Quantum tomography via unambiguous state discrimination

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We study the determination of an unknown mixed state of a d-dimensional quantum system by means of unambiguous state discrimination. We show that optimal and nonoptimal unambiguous state discrimination can be used to reconstruct unknown states of a qubit. This result is extended to the case of a qudit by a sequence of reconstructions in two-dimensional subspaces. The total number of projections scales approximately as $2d^2$ for d large; this is twice as much as in the case of tomography based on mutually unbiased bases or symmetric informationally complete positive-operator-valued measures, and less than the d^3 projections required by standard tomography.

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I. INTRODUCTION

The fact that quantum states are not observable [1] leads us to the fundamental problem of determining unknown quantum states. This has led to the search for methods capable of reliably reconstructing quantum states from noisy experimental data.

A solution to the problem of determining quantum states is quantum tomography [2,3]. This is based on the projection of the system to be reconstructed onto a fixed set of states. When these states are properly chosen, the transition probabilities lead to an invertible set of equations from which the coefficients of the unknown state, in a base fixed in advance, can be obtained. This method requires the preparation of a large ensemble of identical copies of the state to be determined. Different tomographic schemes arise depending on the particular choice of the set of states: standard quantum tomography [4,5], and quantum tomography based on symmetric informationally complete positive operator measures (SIC-POVM's) [6–10], on mutually unbiased bases (MUB's) [11–13], and on recently equidistant states [14].

In this article we show that it is also possible to reconstruct an unknown quantum state by means of unambiguous quantum state discrimination. This process arises in the context of discriminating among a finite set of known nonorthogonal quantum states [15,16], an underlying problem in quantum cryptographic schemes and probabilistic quantum algorithms. Unambiguous discrimination is based on a unitary reduction process, where a unitary transformation acting on an enlarged Hilbert space is concatenated to a von Neumann projection acting on an ancillary system. We show that when unambiguous discrimination is applied on an unknown state, the postmeasurement conditional states provide enough information to reconstruct the initially unknown state.

This article is organized as follows: in Secs. II and III we study the tomography of a single-qubit state, in Sec. IV we extend our result to the case of qudits, and in Sec. V we comment and conclude.

II. QUBIT TOMOGRAPHY VIA OPTIMAL UNAMBIGUOUS STATE DISCRIMINATION

We consider a two-dimensional system described by a mixed state ρ given by

$$\rho_s = \rho_{00}|0\rangle_s\langle 0| + \rho_{01}|0\rangle_s\langle 1| + \rho_{10}|1\rangle_s\langle 0| + \rho_{11}|1\rangle_s\langle 1|. \quad (1)$$

States $|0\rangle_s$ and $|1\rangle_s$ are an orthonormal base of the Hilbert space $\mathcal{H}_s$ of the system s and the coefficients of ρ are such that $\rho_{00} + \rho_{11} = 1$, $\rho_{01} = \rho_{10}^*$, and $|\rho_{01}|^2 \leqslant \rho_{00}\rho_{11}$. In order to specify a particular state describing the system s the four coefficients ρ_{ij} (with $i,j = 0,1$) must be completely determined. The real coefficients ρ_{00} and ρ_{11} can be readily obtained by projecting ρ onto states of the base. Since the coefficients ρ_{00} and ρ_{11} can be easily determined, we are left with the task of determining the coefficient ρ_{01}. This coefficient is a complex number and thus we need to determine its modulus $|\rho_{01}|$ as well as its phase θ_{01}.

