electromagnetic wave is right polarized, than the constant magnetic field suppresses the ionization of a bound level. In contrast, the left-polarized electromagnetic wave counteracts the constant magnetic field; as a result, the ionization rate can grow. Since the Coulomb interaction increases the ionization of a bound level, in the adiabatic limit where the inequality $\gamma \ll 1$ holds, the ratio Γ/Γ_0 may remain almost constant up to rather large magnitudes of ω_H/ω (effect of the "plateau"), as shown in Figs. 11 and 12.

The formulas obtained in this paper allow one to obtain simple estimates for the ionization rate for not only negative ions but also for neutral atoms and positive ions in an elliptically polarized laser beam and an additional constant magnetic field present.

ACKNOWLEDGMENT

I am deeply grateful to A. L. Mitler for performing numerical calculations and proof of the existence and uniqueness of a saddle point.

APPENDIX: PROOF OF THE EXISTENCE AND UNIQUENESS OF A SADDLE POINT

Under conditions $-1 \leqslant g < +1$, $\omega_0 > 0$ there exists a unique root z_0 of Eq. (15) for each $\gamma > 0$.

In order to prove this assertion we rewrite the left-hand side of Eq. (15) as

$$\begin{aligned} \mathrm{equ}(z,\omega_0) &\equiv \sinh^2 z - \frac{(g+\omega_0)^2}{(1-\omega_0^2)^2}\,[\cosh z - \omega_0 \sinh z\, \coth(\omega_0 z)]^2 \\ &= \sinh^2(z)\,[(1+|\alpha| L(z,\omega_0)]\,[1-|\alpha| L(z,\omega_0)]\,, \end{aligned} \tag{A1}$$

where

$$L(z,\omega_0) = \frac{\coth(z) - \omega_0\, \coth(\omega_0 z)}{1-\omega_0}, \qquad \alpha = \frac{g+\omega_0}{1+\omega_0}, \quad 0 < |\alpha| < 1. \tag{A2}$$

The existence of the root follows from the limits $\mathrm{equ}(0,\omega_0) = 0$ and $\mathrm{equ}(\infty,\omega_0) = \infty$ of the continuous function $\mathrm{equ}(z,\omega_0)$.

Since

$$L'_z(z,\omega_0) = \frac{1}{1-\omega_0}\left[\frac{\omega_0^2}{\sinh^2(\omega_0 z)} - \frac{1}{\sinh^2(z)}\right] > 0, \tag{A3}$$

the function $L(z,\omega_0)$ monotonously increases from $L(0,\omega_0) = 0$ to $L(\infty,\omega_0) = 1$. Our goal is to demonstrate that $\mathrm{equ}(z,\omega_0)$ is also a monotone increasing function of z. It is sufficient to prove this for the function

$$\mathrm{equ}\_(z,\omega_0) \equiv \sinh^2(z)\,[1 - |\alpha| L(z,\omega_0)]\,. \tag{A4}$$

We have

$$[\mathrm{equ}\_(z,\omega_0)]'_z = \sinh(z)\cosh(z)[2 - |\alpha|\, K(z,\omega_0)], \tag{A5}$$

where

$$K(z,\omega_0) = 2\, L(z,\omega_0) + \tanh(z)\, L'_z(z,\omega_0). \tag{A6}$$

After some algebra one can rewrite the last expression as

$$K(z,\omega_0) = \tanh(z)[1 + \omega_0 + (1-\omega_0)\, L^2(z,\omega_0)]. \tag{A7}$$

If $\omega_0 < 1$ then $K(z,\omega_0)$ is a monotone increasing function of z, and $0 < K(z,\omega_0) < K(\infty,\omega_0) = 2$. Hence, $[\mathrm{equ}\_(z,\omega_0)]'_z > 0$, so both $\mathrm{equ}\_(z,\omega_0)$ and $\mathrm{equ}(z,\omega_0)$ are monotone increasing with z.

