Quantum-memory-assisted entropic uncertainty relation under noise

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The measurement outcomes of two incompatible observables on a particle can be precisely predicted when it is maximally entangled with a quantum memory. In this work, we explore the behavior of the quantum-memory-assisted entropic uncertainty relation under the influence of local unital and nonunital noisy channels. For a class of Bell-diagonal states, we demonstrate that while the unital noises only increase the amount of uncertainty, the amplitude-damping nonunital noises may reduce the amount of uncertainty in the long-time limit. The mechanism behind this phenomenon is also explored by using two dissimilar methods.

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I. INTRODUCTION

One of the most remarkable features of quantum mechanics is the restriction of our ability to simultaneously predict the measurement outcomes of two incompatible observables with certainty, which is called Heisenberg's uncertainty relation [1]. Nowadays, the more modern approach to characterize the uncertainty relation is to use entropic measures rather than standard deviations [2]. If we denote the probability of the outcome x by $p(x)$ when a given quantum state ρ is measured by an observable X, the Shannon entropy $H(X) = -\sum_x p(x)\log_2 p(x)$ characterizes the amount of uncertainty about X before we learn its measurement outcomes [3]. For two noncommuting observables Q and R, the entropic uncertainty relation can be expressed as $H(Q) + H(R) \geqslant \log_2 \frac{1}{c}$ [2], where $c = \max_{\alpha,\beta}|\langle\phi_\alpha|\varphi_\beta\rangle|^2$ with $|\phi_\alpha\rangle$ and $|\varphi_\beta\rangle$ the eigenstates of Q and R, respectively. Since c is independent of the states of the system to be measured, the widely studied entropic uncertainty relation provides us with a more general framework of quantifying uncertainty than the standard deviations (see the review in [4]).

However, the entropic uncertainty relation may be violated if a particle is initially entangled with another one [5]. In the extreme case, an observer holding particle B (quantum memory), maximally entangled with particle A to be measured by two incompatible observables Q and R, is able to precisely predict the outcomes. A stronger entropic uncertainty relation, conjectured by Renes and Boileau [6] and later proved by Berta *et al*. [7], is then

$$S(Q|B) + S(R|B) \geqslant \log_2 \frac{1}{c} + S(A|B), \tag{1}$$

where $S(A|B) = S(\rho_{AB}) - S(\rho_B)$ is the conditional von Neumann entropy with $S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$ the von Neumann entropy [3]. $S(X|B)$ with $X \in (Q,R)$ is the conditional von Neumann entropy of the postmeasurement state $\rho_{XB} = \sum_x(|\psi_x\rangle\langle\psi_x| \otimes \mathbb{1})\rho_{AB}(|\psi_x\rangle\langle\psi_x| \otimes \mathbb{1})$ after quantum system A is measured by X, where $\{|\psi_x\rangle\}$ are the eigenstates of the observable X and $\mathbb{1}$ is the identity operator. Although the proof of this quantum-memory-assisted entropic uncertainty relation is rather complex, the meaning is clear: the entanglement of systems A and B may lead to a negative conditional entropy $S(A|B)$ [8], which will in turn beat the lower bound $\log_2 \frac{1}{c}$. Especially when A and B are maximally entangled, the simultaneous measurement of Q and R can be precisely predicted [7,9]. Recently, two parallel experiments [10,11] have confirmed the quantum-memory-assisted entropic uncertainty relation.

Quantum objects are inevitably in contact with environments and a consequence of the interaction is decoherence or dissipation [3,12]. So several questions naturally arise: How do environmental noises influence the quantum-memory-assisted entropic uncertainty relation? Will the noisy channels surely and only increase the amount of uncertainty because of disentanglement? Is quantum correlation the only key factor for this uncertainty relation under noise? To answer these questions, we consider in this paper two categories of noises: unital and nonunital noisy channels. Intuition tells us that the uncertainty will increase due to the noise-induced disentanglement. For a class of Bell-diagonal states, we demonstrate that this is true for local unital noises, but it may fail for a local amplitude-damping noise, a typical nonunital noisy channel, in the long-time limit. The mechanism behind this phenomenon is explored.