We now define two nonorthogonal states

$$|\alpha_0\rangle_s = a|0\rangle_s + b|1\rangle_s, \quad |\alpha_1\rangle_s = a|0\rangle_s - b|1\rangle_s, \quad (2)$$

with a and b real coefficients such that $a^2 + b^2 = 1$ and $a,b \neq 0$. States $|0\rangle_s$ and $|1\rangle_s$ can be expressed as linear combinations of the states $|\alpha_0\rangle_s$ and $|\alpha_1\rangle_s$, that is,

$$|0\rangle_s = \frac{1}{2a}\left(|\alpha_0\rangle_s + |\alpha_1\rangle_s\right), \quad |1\rangle_s = \frac{1}{2b}\left(|\alpha_0\rangle_s - |\alpha_1\rangle_s\right). \quad (3)$$

This allows us to write the unknown state ρ of Eq. (1) in terms of the states $|\alpha_0\rangle_s$ and $|\alpha_1\rangle_s$ as

$$\rho = \sum_{i,j} \tilde{\rho}_{ij}|\alpha_i\rangle_s\langle\alpha_j|, \quad (4)$$

where the coefficients $\tilde{\rho}_{ij}$ are given by

$$\begin{aligned}
\tilde{\rho}_{00} &= \frac{1}{4}\left(\frac{\rho_{00}}{a^2} + \frac{\rho_{11}}{b^2} + \frac{2|\rho_{01}|\cos(\theta_{01})}{ab}\right),\\
\tilde{\rho}_{01} &= \frac{1}{4}\left(\frac{\rho_{00}}{a^2} - \frac{\rho_{11}}{b^2} - \frac{2i|\rho_{01}|\sin(\theta_{01})}{ab}\right),\\
\tilde{\rho}_{10} &= \frac{1}{4}\left(\frac{\rho_{00}}{a^2} - \frac{\rho_{11}}{b^2} + \frac{2i|\rho_{01}|\sin(\theta_{01})}{ab}\right),\\
\tilde{\rho}_{11} &= \frac{1}{4}\left(\frac{\rho_{00}}{a^2} + \frac{\rho_{11}}{b^2} - \frac{2|\rho_{01}|\cos(\theta_{01})}{ab}\right).
\end{aligned} \quad (5)$$

As we can see in this expression, the projectors $|\alpha_0\rangle_s\langle\alpha_0|$ and $|\alpha_1\rangle_s\langle\alpha_1|$ are multiplied by the coefficients $\tilde{\rho}_{00}$ and $\tilde{\rho}_{11}$, respectively, which contain the real part of the unknown coefficient ρ_{01}. It is also clear that the operators $|\alpha_0\rangle_s\langle\alpha_1|$ and $|\alpha_1\rangle_s\langle\alpha_0|$ are associated with the imaginary part of ρ_{01}. Since no observable contains nonorthogonal states in its spectral decomposition, these coefficients cannot be directly measured

Nevertheless, it is possible to generate a new state $\tilde{\rho}$ such that its coefficients in the base $\{|0\rangle_s,|1\rangle_s\}$ are the coefficients $\tilde{\rho}_{ij}$ of Eq. (4), albeit probabilistically. In order to generate $\tilde{\rho}$ we resort to quantum state discrimination. This problem arises whenever it is necessary to distinguish among a set of known nonorthogonal states. Since this class of states cannot be univocally identified by von Neumann measurements, quantum state discrimination corresponds in general to a measurement optimization problem [17–19]. Here we are in particular interested in unambiguous state discrimination [20–24]. This discrimination strategy allows us to identify perfectly linearly independent nonorthogonal states but at the expense of considering the possibility of an inconclusive event. This strategy requires the addition of a two-dimensional ancillary system a and the capacity of implementing a joint unitary transformation U_{sa} onto systems a and s as well as local von Neumann measurements on both systems.

The joint unitary transformation U_{sa} is defined by

$$U_{sa}(|\alpha_i\rangle_s|A\rangle_a) = \sqrt{p_i}|i\rangle_s|0\rangle_a + \sqrt{1-p_i}|\phi\rangle_s|1\rangle_a, \quad (6)$$

