

Qubits as spectrometers of dephasing noise

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We present a procedure for direct characterization of the dephasing noise acting on a single qubit by making repeated measurements of the qubit coherence under suitably chosen sequences of controls. We show that this allows a numerical reconstruction of the short-time noise correlation function and that it can be combined with a series of measurements under free evolution to allow a characterization of the noise correlation function over many orders of magnitude range in time scale. We also make an analysis of the robustness and reliability of the estimated correlation functions. Application to a simple model of two uncorrelated noise fluctuators using decoupling pulse sequences shows that the approach provides a useful route for the experimental characterization of dephasing noise and its statistical properties in a variety of condensed phase and atomic systems.

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I. INTRODUCTION

A key step in the design of any quantum information processing device is to gain a quantitative understanding of the decoherence-inducing noise processes present in the system under study. Knowledge of the statistical properties of this noise both informs and constrains theoretical models of the system, aiding in the design process. In some experimental qubit realizations, the technological implementation may allow for a direct measurement of the noise. In the superconducting flux qubit, for example, this is as simple as measuring the magnetic field fluctuations with a superconducting quantum interference device (SQUID). Statistical properties of the noise may be readily computed from such a measurement and this has been very effectively used to measure the spectrum of, for example, the ubiquitous $1/f$ noise [1]. Frequently, however, the noise acting on a quantum system is inaccessible to such direct measurements and the only recourse is instead to describe the noise indirectly through its effects on measurable quantities, such as a qubit decoherence rate.

It has been recognized for some time that appropriately selected indirect measurements may provide insight into specific features of the noise source [2,3]. For example, Schoelkopf *et al.* showed in 2002 that a single qubit is a valuable resource for measuring those characteristics of the power spectrum of an external noise source giving rise to bit flips (i.e., dissipative characteristics of the noise source) [2]. In many implementations, however, dephasing is the dominant decoherence mechanism [4] and the techniques described by the authors of Ref. [2] are not directly applicable. Subsequent work has recognized the possibility of using pulsed spectroscopies as general diagnostic tools of spectral noise, including both dissipative and dephasing components [3,5–7]. In particular, it has been shown that dephasing rates measured under the application of dynamical decoupling pulse sequences can allow differentiation between weak and strong environmental fluctuators [8].

In this work we demonstrate that a controllable single qubit may be used as a sensitive tool to measure dephasing noise that allows direct access to the noise correlation function. We first present a general procedure for estimating the correlation function of such noise using sequences of coherence measurements on a single qubit undergoing free evolution, and then show that the resulting information may be extended to a significantly larger range of time scales by use of pulse sequences that extend the coherence of a qubit by several orders of magnitude [9].

A major feature of our approach is the direct nature of the procedure. No fitting to parametric forms of correlation functions or spectral densities is required. Finally, we note that there has been some recent work [10–13] focusing on the measurement of the qubit power spectral density. We emphasize that our reconstruction of the noise characteristics is explicitly local in the time domain, so our discussion is more naturally suited to the correlation function picture. If necessary, one could apply the Wiener-Khinchin theorem [14] to transform the measured correlation function to a power spectral density.

II. MODEL

We consider a single qubit subjected to a classical source of dephasing noise, described by the Hamiltonian

$$H(t) = \frac{1}{2}\vec{a}(t) \cdot \vec{\sigma} + \frac{1}{2}[\eta_0 + \eta(t)]\sigma_z. \quad (1)$$

Here $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices and $\vec{a}(t)$ is a control field. For later convenience, we have separated the second term into a constant offset field η_0 and a zero-mean stochastic process $\eta(t)$. Such a Hamiltonian could arise, for instance, for the spin degree of freedom of an electron in a fluctuating magnetic field, a common source of environmental noise for many atomic qubit systems (e.g., dopant spins in silicon) [15]. We assume that the qubit can be initialized in an arbitrary pure state and that it can be measured in any basis. We will additionally assume the noise process to be wide-sense stationary, allowing us to write the correlation function as a

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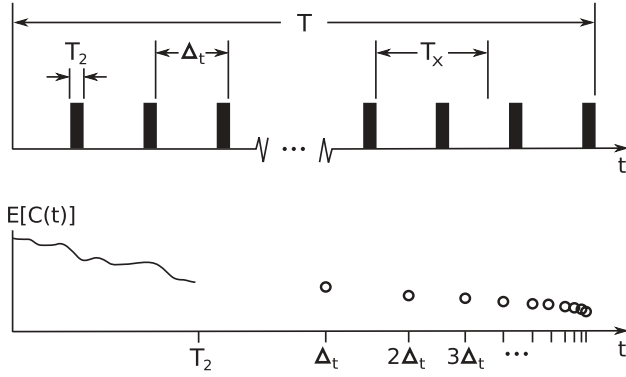