The paper is organized as follows. In Sec. II, for a class of Bell-diagonal states, we present a condition for how the uncertainty of two incompatible observables can reach the lower bound. In Sec. III, we study in detail the local unital and nonunital noise effects on the quantum-memory-assisted entropic uncertainty relation, and two dissimilar explanations of the presented phenomena are discussed in Sec. IV. Finally, we list several open questions and draw our conclusions in Sec. V.

II. QUANTUM-MEMORY-ASSISTED ENTROPIC UNCERTAINTY RELATION FOR BELL-DIAGONAL STATES

We focus on the uncertainty game model illustrated in Ref. [7]: Bob sends qubit A, initially entangled with another qubit B (quantum memory), to Alice. Then, Alice measures either Q or R and announces her measurement choice to

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Bob. Equation (1) captures Bob's uncertainty about Alice's measurement outcome. We assume the two-qubit system to be initially prepared in a class of states with the maximally mixed subsystems [$\rho_{A(B)} = \mathbb{1}^{A(B)}/2$] [13]:

$$\rho_{AB} = \frac{1}{4}\left(\mathbb{1}^A \otimes \mathbb{1}^B + \sum_{j=1}^{3} C_{\sigma_j}\sigma_j^A \otimes \sigma_j^B\right), \quad (2)$$

where σ_j with $j \in \{1,2,3\}$ are the standard Pauli matrices, and the coefficients $C_{\sigma_j} = \mathrm{tr}_{AB}(\rho_{AB}\sigma_j^A \otimes \sigma_j^B)$ satisfy $0 \leqslant |C_{\sigma_j}| \leqslant 1$. A state of this type is also a called Bell-diagonal state, because it can be diagonalized as a convex combination of four Bell states: $\rho_{AB} = \lambda_{\Phi^+}|\Phi^+\rangle\langle\Phi^+| + \lambda_{\Phi^-}|\Phi^-\rangle\langle\Phi^-| + \lambda_{\Psi^+}|\Psi^+\rangle\langle\Psi^+| + \lambda_{\Psi^-}|\Psi^-\rangle\langle\Psi^-|$, with eigenstates $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$, and corresponding eigenvalues $\lambda_{\Phi^\pm} = (1 \pm C_{\sigma_1} \mp C_{\sigma_2} + C_{\sigma_3})/4$ and $\lambda_{\Psi^\pm} = (1 \pm C_{\sigma_1} \pm C_{\sigma_2} - C_{\sigma_3})/4$, respectively. By considering the positivity requirement $\lambda_{\Phi^\pm},\lambda_{\Psi^\pm} \geqslant 0$, all Bell-diagonal states should be confined geometrically within a tetrahedron in a three-dimensional space spanned by $(C_{\sigma_1},C_{\sigma_2},C_{\sigma_3})$ [13] (see Fig. 1), providing an intuitive geometric picture for exploring the quantum-memory-assisted entropic uncertainty relation.

Before investigating the noise effect, we first consider how to consistently reach the lower bound of Eq. (1). We employ the set of Pauli observables $\{\sigma_j\}$ with $j \in \{1,2,3\}$. The conditional von Neumann entropy after qubit A was measured by one of the Pauli observables can be expressed as $S(\sigma_j|B) = H_{\mathrm{bin}}(\frac{1+C_{\sigma_j}}{2})$, where $H_{\mathrm{bin}}(p) = -p\log_2 p - (1-p)\log_2(1-p)$ is the binary entropy [3]. Therefore, if we choose two of the Pauli observables $Q = \sigma_j$ and $R = \sigma_k$ $(j \neq k)$ for measurement, the left-hand side of Eq. (1) can be written as

$$U = H_{\mathrm{bin}}\left(\frac{1+C_{\sigma_j}}{2}\right) + H_{\mathrm{bin}}\left(\frac{1+C_{\sigma_k}}{2}\right). \quad (3)$$