where states $|0\rangle_a$ and $|1\rangle_a$ form a base of the Hilbert space $\mathcal{H}_a$ of the ancilla system a and state $|A\rangle_a$ is an arbitrary initial state of the ancilla system. It has been shown that this transformation exists if and only if the states to be discriminated are linearly independent. The transformation U_{sa} is followed by a projection of the ancilla system onto states $|0\rangle_a$ and $|1\rangle_a$. A projection of the ancilla onto state $|0\rangle_a$ maps nonorthogonal states $|\alpha_0\rangle_s$ and $|\alpha_1\rangle_s$ onto orthogonal states $|0\rangle_s$ and $|1\rangle_s$ with probabilities p_0 and p_1 respectively. Thereby, the nonorthogonal states $|\alpha_0\rangle_s$ and $|\alpha_1\rangle_s$ can be identified by a second von Neumann measurement on the system s. Otherwise, states $|\alpha_0\rangle_s$ and $|\alpha_1\rangle_s$ are projected onto state $|\phi\rangle_s$, from which no further discrimination can be accomplished. The process of unambiguous state discrimination has been studied in the context of quantum teleportation via partially entangled states [25], entanglement swapping [26], entanglement concentration [27], dense coding [28–30], and the quantum eraser [31].

In what follows we consider states $|\alpha_0\rangle_s$ and $|\alpha_1\rangle_s$ to be equally generated. In this case $p_0 = p_1 = p$ and the optimal success probability p is simply given by

$$p = 1 - |{}_s\langle\alpha_0|\alpha_1\rangle_s|. \quad (7)$$

In the case of unambiguous discrimination of more than two states few analytical solutions are known [32–35]. Nevertheless, it has been demonstrated that unambiguous state discrimination is possible only in the case of linearly independent sets [36].

We now assume that the unitary transformation U_{sa} of Eq. (6) can be experimentally realized [37–41] and applied to the state $\rho_s|A\rangle_a\langle A|$. This generates a new bipartite state ρ_{sa} given by

$$\begin{aligned}\rho_{sa} =\ & p\tilde{p}\left(\sum_{i,j}\frac{\tilde{\rho}_{ij}}{\tilde{p}}|i\rangle_s\langle j|\right)|0\rangle_a\langle 0| \\ & + \sqrt{p(1-p)}\left(\sum_{i,j}\tilde{\rho}_{ij}|i\rangle_s\langle\phi|\right)|0\rangle_a\langle 1| \\ & + \sqrt{p(1-p)}\left(\sum_{i,j}\tilde{\rho}_{ij}|\phi\rangle_s\langle j|\right)|1\rangle_a\langle 0| \\ & + (1-p)\left(\sum_{i,j}\tilde{\rho}_{ij}\right)|\phi\rangle_s\langle\phi||1\rangle_a\langle 1|, \end{aligned} \quad (8)$$

where

$$\tilde{p} = \frac{1}{2}\left(\frac{\rho_{00}}{a^2} + \frac{\rho_{11}}{b^2}\right). \quad (9)$$

After projecting the ancilla system a onto the state $|0\rangle_a$ the system s is described with probability $p\tilde{p}$ by the state

$$\tilde{\rho}_{s,0} = \sum_{i,j}\frac{\tilde{\rho}_{ij}}{\tilde{p}}|i\rangle_s\langle j|. \quad (10)$$

From this state it is possible to determine the value of the real part of the coefficient ρ_{01} as

$$|\rho_{01}|\cos(\theta_{01}) = 2ab\tilde{p}[\mathrm{Tr}(|0\rangle_s\langle 0|\tilde{\rho}_{s,0}) - 1/2], \quad (11)$$

where a and b are known coefficients fixed beforehand, $\tilde{p}$ is a known quantity since ρ_{00} and ρ_{11} have been previously determined, and the value of $\mathrm{Tr}(|0\rangle_s\langle 0|\tilde{\rho}_{s,0})$ can be calculated from the statistic of the projections toward the state $|0\rangle_s$. In order to obtain this quantity it is also necessary to register the statistic of the projections toward the state $|1\rangle_s$ for normalization purposes. However, since the total probability, conditional to projections onto the state $|0\rangle_a$, is given by the known quantity $p\tilde{p}$ this is not, in principle, necessary.