FIG. 1. (a) Illustration of the various time scales involved in the measurement of the noise correlation function. The schematic shows a series of individual measurements (tall, black rectangles) distributed along a horizontal time axis. For simplicity, we have shown each measurement as lasting a time approximately equal to T_2 , the coherence time of the qubit. Qubit initialization steps cause a delay Δ_t between measurements and the measurements are repeated until a final time T . These times define the ranges which are accessible to direct measurement of the noise correlations (see text) and T_x denotes one of the time scales on which correlations are inaccessible to measurement. (b) Schematic reconstruction of a noise correlation function. For times shorter than T_2 , a continuous estimate is made by inversion of dynamical decoupling sequences (Sec. IV). For long times, a discrete estimate is made at times equal to integer multiples of the measurement time Δ_t , from measurements under free qubit evolution (Sec. III). The time axis is scaled logarithmically here for greater clarity. In some systems, the short- and long-time estimates will overlap (i.e., $T_2 > \Delta_t$).

function of only a single time interval

$$C(t) = \langle [\eta_0 + \eta(t)][\eta_0 + \eta(0)] \rangle = \eta_0^2 + C_\eta(t). \quad (2)$$

Here we have explicitly separated the full correlation function into its constant offset contribution η_0^2 and its zero-mean stochastic contribution $C_\eta(t) = \langle \eta(t)\eta(0) \rangle$.

Experimental constraints will necessarily introduce time scales that limit what may be learned about the statistical properties of the noise from measurements on the qubit. The proposed experiment will consist of sequences of control pulses and measurements, each of which impose a specific time-scale constraint. The longest of these time scales is the total length of time T over which the entire experiment is performed. No matter what measurements are made during T , correlations long compared to this time cannot affect these measurements and are therefore not accessible. A second constraint is that the state of the qubit must be reinitialized between each measurement, introducing a delay Δ_t between experiments, or conversely, a ceiling on the maximum repetition rate to a single measurement in time Δ_t . The shortest time scale is that of the individual measurements. Since the coherence of the qubit may be extended to a maximum time T_2 by the application of control pulses, beyond which the coherence has decayed to a point where it is no longer measurably different from zero, we take this coherence time as the longest time available for a measurement of the qubit.

Taken together, these experimental constraints provide a natural separation of the problem into short time scales within the coherence time $t \leq T_2$, and long time scales lying in the range $\Delta_t \leq t \leq T$. Noise correlations occurring on time scales intermediate between T_2 and Δ_t are inaccessible to measurement if $T_2 < \Delta_t$, as illustrated in Fig. 1. This puts limitations on the conversion of the directly determined noise correlations to a noise spectral density (see the Appendix) and clearly implies that it is advantageous to make the measurement repetition time as short as possible.

In the following section, we construct the experimental procedures necessary to extract the correlation function for (i) times long compared to the measurement separation time Δ_t , and (ii) times short compared to the maximum coherence time T_2 . While both methods yield direct estimates of the correlation function, the two approaches differ in the type and sequence of both the control pulses and qubit measurements that are employed.

III. LONG-TIME CORRELATIONS

To estimate the correlation function on time scales longer than the intermeasurement time Δ_t we construct a sequence of qubit measurements following different times of free evolution δ_t according to the following protocol:

- (1) initialize qubit in the $+1$ eigenstate of σ_x ;
- (2) allow qubit to evolve for a short time δ_t ;
- (3) measure qubit in the basis of eigenstates of σ_y .

We shall refer to these as free evolution (FE) measurements. We will assume that the j th run of this experiment begins at time $t_j = j\Delta_t$ (i.e., the runs are equally spaced in time). This is a convenient but not essential requirement. During step (2) above, the qubit acquires a relative phase

$$\phi_j = \int_{t_j}^{t_j + \delta_t} [\eta_0 + \eta(s)] ds.$$

Upon measurement in the σ_y basis, the measurement probabilities are

$$P_y^\pm(j) = \frac{1}{2}(1 \mp \sin \phi_j) \simeq \frac{1}{2}(1 \mp \phi_j). \quad (3)$$

Here we have taken a small angle approximation, which is valid if the total evolution time is chosen sufficiently small [i.e., such that $\delta_t \sqrt{C(0)} \ll 1$]. The probabilities in Eq. (3) are approximately linear in the accumulated phase and may be used to estimate the noise correlation function as follows.