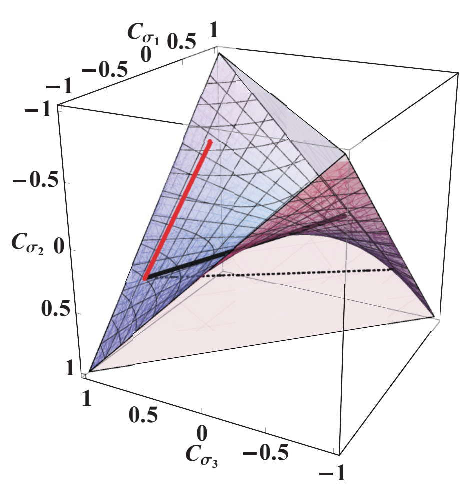

FIG. 1. (Color online) The geometry of Bell-diagonal states with the blue (gray) tetrahedron representing the set of all Bell-diagonal states, where the meshed surface means the valid Bell-diagonal states meeting the SPMC condition $C_{\sigma_2} = -C_{\sigma_1}C_{\sigma_3}$. The black solid, red (gray) solid, and black dotted lines represent trajectories of Bell-diagonal states $(C_{\sigma_1},C_{\sigma_2},C_{\sigma_3}) = (-0.5,0.4,0.8)$ under local bit-flip, phase-flip, and bit-phase-flip noises, respectively.

On the other hand, the complementarity c of the observables σ_j and σ_k is always equal to $1/2$ and the reduced density matrix of the Bell-diagonal state is a maximally mixed state, i.e., $S(\rho_B) = 1$. Therefore, the right-hand side of Eq. (1) reduces to $S(\rho_{AB})$ and takes the form

$$U_b = -\sum_{\chi=\Phi,\Psi;\varepsilon=\pm} \lambda_{\chi^\varepsilon}\log_2\lambda_{\chi^\varepsilon}. \quad (4)$$

In general, Eq. (4) provides a lower bound of uncertainty of Eq. (3). Having in mind the choice of our measurements σ_j and σ_k, we find it convenient to verify that, if the initial Bell-diagonal state meets the condition

$$C_{\sigma_i} = -C_{\sigma_j}C_{\sigma_k}, \ (i \neq j \neq k), \quad (5)$$

$U \equiv U_b$ will be strictly satisfied in Eq. (1), implying a direct measurement of the degree of uncertainty by the joint entropy $S(\rho_{AB})$ of the whole system. In what follows, we name Eq. (5) as the state preparation and measurement choice (SPMC) condition for this new entropic uncertainty relation.

III. NOISE EFFECTS ON THE QUANTUM-MEMORY-ASSISTED ENTROPIC UNCERTAINTY RELATION

We assume that qubit A will experience a local noisy channel when sent to Alice, but qubit B is a quantum memory free from noise. The evolved state of the whole system can be characterized by the quantum map $\mathcal{M}(\rho_{AB}) = \sum_\mu(\kappa_\mu \otimes \mathbb{1})\rho_{AB}(\kappa_\mu \otimes \mathbb{1})^\dagger$ with $\{\kappa_\mu\}$ the local Kraus operators satisfying $\sum_\mu \kappa_\mu^\dagger\kappa_\mu = \mathbb{1}$. In the following sections, we will discuss two categories of noises, i.e., unital and nonunital local noisy channels [14].

A. Unital noise

We first consider several paradigmatic types of local unital noisy channels: bit-flip, bit-phase-flip, and phase-flip (equivalent to phase damping), satisfying the unital condition [14]:

$$\Lambda_u^A\left(\tfrac{1}{2}\mathbb{1}^A\right) = \tfrac{1}{2}\mathbb{1}^A, \quad (6)$$

with $\Lambda_u^A(\rho_A) = \sum_\mu \kappa_\mu\rho_A\kappa_\mu^\dagger$. The corresponding Kraus operators are denoted by $\kappa_0^l = \sqrt{1-\eta_l}\mathbb{1}$, $\kappa_1^l = \sqrt{\eta_l}\sigma_l$, with $l = 1,2,3$ representing bit-flip, bit-phase-flip, and phase-flip channels, respectively (in the following, bit-flip, bit-phase-flip, and phase-flip noises are also called Σ_1, Σ_2, and Σ_3 noises, respectively), and η_l represents the probability of the noise taking place. It is convenient to verify that the state of qubits A and B initially prepared in a Bell-diagonal state $(C_{\sigma_1},C_{\sigma_2},C_{\sigma_3})$ will still be of a Bell-diagonal type when passing through one of the three noisy channels: $\mathcal{M}(\rho_{AB}) = \sum_{\chi=\Phi,\Psi;\varepsilon=\pm}\lambda'_{\chi^\varepsilon}|\chi^\varepsilon\rangle\langle\chi^\varepsilon|$, where $\lambda'_{\Phi^\pm} = [1 \pm C'_{\sigma_1} \mp C'_{\sigma_2} + C'_{\sigma_3}]/4$ and $\lambda'_{\Psi^\pm} = [1 \pm C'_{\sigma_1} \pm C'_{\sigma_2} - C'_{\sigma_3}]/4$ [15] with the three parameters