Now consider the case $\omega_0 > 1$.

$$\begin{aligned} [K(z,\omega_0)]'_z &= (\{(1+\omega_0)\, \tanh(z)[1 - \beta L^2(z,\omega_0)]\})'_z \\ &= (1+\omega_0)\{\mathrm{sech}^2(z)\,[1-\beta L^2(z,\omega_0)] \\ &\quad - 2\beta \tanh(z)\, L(z,\omega_0) L'(z,\omega_0)\} \\ &= (1+\omega_0)\,\mathrm{sech}^2(z)\{1 - \beta L(z,\omega_0)[L(z,\omega_0) \\ &\quad + \sinh(2z)\, L'(z,\omega_0)]\}, \end{aligned} \tag{A8}$$

where

$$\beta = \frac{\omega_0 - 1}{\omega_0 + 1}, \quad 0 < \beta < 1.$$

In this case $L'(z,\omega_0) < \mathrm{sech}^2(z)/(\omega_0 - 1)$, so we have

$$[K(z,\omega_0)]'_z > (1+\omega_0)\,\mathrm{sech}^2(z)\left\{1 - \beta L(z,\omega_0)\left[L(z,\omega_0) + \frac{\sinh(2z)\,\mathrm{sech}^2(z)}{(\omega_0 - 1)}\right]\right\}$$
$$= (1+\omega_0)\,\mathrm{sech}^2(z)\left[1 - \frac{\omega_0^2\coth^2(\omega_0 z) - \coth^2(z)}{\omega_0^2 - 1}\right]. \qquad \text{(A9)}$$

It can be seen that the derivative

$$\left[\frac{\omega_0^2\coth^2(\omega_0 z) - \coth^2(z)}{\omega_0^2 - 1}\right]'_z > 0, \qquad \text{(A10)}$$

so the expression in parentheses is bounded from above by its limit value as $z \to \infty$, which is 1. Hence, $[K(z,\omega_0)]'_z > 0$, and, as in the previous case, $0 < K(z,\omega_0) < 2$. We conclude that $\mathrm{equ}(z,\omega_0)$ is a monotone increasing function of z for each value $\omega_0 > 0$.

For $g = +1$ and $0 < \omega_0 < 1$ the function $\mathrm{equ}(z,\omega_0)$ monotonously increases from $\mathrm{equ}(0,\omega_0) = 0$ to $\mathrm{equ}(\infty,\omega_0) = \infty$ and Eq. (15) also has a unique root. But the case $g = +1$ and $\omega_0 > 1$ is the peculiar one. In this case the function $\mathrm{equ}(z,\omega_0)$ has the limits: $\mathrm{equ}(0,\omega_0) = 0$ and $\mathrm{equ}(\infty,\omega_0) = 1/(\omega_0 - 1)$. Therefore, if $g = +1$ and $\omega_0 > 1$, a unique root z_0 of Eq. (15) exists only for $\gamma < \gamma_{\mathrm{cr}}$, where $\gamma_{\mathrm{cr}} = 1/\sqrt{\omega_0 - 1}$. For $\gamma \geqslant \gamma_{\mathrm{cr}}$ Eq. (15) has no roots.