Making a large number N of repetitions of the above procedure will yield N measurement results. We put these results in a vector \vec{r} , with $r_j = \pm 1$ the result of the j th measurement. As shown in the Appendix, this measurement result vector may be used to estimate the full noise correlation function at times $t_k = k\Delta_t$:

$$E[C(k\Delta_t)] = \frac{1}{\delta_t^2(N-k)} \sum_{i=1}^{N-k} r_i r_{i+k}. \quad (4)$$

The expected error of this estimate will scale inversely with the number of measurements performed for each interval,

$$\text{var}\{E[C(t_k)]\} \propto \frac{1}{N-k}.$$

So for a given k this error can be reduced by increasing N , the total number of FE measurements.

Note that if we were to measure in the σ_x basis instead of σ_y , the measurement probabilities would then depend quadratically on the phase $P_x^+(j) = \cos^2 \phi_j$ and $P_x^-(j) = \sin^2 \phi_j$. Such probabilities are independent of the sign of the acquired phase and are therefore not useful for determining the long correlations. However, repeated measurements in σ_x are nevertheless able to give useful information about the zero-time correlation $C(0) = \langle \eta^2 \rangle$. In particular, by making a series of σ_x measurements and recording the values in a vector \vec{r} , we may obtain an estimate of the variance of η [i.e., $\langle \eta^2 \rangle = (1 - \langle \sigma_x \rangle) / (2\delta_t^2)$].

IV. SHORT-TIME CORRELATIONS

Noise correlations on time scales between $t = 0$ and $t = T_2$ cannot be investigated by the method described in the previous section as these time scales are generally shorter than the minimum time between measurements (Δ_t). For these short time scales we show instead that judiciously constructed sequences of control pulses allow us to make direct measurement of the overlap integral of the noise correlation function with a filter function that is defined in terms of the applied control field.

We begin by transforming the Hamiltonian (1) into an interaction picture which removes the explicit dependence on the control fields

$$H_T(t) = \frac{1}{2}[\eta_0 + \eta(t)]U_a^\dagger(t)\sigma_z U_a(t).$$

Here $U_a(t) = e^{-i \int_0^t \tilde{a}(s) \cdot \tilde{\sigma} ds / 2}$ is the unitary operator deriving from just the control field. The simplest dynamical decoupling procedures typically limit the control fields to π pulses polarized along σ_x [16]. Since $\sigma_x \sigma_z \sigma_x = -\sigma_z$, the Hamiltonian in the interaction picture remains proportional to σ_z and becomes

$$H_T(t) = \frac{1}{2}y(t)[\eta_0 + \eta(t)]\sigma_z, \quad (5)$$

where we have introduced the pulse function $y(t)$, defined by

$$y(t) = \begin{cases} 1 & \text{after even number of } \pi \text{ pulses,} \\ -1 & \text{after odd number of } \pi \text{ pulses.} \end{cases}$$

Though we do not consider it here, the effects of nonzero pulse widths may be included to first order [17] through the modification

$$y(t) = \begin{cases} 1 & \text{after even number of } \pi \text{ pulses,} \\ 0 & \text{during application of } \pi \text{ pulses,} \\ -1 & \text{after odd number of } \pi \text{ pulses.} \end{cases}$$

The pulse function describes the fact that, from the perspective of the qubit, each π pulse acts to change the sign of the noise. When averaged over all possible noise trajectories $\eta(t)$, evolution under this Hamiltonian results in dephasing of the qubit, which may be quantified by the decay of the expectation value of the coherence $\sigma_+ = (\sigma_x + i\sigma_y)/2$. Taking

the average over all possible noise trajectories yields

$$\begin{aligned} \langle \langle \sigma_+(t) \rangle \rangle &= \langle \text{Tr} \left(e^{i \int_0^t H_T(s) ds} \sigma_+ e^{-i \int_0^t H_T(s) ds} \rho_0 \right) \rangle \\ &= \left\langle \exp \left(i \int_0^t [\eta_0 + \eta(s)] y(s) ds \right) \right\rangle \text{Tr}(\sigma_+ \rho_0) \\ &= \exp \left(- \sum_l \chi^{(l)}(t) \right) \text{Tr}(\sigma_+ \rho_0), \end{aligned} \quad (6)$$