$$C'_{\sigma_l} = C_{\sigma_l}, \quad C'_{\sigma_m} = (1-2\eta_l)C_{\sigma_m} \ (m \neq l). \quad (7)$$

Here $l = 1$, 2, and 3 represent qubit A suffering from Σ_1, Σ_2, and Σ_3 noisy channels, respectively. Equation (7) implies a subtle relation between the noise and the quantum-memory-assisted entropic uncertainty relation, which leads to the following theorem.

Theorem 1. Assuming qubit A will experience one of the three noises (Σ_1, Σ_2, or Σ_3), for quantum states initially prepared in Bell-diagonal states meeting the SPMC condition $C_{\sigma_i} = -C_{\sigma_j}C_{\sigma_k}$, the quantum-memory-assisted entropic uncertainty of the observables σ_j and σ_k can consistently reach the lower bound, if no Σ_i noise takes place.

Proof. Given the Σ_i noise takes place, we may have $C'_{\sigma_i} = C_{\sigma_i}$, $C'_{\sigma_j} = (1-2\eta_i)C_{\sigma_j}$, and $C'_{\sigma_k} = (1-2\eta_i)C_{\sigma_k}$ according to Eq. (7). Assuming that the SPMC condition can still be consistently satisfied under Σ_i noise, i.e., $C'_{\sigma_i} = -C'_{\sigma_j}C'_{\sigma_k}$, we may get the relation $C_{\sigma_i} = -C_{\sigma_j}C_{\sigma_k}(1-2\eta_i)^2$, which requires $(1-2\eta_i)^2 = 1$, i.e., $\eta_i = 0$ or 1 (just two extreme cases). ■

For an illustration, the geometric picture of Bell-diagonal states satisfying the SPMC condition $C_{\sigma_2} = -C_{\sigma_1}C_{\sigma_3}$ is depicted as the meshed surface in Fig. 1. On this surface, the uncertainty of the observables σ_1 and σ_3 is equal to $S(\rho_{AB})$. The black solid, red (gray) solid, and black dotted lines represent, respectively, trajectories of the Bell-diagonal state, initially prepared in $(C_{\sigma_1},C_{\sigma_2},C_{\sigma_3}) = (-0.5,0.4,0.8)$, under Σ_1, Σ_3, and Σ_2 noises. Apparently, Σ_1 and Σ_3 noises will not break the SPMC condition, since their trajectories are always on the surface, whereas Σ_2 noise, due to the departure from the surface, will definitely break the SPMC condition (except for two points). As a result, the SPMC condition will be consistently satisfied if no Σ_2 noise takes place.

To investigate the quantum-memory-assisted entropic uncertainty relation under a general local unital channel, we prove the following theorem.

Theorem 2. The lower bound of quantum-memory-assisted entropic uncertainty U_b will not decrease under local unital noise if a bipartite system is initially prepared with the maximally mixed subsystems (e.g., Bell-diagonal state).