We still need to determine the value of the imaginary part of ρ_{01}. This can be done by measuring the state $\tilde{\rho}_{s,0}$ in the base

$$|\pm\rangle_s = \frac{1}{\sqrt{2}}(|0\rangle_s \pm i|1\rangle_s), \quad (12)$$

which leads us to the equation

$$|\rho_{01}|\sin(\theta_{01}) = 2ab\tilde{p}[\mathrm{Tr}(|+\rangle_s\langle +|\tilde{\rho}_{s,0}) - 1/2]. \quad (13)$$

This determines the value of the of the imaginary part of ρ_{01}. Thereby, we have determined completely the unknown state ρ by means of unambiguous state discrimination. This last stage, the determination of the imaginary part of ρ_{01}, can also by accomplished by resorting to a second transformation U_{sa} which discriminates states

$$|\alpha_0\rangle_s = a|0\rangle_s + ib|1\rangle_s, \quad |\alpha_1\rangle_s = a|0\rangle_s - ib|1\rangle_s. \quad (14)$$

Thereby, all projective measurements are performed on states $|0\rangle_s$ and $|1\rangle_s$ of system s. However, as we shall see in the next section, a single transformation suffices to determine the coefficient ρ_{01} completely.

III. QUBIT TOMOGRAPHY VIA NONOPTIMAL UNAMBIGUOUS STATE DISCRIMINATION

In the previous section we have shown that unambiguous state discrimination is well suited for determining the unknown state of a two-dimensional quantum system. In the process, the determination of the imaginary part of the coefficient ρ_{01} required projective measurements on a new base or the use of a second unitary transformation. This second choice generates a new state such that its diagonal coefficients are functions of the imaginary part of ρ_{01}.

Here we show that a single discrimination process is enough to determine ρ_{01} completely. Instead of optimal state discrimination we resort to a nonoptimal unambiguous discrimination. In this process the projection of the ancilla system a onto state $|1\rangle_a$ maps system s onto two different nonorthogonal states $|\phi_0\rangle_s$ and $|\phi_1\rangle_s$, the success probability being given by

$$p = 1 - \frac{|_s\langle\alpha_0|\alpha_1\rangle_s|}{|_s\langle\phi_0|\phi_1\rangle_s|}, \tag{15}$$

under the constraint that both inner products have the same complex phase, the states to be discriminated are equally probable, and $|_s\langle\alpha_0|\alpha_1\rangle_s| \leqslant |_s\langle\phi_0|\phi_1\rangle_s|$. The optimal probability is recovered by considering states $|\phi_0\rangle_s$ and $|\phi_1\rangle_s$ to be linearly dependent, that is, $|_s\langle\phi_0|\phi_1\rangle_s| = 1$. The unitary transformation V_{sa} implementing this process is defined as

$$V_{sa}(|\alpha_i\rangle_s|A\rangle_a) = \sqrt{p}|i\rangle_s|0\rangle_a + \sqrt{1-p}|\phi_i\rangle_s|1\rangle_a. \tag{16}$$

Applying this transformation onto state $\rho_s|A\rangle_a\langle A|$, we obtain the joint state ρ_{sa} of system s and a given by

$$\begin{aligned}\rho_{sa} = {}& p\tilde{p}\left(\sum_{i,j}\frac{\tilde{\rho}_{ij}}{\tilde{p}}|i\rangle_s\langle j|\right)|0\rangle_a\langle 0| \\ &+ \sqrt{p(1-p)}\left(\sum_{i,j}\tilde{\rho}_{ij}|i\rangle_s\langle\phi_j|\right)|0\rangle_a\langle 1| \\ &+ \sqrt{p(1-p)}\left(\sum_{i,j}\tilde{\rho}_{ij}|\phi_i\rangle_s\langle j|\right)|1\rangle_a\langle 0| \\ &+ (1-p)\tilde{\tilde{p}}\left(\sum_{i,j}\frac{\tilde{\rho}_{ij}}{\tilde{\tilde{p}}}|\phi_i\rangle_s\langle\phi_j|\right)|1\rangle_a\langle 1|,\end{aligned} \tag{17}$$

where

$$\tilde{p} = \frac{1}{2}\left(\frac{\rho_{00}}{a^2} + \frac{\rho_{11}}{b^2}\right), \tag{18}$$

as in the previous case of the transformation U_{sa} of Eq. (6), and

$$\tilde{\tilde{p}} = \tilde{\rho}_{00} + \tilde{\rho}_{11} + 2\mathrm{Re}(\tilde{\rho}_{01}\langle\phi_1|\phi_0\rangle). \tag{19}$$