where we have made use of a cumulant expansion in the last line [18] and the terms $\chi^{(l)}(t)$ will be defined below. Note the two different sources of averaging for the qubit coherence. On the right-hand side we have indicated averages over random variables (i.e., the stochastic average over all consistent trajectories of the noise term, with a single expectation value, $\langle \cdot \rangle$). The double expectation $\langle \langle \cdot \rangle \rangle$ of the coherence operator on the left-hand side represents both this stochastic average over the noise realizations and the quantum average over the initial qubit states, denoted by the usual Tr operation on the right-hand side. For our purposes here, we are interested in experiments in which the qubit is initialized into the $+1$ eigenstate of σ_x , a pulse sequence is applied, and the coherence is measured at a time τ . In this case, $\text{Tr}(\sigma_+ \rho_0) = 1/2$. For notational convenience, we will drop the explicit dependence on t from the cumulant expansion, assuming that it is always implicitly evaluated at $t = \tau$, at the end of the pulse sequence when the coherence has maximally refocused. The first term in the cumulant series, the zeroth-order cumulant, is

$$\chi^{(0)} = -i\eta_0 \int_0^\tau y(t_1) dt_1.$$

This term, which is purely imaginary, represents the coherent precession of the qubit due to the offset field. The zero-order cumulant vanishes if we select any of the many refocusing pulse sequences for which $\int_0^\tau y(s) ds = 0$. The next term in the expansion vanishes

$$\chi^{(1)} = -i \int_0^\tau \langle \eta(t_1) \rangle y(t_1) dt_1 = 0,$$

because the stochastic term $\eta(t)$ has zero mean by construction. In fact, all odd-order cumulants are purely imaginary and will vanish provided the unconditioned probability of a given noise realization $P[\eta(t)]$ is symmetric so that negative and positive contributions to the integral of $y(t_1)$ cancel. All even-order cumulants $l = 2$ and greater are purely real and therefore contribute to the decay of the coherence. These are therefore the terms that are responsible for dephasing of the qubit. The dominant decoherence-causing term in the expansion (6) is the $l = 2$ cumulant

$$\chi^{(2)}(\tau) = \int_0^\tau dt_1 \int_0^\tau dt_2 \langle \eta(t_1) \eta(t_2) \rangle y(t_1) y(t_2). \quad (7)$$

If the noise is Gaussian, as it is in many physical systems, then the all higher-order cumulants vanish identically and the series may be truncated without approximation. For non-Gaussian noise sources the second-order cumulant will dominate if (i) the correlation time is short compared to the time t and/or (ii) the product of the coupling and evolution time τ is much less than unity. In the first case, the central limit theorem implies that the accumulated phase will tend to

a Gaussian distribution in the long-time limit. In the second case, higher-order cumulants are suppressed by powers of $\langle \eta^2 \rangle \tau^2$. We will not consider the case of highly non-Gaussian noise, although we expect that the explicit inclusion of higher-order cumulants into our approach will extend the applicability of this method to such situations.

Making use of the assumption of wide-sense stationarity that we imposed earlier, Eq. (2), allows us to relate this expression to the stochastic part of the noise correlation function $C_\eta(t_2 - t_1) = \langle \eta(t_1)\eta(t_2) \rangle$. By changing variables from t_1, t_2 to $u = t_2 - t_1$, $v = t_2 + t_1$, Eq. (7) may be expressed as a single-variate integral over the stochastic part of the correlation function

$$\chi^{(2)}(\tau) = \int_0^\tau du C_\eta(u) \mathcal{F}(u), \quad (8)$$

with $\mathcal{F}(u)$ given as a second single-variate integral over a quadratic function of the control pulse sequence

$$\mathcal{F}(u) = \int_u^{2\tau-u} dv y\left(\frac{v+u}{2}\right) y\left(\frac{v-u}{2}\right). \quad (9)$$

Equation (8) is known as the coherence integral and defines the correlation filter function (CFF), $\mathcal{F}(t)$. The CFF specifies the regions of the correlation function that contribute to dephasing under a particular pulse sequence and is only defined for $u \in [0, \tau]$. Some examples of CFFs resulting from different pulse sequences are shown in Fig. 2.

We note that discussion of spin coherence decay is usually given in terms of the noise spectral density (see, e.g., Ref. [19]), which is the Fourier transform of the correlation function [20]. As explained earlier, the current formulation in terms of the time-domain correlation function is preferred here since it allows for consistency with the method of elucidating long-time correlations presented in the previous section.