Proof. Consider a bipartite system AB with dimension $d_{AB} = d_A \times d_B$. If the initial state is prepared in the maximally mixed subsystems, i.e., $\rho_{A(B)} = \frac{\mathbb{1}^{A(B)}}{d_{A(B)}}$, then quantum memory B free from noise will keep maximally mixed: $S(\rho_B) \equiv \log_2 d_B$ [3]. Therefore $U_b = \log_2 \frac{1}{c} + S(A|B) = \log_2 \frac{1}{c} + S(\rho_{AB}) - S(\rho_B)$ is fully dependent on $S(\rho_{AB})$. With the help of the monotonicity of the relative entropy for quantum maps, then $S[\mathcal{M}(\rho_{AB})||\mathcal{M}(\frac{\mathbb{1}^{AB}}{d_{AB}})] \leqslant S[\rho_{AB}||\frac{\mathbb{1}^{AB}}{d_{AB}}]$. [16] Then we have $-S[\mathcal{M}(\rho_{AB})] - \mathrm{tr}[\mathcal{M}(\rho_{AB})\log_2 \mathcal{M}(\frac{\mathbb{1}^{AB}}{d_{AB}})] \leqslant -S(\rho_{AB}) - \mathrm{tr}[\rho_{AB} \log_2 \frac{\mathbb{1}^{AB}}{d_{AB}}] = -S(\rho_{AB}) + \log_2 d_{AB}$. Since the local noisy channel Λ_u^A is unital, the map $\mathcal{M}_{lu} = [\Lambda_u^A \otimes \mathbb{1}^B]$ is still unital. The proof is straightforward: $\mathcal{M}_{lu}(\frac{\mathbb{1}^{AB}}{d_{AB}}) = \sum_\mu (\kappa_\mu \otimes \mathbb{1})\frac{\mathbb{1}^{AB}}{d_{AB}}(\kappa_\mu \otimes \mathbb{1})^\dagger = \sum_\mu (\kappa_\mu \frac{\mathbb{1}^A}{d_A}\kappa_\mu^\dagger) \otimes \frac{\mathbb{1}^B}{d_B} = \frac{\mathbb{1}^A}{d_A} \otimes \frac{\mathbb{1}^B}{d_B} = \frac{\mathbb{1}^{AB}}{d_{AB}}$. Then $\mathrm{tr}[\mathcal{M}_{lu}(\rho_{AB})\log_2 \mathcal{M}_{lu}(\frac{\mathbb{1}^{AB}}{d_{AB}})] = \mathrm{tr}[\mathcal{M}_{lu}(\rho_{AB}) \log_2 \frac{\mathbb{1}^{AB}}{d_{AB}}] = -\log_2 d_{AB}$. Finally, we may obtain

$$S[\mathcal{M}_{lu}(\rho_{AB})] \geqslant S(\rho_{AB}), \tag{8}$$

which implies that U_b will not decrease under local unital noisy channels. ■

As entanglement will not increase under local noisy channels [16], the noise-induced disentanglement can be employed to account for the nondecreasing of uncertainty in Theorem 2. However, entanglement is not the only way to characterize quantum correlations. In order to explore the influence of quantum correlations beyond entanglement on this uncertainty relation, we relate the lower bound of Eq. (1) to the definition of discord: $D = -S(A|B) + \min_{\{B_k\}} \sum_k q_k S(\rho_A^k)$ [17], where $\min_{\{B_k\}} \sum_k q_k S(\rho_A^k)$ (denoted by M in the following) captures the minimal missing information about A after B is measured, and $\rho_A^k = \mathrm{tr}_B\{B_k \rho_{AB} B_k^\dagger\}/q_k$ is the resulting state after the complete measurement $\{B_k\}$ on qubit B, and $q_k = \mathrm{tr}_{AB}\{B_k \rho_{AB} B_k^\dagger\}$. Therefore, we have

$$U \geqslant \log_2 \frac{1}{c} + M - D. \tag{9}$$

For Bell-diagonal states, M can be expressed as [18]

$$M = H_{\mathrm{bin}}\left(\frac{1 + C_{\max}}{2}\right), \tag{10}$$

with $C_{\max} = \max\{|C'_{\sigma_1}|,|C'_{\sigma_2}|,|C'_{\sigma_3}|\}$. According to Eq. (7), if Σ_i noise takes place, as long as $|C_{\sigma_i}| \geqslant |C_{\sigma_j}|,|C_{\sigma_k}|$, we may have $M = H_{\mathrm{bin}}(\frac{1+|C_{\sigma_i}|}{2})$, which is a constant, and this implies that the uncertainty is fully dependent on the quantum correlations between qubit A and quantum memory B. Especially, if initial state is prepared according to SPMC condition $C_{\sigma_j} = -C_{\sigma_i}C_{\sigma_k}$ (or $C_{\sigma_k} = -C_{\sigma_i}C_{\sigma_j}$), the equality in Eq. (9) can be consistently satisfied, which suggests that measuring the uncertainty of the observables σ_i and σ_k (or σ_i and σ_j) can be directly related to quantum discord,

$$D = \mathrm{const.} - U, \tag{11}$$

with const. $= \log_2 \frac{1}{c} + H_{\mathrm{bin}}(\frac{1+|C_{\sigma_i}|}{2})$ a constant.