Since we have considered states $|\alpha_0\rangle_s$ and $|\alpha_1\rangle_s$ with real inner product, the inner product between states $|\phi_0\rangle_s$ and $|\phi_1\rangle_s$ must also be real. Thus, $\tilde{\tilde{p}}$ does not provide information on the imaginary part of ρ_{01}.

Let us now consider the particular choice

$$|\phi_0\rangle_s = A|0\rangle_s + iB|1\rangle_s, \quad |\phi_1\rangle_s = iB|0\rangle_s + A|1\rangle_s, \tag{20}$$

where $A = |A|e^{i\theta_A}$ and $B = |B|e^{i\theta_B}$ are such that $|A|^2 + |B|^2 = 1$. The inner product between these two states is

$$_s\langle\phi_0|\phi_1\rangle_s = 2|A||B|\sin(\theta_A - \theta_B), \tag{21}$$

which is a real quantity. The sign of this quantity must be identical to the sign of the inner product between states $|\alpha_0\rangle_s$ and $|\alpha_1\rangle_s$, which can be met by a proper choice of $\theta_A - \theta_B$. Thereby we obtain

$$\begin{aligned}\tilde{\tilde{p}} = {}& \frac{1}{2}\left(\frac{\rho_{00}}{a^2} + \frac{\rho_{11}}{b^2}\right) \\ &+ \frac{1}{2}\left(\frac{\rho_{00}}{a^2} - \frac{\rho_{11}}{b^2}\right)2|A||B|\sin(\theta_A - \theta_B),\end{aligned} \tag{22}$$

which is a known quantity since it is expressed in terms of quantities fixed beforehand and quantities determined from measurements. Clearly $\tilde{\tilde{p}}$ does not allow us to deduce the imaginary part of ρ_{01} and thus we are led to study the state of system s after the ancilla system is projected onto state $|1\rangle_a$. This state is

$$\tilde{\rho}_{s,1} = \sum_{i,j}\frac{\tilde{\rho}_{ij}}{\tilde{\tilde{p}}}|\phi_i\rangle_s\langle\phi_j|. \tag{23}$$

The probability of projecting this state onto the state $|0\rangle_s$ is given by

$$\tilde{\tilde{p}}\mathrm{Tr}(|0\rangle_s\langle 0|\tilde{\rho}_{s,1}) = [\tilde{\rho}_{00}|A|^2 + \tilde{\rho}_{11}|B|^2 2\mathrm{Im}(\tilde{\rho}_{01}AB^*)] \tag{24}$$

or equivalently

$$\begin{aligned}\tilde{\tilde{p}}\mathrm{Tr}(|0\rangle_s\langle 0|\tilde{\rho}_{s,1}) = {}& \frac{1}{4}\left(\frac{\rho_{00}}{a^2} + \frac{\rho_{11}}{b^2}\right) \\ &+ \frac{|\rho_{01}|\cos(\theta_{01})}{2ab}(|A|^2 - |B|^2) \\ &+ \frac{1}{2}\left(\frac{\rho_{00}}{a^2} - \frac{\rho_{11}}{b^2}\right)|A||B|\sin(\theta_A - \theta_B) \\ &- \frac{|\rho_{01}|\sin(\theta_{01})}{ab}|A||B|\cos(\theta_A - \theta_B).\end{aligned} \tag{25}$$

The last term in the previous expression turns out to be proportional to the imaginary part of $|\rho_{01}|$. Thereby, a single nonoptimal unambiguous discrimination process together with projections to a single base allows us to determine the unknown state ρ completely.