By taking the absolute value of Eq. (6) we can remove the dependence on the zero-order cumulant (since this is purely

imaginary), resulting in

$$|\langle \langle \sigma_+(\tau) \rangle \rangle| \simeq |\exp(-\chi^{(0)} - \chi^{(2)})| = \exp(-\chi^{(2)}). \quad (10)$$

This equation relates the CFF, which is determined by and calculable from the pulse sequence, to the qubit coherence and to the noise correlation function. Since the qubit coherence is measurable while the noise correlation function is unknown, this suggests that a direct estimate of the latter may be obtained from the former by a suitable discretization and inversion of Eq. (10). We now show how this may be done numerically.

To reconstruct the correlation function on short time scales, one must measure the coherence decay for a number of different pulse sequences. Each pulse sequence i determines a filter function $\mathcal{F}_i(t)$ as well as a coherence integral $\chi_i^{(2)} = \int_0^{\tau_i} C_\eta(t) \mathcal{F}_i(t) dt$. We define $|\langle \langle \sigma_+ \rangle \rangle_i|$ as the norm of the coherence measured after the pulse sequence associated with filter function $\mathcal{F}_i(t)$. Now because the norm of the coherence is an experimentally accessible quantity, Eq. (10) then allows us to calculate the coherence integral $\chi_i^{(2)}$ at time τ . The procedure is then repeated for this particular pulse sequence sufficiently many times to gather good statistics. By subsequently repeating the entire procedure for a large number of different pulse sequences $\{i\}$ we will gather a set of measured coherence integrals and filter functions from which an estimate of the short-time correlation function $E[C_\eta(t)]$ may be regressed by making use of the theory of underdetermined least squares [21]. This results in the expression

$$E[C_\eta(t)] = \sum_j \chi_i^{(2)} \mathbb{F}_{ij}^+ \mathcal{F}_j(t), \quad (11)$$

where $\mathbb{F}_{ij} = \int_0^\infty \mathcal{F}_i(t) \mathcal{F}_j(t) dt$ is the filter function overlap matrix which may be constructed analytically for dynamical decoupling sequences, or numerically for more general pulse sequences. This set of equations is underdetermined because we are trying to reconstruct a continuous function by measuring a finite set of real numbers. Consequently, there are in general an infinite number of possible correlation functions that are capable of reproducing the measured coherences and it is necessary to impose an optimality constraint. Our estimate uses the Moore-Penrose pseudoinverse \mathbb{F}^+ , which yields the solution with minimal Euclidean norm [21].

The propagation of uncertainty allows us to compute the variance of the estimator Eq. (11) from the measurements. Suppose for each pulse sequence, corresponding to filter function $\mathcal{F}_i(t)$, one measured the $+1$ eigenstate of σ_x for n_i out of N_i measurements. This corresponds to an estimate of $\chi_i = -\ln(2n_i/N_i - 1)$, with variance $\sigma_{\chi_i}^2 = n_i^2(n_i - N_i)^2/(N_i - 2n_i)^2$. The variance of the estimate of $C(t)$ is therefore

$$\text{Var}\{E[C(t)]\} = \sum_i \sigma_{\chi_i}^2 \sum_j [\mathbb{F}_{ij}^+ \mathcal{F}_j(t)]^2. \quad (12)$$

We note that if one possesses a theoretical model for the noise correlation function it may be experimentally much less demanding to fit to the free parameters of that model. In such a situation one can appeal to the myriad numerical solvers for (possibly overdetermined) nonlinear least squares [22].

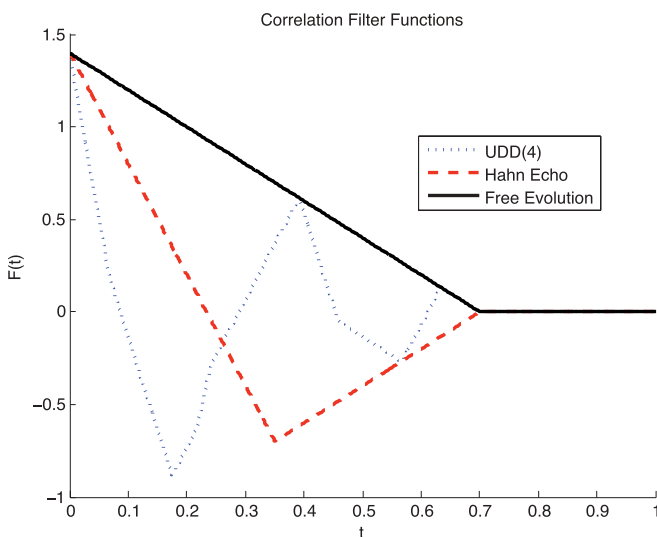


FIG. 2. (Color online) Example correlation filter functions, $\mathcal{F}(t)$. Black, solid line: Free evolution. Red, dashed line: Hahn echo. Blue, dotted line: four-pulse Uhrig [16] sequence, UDD(4).