As an example, we consider the phase-damping channel with the Kraus operators $\kappa_0^{pd} = |0\rangle\langle 0| + e^{-\frac{\Gamma_{pd}t}{2}}|1\rangle\langle 1|$ and $\kappa_1^{pd} = \sqrt{1 - e^{-\Gamma_{pd}t}}|1\rangle\langle 1|$, which is equivalent to the phase-flip channel with $\eta_3 = (1 - e^{-\frac{\Gamma_{pd}t}{2}})/2$ [3]. Qubit A and quantum memory B are initially prepared in a Bell-diagonal state $(C_{\sigma_1},C_{\sigma_2},C_{\sigma_3}) = (-0.5,0.4,0.8)$ satisfying the SPMC condition $C_{\sigma_2} = -C_{\sigma_1}C_{\sigma_3}$, then qubit A, which is sent through the phase-damping channel, will not break the SPMC condition, and the uncertainty of the observables σ_1 and σ_3 can be directly related to quantum discord $D = \mathrm{const.} - U$, with const. $= 1 + H_{\mathrm{bin}}(0.9)$ and $U = H_{\mathrm{bin}}(0.9) + H_{\mathrm{bin}}(0.5 - 0.25e^{-\Gamma_{pd}t/2})$. As shown in Fig. 2(a) and 2(b), the uncertainty will increase in the long-time limit due to the gradually missing quantum correlations (measured by discord and concurrence [19]).

Since $\mathcal{M}(\rho_{AB}) = \sum_{\chi=\Phi,\Psi;\varepsilon=\pm} \lambda'_{\chi^\varepsilon}|\chi^\varepsilon\rangle\langle\chi^\varepsilon|$ holds for bit-flip, bit-phase-flip, and phase-flip (phase-damping) noises, we may conclude that these unital channels are also semiclassical according to the definition presented in Ref. [20]. Therefore, quantum correlations never increase under the above local unital channels. Here, the decrease of quantum correlations including entanglement and discord makes the outcomes of two incompatible observables more uncertain. Will this phenomenon still appear under the influence of nonunital channels?

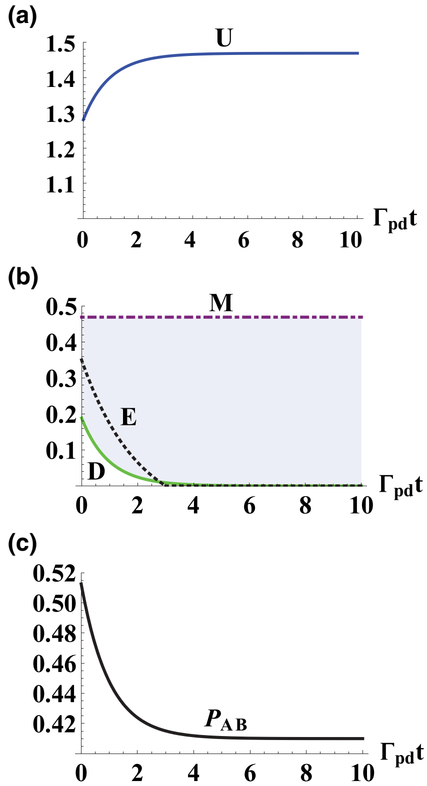

FIG. 2. (Color online) (a) Uncertainty of the observables σ_1 and σ_3 with initial state $(C_{\sigma_1},C_{\sigma_2},C_{\sigma_3}) = (-0.5,0.4,0.8)$ under the local phase-damping channel with Γ_{pd} as the damping rate. (b) Entanglement (E), discord (D), and minimal missing information about A after B is measured (M) vs $\Gamma_{pd}t$. (c) Purity of state ρ_{AB} vs $\Gamma_{pd}t$.