Equations (11) and (25) can be simplified by considering some particular choices for the coefficients a, b, A, and B. For instance, the choice $a = \sqrt{\rho_{00}}$ and $b = \sqrt{\rho_{11}}$ leads to $\tilde{p} = 1 = \tilde{\tilde{p}}$ and

$$|\rho_{01}|\cos(\theta_{01}) = 2\sqrt{\rho_{00}\rho_{11}}[\mathrm{Tr}(|0\rangle_s\langle 0|\tilde{\rho}_{s,0}) - 1/2]. \tag{26}$$

In addition, the choice $|A| = |B|$ leads to

$$|\rho_{01}|\sin(\theta_{01}) = \frac{2\sqrt{\rho_{00}\rho_{11}}[1/2 - \mathrm{Tr}(|0\rangle_s\langle 0|\tilde{\rho}_{s,1})]}{\cos(\theta_A - \theta_B)}. \tag{27}$$

IV. QUDIT TOMOGRAPHY VIA NONOPTIMAL UNAMBIGUOUS STATE DISCRIMINATION

The tomographic scheme introduced in the previous section can be used, under some modifications, to determine the state of d-dimensional quantum systems. Here we propose to apply the process of nonoptimal unambiguous state discrimination to nonorthogonal states in two-dimensional Hilbert subspaces, such that the discrimination allows us to obtain one nondiagonal coefficient of the unknown state.

The state to be reconstructed is

$$\rho = \sum_{m,n}\rho_{mn}|m\rangle_s\langle n| \tag{28}$$

with $n,m = 0,\ldots,d-1$ and where ρ_{mn} are the coefficients of ρ in the d^2-dimensional operator orthonormal base $\{|m\rangle_s\langle n|\}$. This state can be cast in the form

$$\rho = \rho_{ii}|i\rangle_s\langle i| + \rho_{jj}|j\rangle_s\langle j| + \rho_{ij}|i\rangle_s\langle j| + \rho_{ji}|j\rangle_s\langle i| + \sum_{p\neq(i,j)}\sum_{q\neq(i,j)}\rho_{pq}|p\rangle_s\langle q|, \quad (29)$$

where we have separated the coefficients associated with states $|i\rangle_s$ and $|j\rangle_s$. We now define two nonorthogonal arbitrary superpositions $|\alpha_0^{(ij)}\rangle_s$ and $|\alpha_1^{(ij)}\rangle_s$ in the subspace spanned by the states $|i\rangle_s$ and $|j\rangle_s$ as

$$\left|\alpha_0^{(ij)}\right\rangle_s = a^{(ij)}|i\rangle_s + b^{(ij)}|j\rangle_s \quad (30)$$

and

$$\left|\alpha_1^{(ij)}\right\rangle_s = a^{(ij)}|i\rangle_s - b^{(ij)}|j\rangle_s. \quad (31)$$

Inverting the previous relationship, the state ρ of Eq. (29) becomes

$$\rho = \sum_{pq=0,1}\tilde{\rho}_{pq}^{(ij)}\left|\alpha_p^{(ij)}\right\rangle_s\left\langle\alpha_q^{(ij)}\right| + \sum_{m,n\neq i,j}\rho_{mn}|m\rangle_s\langle n|, \quad (32)$$