A. Choosing pulse sequences

The particular choice of pulse sequences will drastically affect the quality of the correlation function estimate, both by dictating the range of time over which the correlation function may be measured and by influencing the accuracy of the estimate within that range. For instance, limiting oneself to a series of free evolution experiments will only allow for an estimate of the correlation function at very short times. Decoupling sequences greatly extend the average coherence time, facilitating a concomitant extension of the region of the correlation function that one can estimate with this procedure.

To quantify the sensitivity of a particular set of pulse sequences to the correlation function at a particular time, consider a perturbation of the noise correlation function $C_\eta(t) \rightarrow C_\eta(t) + \lambda_\tau \delta(t - \tau)$. To first order, this perturbation changes the correlation function estimate to

$$E[C_\eta(t)] \rightarrow E[C_\eta(t)] + \lambda_\tau \sum_{i,j} \mathcal{F}_i(\tau) \mathbb{F}_{ij}^+ \mathcal{F}_j(t).$$

Taking the variation of this with λ gives a measure of the effect of the perturbation on the estimated correlation function

$$\frac{\delta E[C_\eta(t)]}{\delta \lambda_\tau} = \sum_{i,j} \mathcal{F}_i(\tau) \mathbb{F}_{ij}^+ \mathcal{F}_j(t).$$

Squaring this quantity and integrating over t provides us with a positive scalar measure of the sensitivity of our reconstruction to variation of the correlation function at $t = \tau$

$$Q(\tau) = \sum_{i,j} \int_0^\infty [\mathcal{F}_i(\tau) \mathbb{F}_{ij}^+ \mathcal{F}_j(t)]^2 dt. \quad (13)$$

This quality function depends only on the filter functions $\mathcal{F}_i(t)$ and the overlap matrix \mathbb{F} . An examination of the quality function for various sets of filter functions has empirically shown that the estimated correlation function becomes unreliable at times for which

$$Q(t) < \max[Q(t)]/5. \quad (14)$$

V. NUMERICAL SIMULATIONS

To illustrate the efficacy of our approach, we apply the procedure to a single qubit dephasing under the action of two mutually uncorrelated random telegraph (RT) fluctuators, using Monte Carlo techniques to simulate a statistically consistent noise trajectory for each measurement. Each fluctuator is capable of existing in either of two states $\pm \eta_i$, and will randomly transition from one state to the other at a rate γ_i . To capture both the short- and long-time correlations that may be characterized by this spectrometry, we choose a weak, fast fluctuator with parameters $\eta_1 = 1, \gamma_1 = 10$ and a strong, slow fluctuator with parameters $\eta_2 = 10, \gamma_2 = 0.01$. For simplicity, we set the offset field to zero, $\eta_0 = 0$. The resulting noise correlation function may be calculated to be $C(t) = \eta_1^2 e^{-2\gamma_1 |t|} + \eta_2^2 e^{-2\gamma_2 |t|}$. We simulated a sequence of $N_s = 5 \times 10^4$ experiments to measure short-time correlations and $N_l = 10^6$ experiments to measure the long-time correlations. The long-time correlation experiments were brief free evolution experiments performed at integer multiples of the time unit. The pulse sequences chosen to investigate the

TABLE I. Pulse sequences used in numerical simulations to measure short-time noise correlations as discussed in Sec. V. For each pulse sequence and each of the 10 final times, one thousand noise trajectories are simulated and their effects on the qubit coherence is measured. We have chosen the Uhrig decoupling sequence [16] as the basis for this simulated experiment because of their demonstrated success [24] in extending coherence. Here FE = free evolution, i.e., no pulses, and UDD(m) = m th-order Uhrig dynamical decoupling sequence.

Sequence	Final time range	Time divisions
FE	0.01–0.1	10
UDD(1)	0.1–1.0	10
UDD(3)	0.1–1.0	10
UDD(5)	0.1–1.0	10
UDD(7)	0.1–1.0	10

short-time coherences are given in Table I. For each of the N_l free evolution measurements, we simulated noise trajectories of length $t_{FE} = 0.04$. For each of the N_s dynamical decoupling measurements a noise trajectory of length $t_n = 1$ was simulated. To accurately capture the long-time correlations, the initial state of each simulated trajectory was conditioned on the final state of the previous trajectory. Each measurement was simulated by first evolving the qubit under the combined action of the applied the pulse sequence and the simulated noise trajectories, then randomly selecting a measurement outcome based on probabilities calculated from the Born rule [23].