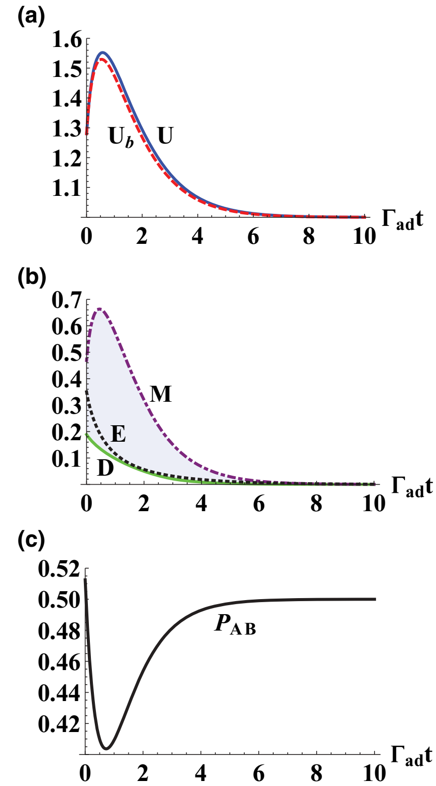

FIG. 3. (Color online) (a) U and U_b of the observables σ_1 and σ_3 with initial state $(C_{\sigma_1},C_{\sigma_2},C_{\sigma_3}) = (-0.5,0.4,0.8)$ under the local amplitude-damping channel with Γ_{ad} as the damping rate. (b) Entanglement (E), discord (D), and minimal missing information about A after B is measured (M) vs $\Gamma_{ad}t$. (c) Purity of state ρ_{AB} vs $\Gamma_{ad}t$.

B. Nonunital noise

To further explore this problem, we consider a nonunital and nonsemiclassical local channel, i.e., the amplitude-damping noise with Kraus operators $\kappa_0^{ad} = e^{-\frac{\Gamma_{ad}t}{2}}|0\rangle\langle 0| + |1\rangle\langle 1|$, $\kappa_1^{ad} = \sqrt{1-e^{-\Gamma_{ad}t}}|1\rangle\langle 0|$ [3,21]. Here $\Lambda^A_{nu}(\frac{1}{2}\mathbb{1}^A) = [e^{-\Gamma_{ad}t}|0\rangle\langle 0| + (2-e^{-\Gamma_{ad}t})|1\rangle\langle 1|]/2$ is not maximally mixed, which implies that the state through a noisy channel to be measured by Alice is not a Bell-diagonal state, and the SPMC condition presented above is no longer satisfied. We may only study the lower bound of uncertainty instead. Given the initial condition $|C_{\sigma_1}| \geqslant |C_{\sigma_2}|$, M can be expressed as

$$M = \min\{M_x, M_z\}, \tag{12}$$

where $M_x = H_{\text{bin}}(\frac{1+u}{2})$ with $u = \sqrt{e^{-\Gamma_{ad}t}[C_{\sigma_1}^2 + 2\cosh(\Gamma_{ad}t) - 2]}$ and $M_z = \frac{H_{\text{bin}}(v_+)+H_{\text{bin}}(v_-)}{2}$ with $v_\pm = (1 \pm C_{\sigma_3})\exp(-\Gamma_{ad}t)/2$ (see the Appendix), which is time dependent and may also be nonmonotonic. Figure 3(a) demonstrates an interesting phenomenon: the uncertainty of two incompatible observables might be reduced under the influence of the amplitude-damping noise in the long-time limit [22]. The key factor for the uncertainty reduction should not be the quantum correlations, which are decreasing in this case [shown in Fig. 3(b)]. Therefore, the quantum correlation is not the only decisive factor for the amount of the uncertainty.

IV. EXPLANATIONS FOR THE ABOVE PHENOMENA

In this section, we first summarize the above phenomena: (i) The uncertainty (or lower bound) will increase under local unital noisy channels while it may be reduced under the nonunital noise channel. (ii) The relation between quantum correlations and uncertainty is subtle, since the reduced uncertainty occurs in the case in which quantum correlations, including discord and entanglement, are also reduced under the above local amplitude-damping channel. In what follows, we present two qualitative methods to explain the above phenomena.