[1] S. V. Popruzhenko, V. D. Mur, V. S. Popov, and D. Bauer, Zh. Eksp. Teor. Fiz. **135**, 1092 (2009) [J. Exp. Theor. Phys. **108**, 947 (2009)].
[2] H. R. Reiss, Phys. Rev. Lett. **102**, 143003 (2009).
[3] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1945 (1964) [Sov. Phys. JETP **20**, 1307 (1965)].
[4] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **50**, 255 (1966) [Sov. Phys. JETP **23**, 168 (1966)].
[5] A. M. Perelomov, V. S. Popov, and M. V. Terent'ev, Zh. Eksp. Teor. Fiz. **50**, 1393 (1966) [Sov. Phys. JETP **23**, 924 (1966)].
[6] A. M. Perelomov, V. S. Popov, and M. V. Terent'ev, Zh. Eksp. Teor. Fiz. **51**, 309 (1966) [Sov. Phys. JETP **24**, 207 (1967)].
[7] V. S. Popov, Zh. Eksp. Teor. Fiz. **118**, 56 (2000) [J. Exp. Theor. Phys. **91**, 48 (2000)].
[8] F. H. M. Faisal, J. Phys. B **6**, L89 (1973).
[9] H. R. Reiss, Phys. Rev. A **22**, 1786 (1980).
[10] J. Ortner and V. M. Rylyuk, J. Phys. B **34**, 3251 (2001).
[11] M. Gavrila, in *Photons and Electron Collisions with Atoms and Molecules*, edited by Philip G. Burke and Charles J. Joachain (Plenum Press, New York, 1997).
[12] A. M. Perelomov and V. S. Popov, Zh. Eksp. Teor. Fiz. **52**, 514 (1967).
[13] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **52**, 223 (1967).
[14] A. I. Nikishov, Trudy FIAN **111**, 152 (1979).
[15] V. S. Popov, V. D. Mur, and S. V. Popruzhenko, Pis'ma Zh. Eksp. Teor. Fiz. **85**, 275 (2007) [JETP Lett. **85**, 223 (2007)].
[16] N. B. Delone and V. P. Krainov, Usp. Fiz. Nauk, **168**, 531 (1998) [Sov. Phys. Usp. **41**, 469 (1998)].
[17] V. S. Popov, Usp. Fiz. Nauk **174**, 921 (2004) [Sov. Phys. Usp. **47**, 855 (2004)].
[18] Yu. N. Demkov and G. F. Drukaryev, Zh. Eksp. Teor. Fiz. **47**, 918 (1964) [Sov. Phys. JETP **22**, 585 (1966)].
[19] Yu. N. Demkov and G. F. Drukaryev, Zh. Eksp. Teor. Fiz. **49**, 257 (1965).
[20] G. F. Drukaryev and B. S. Monozon, Zh. Eksp. Teor. Fiz. **61**, 956 (1971).
[21] L. I. Kotova, A. M. Perelomov, and V. S. Popov, Zh. Eksp. Teor. Fiz. **54**, 1151 (1968).
[22] V. S. Popov and A. V. Sergeev, Pis'ma Zh. Eksp. Teor. Fiz. **63**, 398 (1996) [JETP Lett. **63**, 417 (1996)].
[23] N. L. Manakov and A. G. Fainshtein, Dokl. Akad. Nauk SSSR **244**, 567 (1979) [Sov. Phys. Dokl. **24**, 41 (1979)].
[24] N. L. Manakov and A. G. Fainshtein, Zh. Eksp. Teor. Fiz. **79**, 751 (1980) [Sov. Phys. JETP **52**, 382 (1980)].
[25] V. M. Rylyuk and J. Ortner, Phys. Rev. A **67**, 013414 (2003).
[26] A. Pukhov and J. Meyer-ter-Vehn, Phys. Rev. Lett. **76**, 3975 (1996).
[27] M. Steinberg and J. Ortner, Phys. Rev. B **58**, 15460 (1998).
[28] V. S. Popov, B. M. Karnakov, and V. D. Mur, Zh. Eksp. Teor. Fiz. **113**, 1579 (1998) [JETP **86**, 960 (1998)].
[29] Ya. B. Zel'dovich, N. L. Manakov, and L. P. Rapoport, Usp. Fiz. Nauk **117**, 569 (1975) [Sov. Phys. Usp. **18**, 920 (1975)].
[30] N. L. Manakov, M. A. Preobrazhenskii, L. P. Rapoport, and A. G. Fainshtein, Zh. Eksp. Teor. Fiz. **75**, 1243 (1978) [Sov. Phys. JETP **48**, 626 (1978)].
[31] M. V. Frolov, A. V. Flegel, N. L. Manakov, and Anthony F. Starace, Phys. Rev. A **75**, 063407 (2007).
[32] M. V. Frolov, A. V. Flegel, N. L. Manakov, and Anthony F. Starace, Phys. Rev. A **75**, 063408 (2007)
[33] M. V. Frolov, N. L. Manakov, and Anthony F. Starace, Phys. Rev. A **78**, 063418 (2008)
[34] Ya. B. Zel'dovich, Usp. Fiz. Nauk **110**, 139 (1973) [Sov. Phys. Usp. **16**, 427 (1973)].
[35] N. L. Manakov and L. I. Rapoport, Zh. Eksp. Teor. Fiz. **69**, 842 (1975) [Sov. Phys. JETP **42**, 430 (1975)].
[36] V. D. Mur, S. V. Popruzhenko, and V. S. Popov, Zh. Eksp. Teor. Fiz. **119**, 893 (2001) [J. Exp. Theor. Phys. **92**, 777 (2001)].
[37] V. S. Popov, V. P. Kuznetsov, and A. M. Perelomov, Zh. Eksp. Teor. Fiz. **53**, 331 (1967).
[38] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, 3rd revised ed. (Pergamon, Oxford, 1977).