The resulting numerically reconstructed correlation function is shown in Fig. 3, where it is compared with the exact, analytical correlation function. The inset shows a plot of the corresponding quality function $Q(t)$. Using the heuristic Eq. (14), we are able to disregard the short-time reconstruction of the noise for times, $\log_{10}(t) > -1.3$. We see that the reconstructed time correlation function of the dephasing noise demonstrates remarkable overlap with the analytic correlation function at both short and long times, validating the direct reconstruction approach.

We note that for this example of random telegraph noise, the applicability of the short time correlation function extraction relies on the validity of the truncation of the cumulant expansion Eq. (6). In general, since random telegraph noise is strongly non-Gaussian and the measurement time necessarily short relative to the correlation time, we cannot assume the recovery of Gaussian behavior via the long time limit and it is necessary that the expected evolved phase over the evolution period be small for the truncation to be valid. The pulse sequences were chosen explicitly so that this later criterion is satisfied.

VI. DISCUSSION

By inverting the conventional use of control pulse sequences, we have shown that a single qubit can be a valuable resource for direct measurement of dephasing noise. In particular, we have demonstrated that direct reconstruction of short- and long-time noise correlation functions may be made using a combination of control pulse sequences, coherence measurements and free evolution measurements.

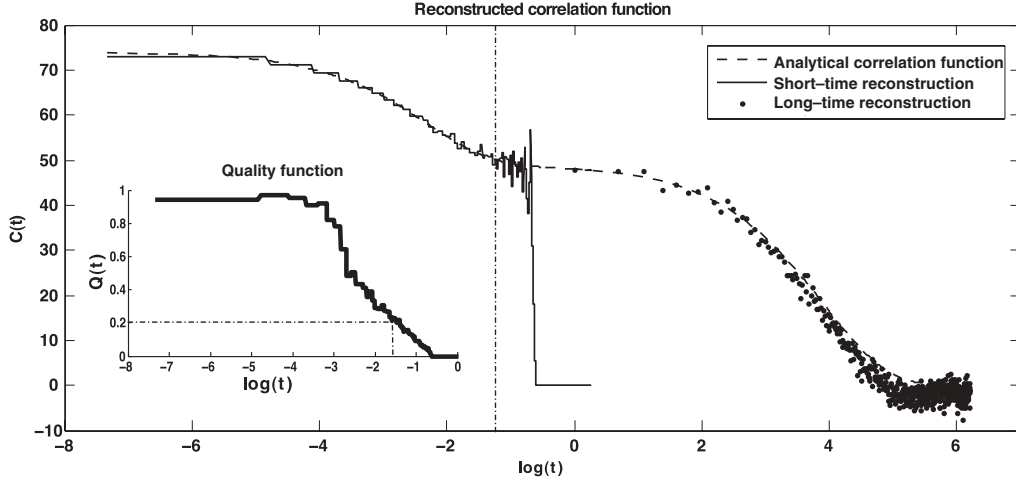


FIG. 3. Combined reconstruction of the correlation function of two mutually uncorrelated RT fluctuators at both short and long times, obtained with the use of the set of pulse sequences given in Table I. The dashed line is the analytical correlation function, the solid line is the short-time reconstruction, and the black dots represent the long-time correlations. Inset is the quality function for the short-time reconstruction. Dashed-dotted lines demarcate low-quality regions. The short-time reconstruction is unreliable at times for which the quality function $Q(t) < 0.2$, corresponding to $\log_{10}(t) > -1.3$. This unreliable portion is separated by the dot-dashed line in the main figure.

Dephasing noise, pervasive in many quantum systems, is still relatively poorly understood. Our direct reconstruction method is general and not dependent on any specific physical features of the probe qubit. It may therefore be applied to any system for which dephasing is the dominant source of noise. It is particularly well suited to the measurement of dephasing noise at interfaces (e.g., for trapped ions or for dopants in semiconductors). One significant possible application of this approach is to the measurement of interface noise experienced by donor qubits in silicon-based devices. As shown in Ref. [4], donor qubits in silicon near an oxide interface demonstrate a marked increase in coherence time as the distance from the oxide is increased. Theoretical models of the noise process causing this decoherence suggest that the presence of fluctuating dangling bonds at the interface is responsible for decoherence [15]. However, these theoretical models require a dangling bond density which is inconsistent with the measured density [15,25]. The ability to make direct measurements of the statistical properties of this noise could aid greatly in developing understanding of its microscopic origin and in construction of new theoretical models to describe the interplay of donor spins and interfaces.