A. Competition between *M* and *D*

In order to understand the physical origin of the above phenomena, we reconsider the treatment in Eq. (9). Apparently, the uncertainty is related to the discrepancy between M and D, not just the quantum correlations only, and it is decided by the competition between quantum correlations and the minimal missing information of a single particle after local measurement on another one. For illustration, the competition between M and D are depicted with blue (gray) shades in the insets of Figs. 2(b) and 3(b). Quantum correlations, including entanglement and discord, will decrease in both cases. However, the most difference is that Eq. (12) is not a monotonic function, which may even decrease under an amplitude-damping channel. That is to say, the missing information by local measurements may be reduced in the long-time limit, which in turn lowers the uncertainty. For the operational interpretations of M and D see Ref. [23]

B. Purity of state ρ_{AB}

Another possible way to explain the above phenomena is to employ the purity of state ρ_{AB}:

$$P_{AB} = \mathrm{tr}_{AB}\big(\rho_{AB}^2\big), \tag{13}$$

which would be anticorrelated with the uncertainty (i.e., a purer state will cause less uncertainty) [24]. As shown in Figs. 2(c) and 3(c), the increase (decrease) of the uncertainty is caused by the reduction (growth) of state purity.

V. CONCLUSION

Before ending this paper, we mention some open problems waiting for solution: Is it possible to find the minimum uncertainty achievable in the presence of nonunital noise? Can we find similar phenomena by other entropy measures, such as smooth entropy [25]? Finally, is it possible to directly utilize the decoherence or dissipation properties illustrated in this paper to perform quantum information tasks such as quantum channel testing?

In summary, we have studied the noise effect on quantum-memory-assisted entropic uncertainty relation on Bell-diagonal states. By investigating different noises, we have demonstrated in this case that local unital noises will surely increase the uncertainty, but under the influence of a nonunital amplitude-damping channel, we found that the uncertainty might even be reduced. Our work is the first step toward the study of the noise effect on the quantum-memory-assisted entropic uncertainty relation and can be immediately investigated by all-optical setups, where the state preparation and measurement can be realized like in Refs. [10,11] and the noisy channels can be simulated according to Refs. [26,27].

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APPENDIX: EXPRESSION FOR EQ. (12)

Although a Bell-diagonal state under a local amplitude-damping channel will no longer be of the Bell-diagonal type, it is still of "X" type,

$$\rho_{AB}(t) = \frac{1}{2}\begin{pmatrix} v_+ & 0 & 0 & w_- \\ 0 & v_- & w_+ & 0 \\ 0 & w_+ & 1-v_+ & 0 \\ w_- & 0 & 0 & 1-v_- \end{pmatrix}, \tag{A1}$$

with $v_\pm = e^{-\Gamma_{ad}t}(1 \pm C_{\sigma_3})/2$ and $w_\pm = e^{-\Gamma_{ad}t/2}(C_{\sigma_1} \pm C_{\sigma_2})/2$. In general, we may employ projectors $\{B_k\} = \{\cos\theta|0\rangle + e^{i\xi}\sin\theta|1\rangle,\ \ e^{-i\xi}\sin\theta|0\rangle - \cos\theta|1\rangle\}$ [17]. If $|w_+ + w_-| \geqslant |w_+ - w_-|$, i.e., $|C_{\sigma_1}| \geqslant |C_{\sigma_2}|$, the optimal measurement is either $\{(|0\rangle + |1\rangle)/\sqrt{2}, (|0\rangle - |1\rangle)/\sqrt{2}\}$, i.e., σ_1 operation, or $\{|0\rangle, -|1\rangle\}$, i.e., σ_3 operation [28]. Therefore, the minimal missing information after local measurement can be expressed as

$$M = \min\{M_x, M_z\}, \tag{A2}$$

where $M_x = H_{\mathrm{bin}}(\frac{1+u}{2})$ with $u = \sqrt{e^{-\Gamma_{ad}t}[C_{\sigma_1}^2 + 2\cosh(\Gamma_{ad}t) - 2]}$ and $M_z = \frac{H_{\mathrm{bin}}(v_+)+H_{\mathrm{bin}}(v_-)}{2}$. With M, D can be calculated straightforwardly.

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