with coefficients $\tilde{\rho}_{pq}^{(ij)}$ given by

$$\begin{aligned}
\tilde{\rho}_{00}^{(ij)} &= \frac{1}{4}\left(\frac{\rho_{ii}}{(a^{(ij)})^2} + \frac{\rho_{jj}}{(b^{(ij)})^2} + \frac{2|\rho_{ij}|\cos(\theta_{ij})}{a^{(ij)}b^{(ij)}}\right),\\
\tilde{\rho}_{ij}^{(ij)} &= \frac{1}{4}\left(\frac{\rho_{ii}}{(a)^{(ij)})^2} - \frac{\rho_{jj}}{(b^{(ij)})^2} - \frac{2i|\rho_{ij}|\sin(\theta_{ij})}{a^{(ij)}b^{(ij)}}\right),\\
\tilde{\rho}_{10}^{(ij)} &= \frac{1}{4}\left(\frac{\rho_{ii}}{(a^{(ij)})^2} - \frac{\rho_{jj}}{(b^{(ij)})^2} + \frac{2i|\rho_{ij}|\sin(\theta_{ij})}{a^{(ij)}b^{(ij)}}\right),\\
\tilde{\rho}_{11}^{(ij)} &= \frac{1}{4}\left(\frac{\rho_{ii}}{(a^{(ij)})^2} + \frac{\rho_{jj}}{(b^{(ij)})^2} - \frac{2|\rho_{ij}|\cos(\theta_{ij})}{a^{(ij)}b^{(ij)}}\right).
\end{aligned} \quad (33)$$

We now define the joint unitary transformation $V_{sa}^{(ij)}$ whose action is given by

$$\begin{aligned}
V_{sa}^{(ij)}\left(\left|\alpha_0^{(ij)}\right\rangle_s|A\rangle_a\right) &= \sqrt{p^{(ij)}}|i\rangle_s|0\rangle_a + \sqrt{1-p^{(ij)}}\left|\phi_i^{(ij)}\right\rangle_s|1\rangle_a,\\
V_{sa}^{(ij)}\left(\left|\alpha_1^{(ij)}\right\rangle_s|A\rangle_a\right) &= \sqrt{p^{(ij)}}|j\rangle_s|0\rangle_a + \sqrt{1-p^{(ij)}}\left|\phi_j^{(ij)}\right\rangle_s|1\rangle_a.
\end{aligned} \quad (34)$$

Let us note that we have defined $V_{sa}^{(ij)}$ by its action on states $|i\rangle_s|A\rangle_a$ and $|j\rangle_s|A\rangle_a$ and thus its action on states of the form $|m\rangle_s|A\rangle_a$ with $m\neq i,j$ is unknown.

The action of $V_{sa}^{(ij)}$ on the state $\rho_s|A\rangle_a\langle A|$ generates the state $\rho_{sa}^{(ij)}$ given by

$$\begin{aligned}
\rho_{sa}^{(ij)} = {}& p^{(ij)}\tilde{p}^{ij}\left(\sum_{p,q=i,j}\frac{\tilde{\rho}_{pq}^{(ij)}}{\tilde{p}^{(ij)}}|p\rangle_s\langle q|\right)|0\rangle_a\langle 0|\\
&+ \sqrt{p^{(ij)}(1-p^{(ij)})}\left(\sum_{p,q=i,j}\tilde{\rho}_{pq}^{(ij)}|p\rangle_s\langle\phi_q|\right)|0\rangle_a\langle 1|\\
&+ \sqrt{p^{(ij)}(1-p^{(ij)})}\left(\sum_{p,q=i,j}\tilde{\rho}_{pq}^{(ij)}|\phi_p\rangle_s\langle q|\right)|1\rangle_a\langle 0|\\
&+ (1-p^{(ij)})\tilde{\tilde{p}}^{(ij)}\left(\sum_{p,q=i,j}\frac{\tilde{\rho}_{pq}^{(ij)}}{\tilde{\tilde{p}}^{(ij)}}|\phi_p\rangle_s\langle\phi_q|\right)|1\rangle_a\langle 1|\\
&+ V_{sa}^{(ij)}\left(\sum_{m,n\neq i,j}\rho_{mn}|m\rangle_s\langle n|\right)|A\rangle_a\langle A|\left(V_{sa}^{(ij)}\right)^\dagger.
\end{aligned} \quad (35)$$