Lastly, we note that the direct reconstruction method described here is not restricted to use with a single qubit and may also be used for measurement of dephasing noise acting on ensembles of qubits. In this situation, however, unless the measurements can be spatially resolved, the reconstructed correlation function will be necessarily restricted to noise that is spatially correlated across the sample, such as that deriving from fluctuations in the applied magnetic field of a magnetic resonance experiment.

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APPENDIX: ESTIMATE OF LONG-TIME CORRELATION FUNCTION FROM FREE EVOLUTION MEASUREMENTS

The correlation function at long times is sampled by a series of free evolution measurements, yielding a results vector \vec{r} . We define the correlator of this result vector as

$$C_k = \frac{1}{N-k} \sum_{i=1}^{N-k} r_i r_{i+k}. \quad (\text{A1})$$

The expected value of this correlator may be calculated from the measurement probabilities given in Eq. (3) according to

$$\begin{aligned} \langle C_k \rangle &\simeq \frac{1}{N-k} \sum_i \langle r_i r_{i+k} \rangle \\ &\simeq \sum_{i=1}^{N-k} \sum_{m,n=\pm 1} \left\langle \frac{[(-1)^m + \phi_i][(-1)^n + \phi_{i+k}]}{4(N-k)} \right\rangle \\ &= \frac{1}{N-k} \sum_{i=1}^{N-k} \langle \phi_i \phi_{i+k} \rangle. \end{aligned} \quad (\text{A2})$$

The covariance of the acquired phases ϕ may be simplified as

$$\begin{aligned} \langle \phi_i \phi_{i+k} \rangle &= \int_{t_i}^{t_i+\delta_t} dt_1 \int_{t_{i+k}}^{t_{i+k}+\delta_t} dt_2 [\eta_0 + \eta(t_1)] [\eta_0 + \eta(t_2)] \\ &= \delta_t^2 \eta_0^2 + \int_{t_i}^{t_i+\delta_t} dt_1 \int_{t_{i+k}}^{t_{i+k}+\delta_t} dt_2 \langle \eta(t_1) \eta(t_2) \rangle \\ &= \delta_t^2 \eta_0^2 + \int_{k\Delta-\delta_t}^{k\Delta+\delta_t} C_\eta(u) \mathcal{F}_k(u) du. \end{aligned} \quad (\text{A3})$$

In the last equality we have changed variables in the integral from t_1 and t_2 to $v = t_2 + t_1$ and $u = t_2 - t_1$ and then integrated over v . The filter function $\mathcal{F}(u)$ that appears in the integral is defined as

$$\mathcal{F}_k(u) = \begin{cases} \sqrt{2}(u - k\Delta_t + \delta_t) & u \in [k\Delta_t - \delta_t, k\Delta_t] \\ \sqrt{2}(1 - u + k\Delta_t) & u \in [k\Delta_t, k\Delta_t + \delta_t] \\ 0 & \text{otherwise.} \end{cases}$$

However, from small δ_t we can assume that $C_\eta(t)$ is constant over the range $t \in [k\Delta_t - \delta_t, k\Delta_t + \delta_t]$ and so comes out of the integral. We can now rewrite the correlation function as

$$\langle \phi_i \phi_{i+k} \rangle = \delta_t^2 \eta_0^2 + \delta_t^2 C_\eta(k\Delta_t).$$

Combining this with Eq. (A3) we see that

$$\langle C_k \rangle = \delta_t^2 \eta_0^2 + \delta_t^2 C_\eta(k\Delta_t).$$

Because the best estimate of $\langle C_k \rangle$ is the sample correlation C_k , given in Eq. (A1), we are left with

$$\delta_t^2 \eta_0^2 + \delta_t^2 C_\eta(k\Delta_t) \simeq \frac{1}{N-k} \sum_i^{N-k} r_i r_{i+k}.$$

This may be solved for the full correlation function $C(k\Delta_t)$ as

$$C(k\Delta_t) \simeq \frac{1}{\delta_t^2(N-k)} \sum_i^{N-k} r_i r_{i+k}.$$

In the very-long-time limit we expect that the stochastic part becomes completely uncorrelated

$$\lim_{t \rightarrow \infty} C_\eta(t) \equiv \lim_{t \rightarrow \infty} \langle \eta(t)\eta(0) \rangle = 0.$$

The stochastic part of the correlation function may thus be recovered from the full correlation function by subtracting

$$C_\eta(t) = C(t) - \lim_{t \rightarrow \infty} C(t).$$

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