The last term in the previous expression corresponds to the action of the transformation $V_{sa}^{(ij)}$ on terms of the form $|m\rangle_s\langle n||A\rangle_a\langle A|$ with $m,n\neq i,j$. The determination of the coefficient ρ_{ij} follows from the projection of the ancilla system onto states $|0\rangle_a$ and $|1\rangle_a$ and the projection of the systems s onto the state $|i\rangle_s$. Since we do not want to introduce more unknown coefficients into the equation systems which determine ρ_{ij} we demand that

$$V_{sa}^{(ij)}|m\rangle_s|A\rangle_a = |\Xi_m\rangle_s|2\rangle_a, \quad (36)$$

where $m\neq i,j$ and states $|\Xi_m\rangle_s$ are mutually orthogonal. State $|2\rangle_a$ is orthogonal to states $|0\rangle_a$ and $|1\rangle_a$. Thereby, a projection onto states $|0\rangle_a$ and $|1\rangle_a$ does not mix up coefficients ρ_{mn} with the coefficient ρ_{ij}, and this can be determined as in the previous section.

Let us now quantify the cost of this scheme in terms of the number of projections required to implement the determination of a d-dimensional quantum system: first, d projectors to determine the diagonal coefficients; second, one unitary transformation for each one of the $d(d-1)/2$ nondiagonal coefficients. Each transformation is followed by a projection of the ancilla onto states $|0\rangle_a$ and $|1\rangle_a$, followed by a projection of system s onto state $|i\rangle_s$. Thus, the total number of projectors is

$$d + \frac{d^2-d}{2}4 = 2d^2 - d, \quad (37)$$

which is well below the number for standard quantum tomography with $(d^2-1)(d-1)$ projections and is above MUB-based tomography with d^2-1 projections. In the limit of large d standard quantum tomography requires $(d-1)/2$ times more projectors than the scheme presented here, which in turn require twice as many projectors as MUB-based tomography.

Let us now consider briefly the case of reconstructing an unknown pure quantum state

$$|\psi\rangle = \sum_{m=0}^{d-1}|c_m|e^{i\theta_m}|m\rangle, \quad (38)$$

which is a particular situation of quantum tomography with *a priori* information. In this case the density matrix is simply given by

$$\rho = \sum_{m,n}|c_m c_n|e^{i\theta_{n,m}}|m\rangle\langle n|, \quad (39)$$

with $\theta_{n,m} = \theta_n - \theta_m$. We can cast this operator in the form

$$\rho = \sum_{n=}^{d-1}|c_n|^2|n\rangle\langle n| + \sum_{k=1}^{d-1}|c_k c_{k+1}|e^{i\theta_{k,k+1}}|k\rangle\langle k+1| + \cdots, \quad (40)$$

which indicates that the determination of the coefficients along the main and upper diagonals is enough to characterize the pure state. This requires a total of $5d - 4$ projectors.

In the particular case of SIC-POVMs and MUBs, the additional information for reconstructing a pure quantum state has not led to a reduction in the number of projectors to be measured. This is due to the fact that a generic pure state has nonvanishing components on all projectors forming a SIC-POVM and also on all MUBs. However, it has been shown [42] that almost any pure quantum state can be reconstructed with a POVM formed by $2d$ rank-1 operators, and that the determination of all pure quantum states requires a POVM composed of $3d - 1$ rank-1 operators at most [43], but it is unknown if this is the minimal number of operators. Comparing with this latter case, our scheme requires $5/3$ more measurements, for d large. It has also been shown [44] that the determination of all pure quantum states requires five orthonormal bases only, independently of the dimension of the underlying Hilbert space. This approach requires thus $5d$ measurements, a number close to that for the scheme here presented.

V. CONCLUSIONS

We have presented a tomographic method via nonoptimal unambiguous state discrimination for arbitrary states in finite dimensions. First, we have shown how to use optimal and nonoptimal unambiguous state discrimination for tomographic purposes in a two-dimensional system. Then we applied the previous results to determine the state of a d-dimensional system by a sequence of reconstructions of two-dimensional subspaces. Considering the total number of measurements to be carried out, the tomographic method based on state discrimination is shown to be more efficient than standard quantum tomography but with half the efficiency of MUB-based tomography. It might be possible, however, to reduce the number of projections needed by the scheme here presented by considering the unambiguous discrimination of more than two nonorthogonal states, such as, for instance, symmetric states.

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