Multipartite quantum nonlocality under local decoherence

R. Chaves,[1,2] D. Cavalcanti,[3] L. Aolita,[1] and A. Acín[1,4]

[1]*ICFO-Institut de Ciències Fotòniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain*
[2]*Instituto de Física, Universidade Federal do Rio de Janeiro. Caixa Postal 68528, 21941-972 Rio de Janeiro, RJ, Brazil*
[3]*Centre for Quantum Technologies, University of Singapore, Singapore*
[4]*ICREA-Institució Catalana de Recerca i Estudis Avançats, Lluis Companys 23, 08010 Barcelona, Spain*

(Received 11 April 2012; published 16 July 2012)

We study the nonlocal properties of two-qubit maximally entangled and N-qubit Greenberger-Horne-Zeilinger states under local decoherence. We show that the (non)resilience of entanglement under local depolarization or dephasing is not necessarily equivalent to the (non)resilience of Bell-inequality violations. Apart from entanglement and Bell-inequality violations, we consider also nonlocality as quantified by the nonlocal content of correlations and provide several examples of anomalous behaviors, both in the bipartite and multipartite cases. In addition, we study the practical implications of these anomalies on the usefulness of noisy Greenberger-Horne-Zeilinger states as resources for nonlocality-based physical protocols given by communication complexity problems.

DOI: 10.1103/PhysRevA.86.012108 PACS number(s): 03.65.Ud, 03.67.−a, 03.65.Yz

I. INTRODUCTION

Although closely connected, entanglement and nonlocality constitute two substantially different concepts. Entanglement refers to whether a state can be decomposed as a convex combination of product quantum states and is therefore inherent to the Hilbert-space structure of quantum theory [1]. Operationally, a state is entangled whenever it cannot be prepared by local quantum operations and classical communication. Nonlocality on the other hand refers to correlations between distant measurements—whatever the underlying theory—that cannot be explained by local hidden-variable models [2].

Correlations describable in terms of local hidden variables necessarily satisfy a set of linear constraints known as *Bell inequalities* [2], which can be tested in the laboratory. Thus, the violation of any Bell inequality reveals the presence of nonlocality; whereas its nonviolation does not have any implication on the local or nonlocal nature of the corresponding correlations (unless all Bell inequalities are proven to be satisfied). In turn, every pure quantum state is entangled if, and only if, it violates some Bell inequality [3]. Additionally, if an arbitrary quantum state is nonlocal, it is also entangled [4]. The converse however has long been known not to be true: There exist mixed entangled states that admit local hidden-variable descriptions [4,5].

From an applied point of view, entanglement has been identified over the last two decades as the key resource in a variety of physical tasks (see Ref. [1] and references therein). These go from teleportation, dense coding, secure quantum key distribution (QKD) and quantum communication, to quantum computing, for instance. In more recent years, it was nonlocality that was also raised to the status of a physical resource. An example thereof was given by the advent of device-independent applications, such as QKD [6] or randomness generation [7]. There, correlations violating some Bell inequality suffice to establish a secret key or a perfectly random bit, regardless of the physical means by which they are established. Another prominent example is distributed-computing scenarios such as those of *communication complexity problems* (CCPs) [8,9]. There, N distant users, assisted by some initial correlations and a restricted amount of public communication, must locally calculate the value of a given function f with some probability of success. It was shown in Ref. [9] that, for a broad family of N-partite Bell inequalities, one can associate to every inequality a CCP that can be solved more efficiently (with higher probability) than by any classical protocol if, and only if, it is assisted by correlations that violate the inequality. Furthermore, the authors showed that, for any fixed N, the quantum gain (in success probability) is proportional to the amount of violation, thus automatically yielding a direct operational interpretation for the violation of this type of Bell inequality.

However, under realistic situations where actual applications take place, systems are unavoidably subject to noise. It is therefore important to probe the resilience of physical resources in the presence of noise. This becomes particularly necessary for many-particle systems, where the detrimental effects of the interaction with the environment typically accumulate exponentially with the number of system components. Nevertheless, while the open-system dynamics of multipartite entanglement has been extensively studied [10–13], the scaling behavior of the nonlocal properties of quantum states under decoherence is barely understood. To the best of our knowledge, some cases of such behavior were systematically studied in Ref. [14], but focused only on the critical noise strengths (or, alternatively, the times) for which the violations of a specific family of multisetting N-partite Bell inequalities vanish. We know, however, from the study of entanglement, that such critical values on their own can be very misleading as figures of merit of any robustness. Situations are known where correlations take longer to vanish but still, for a given fixed time, decay exponentially with N [11]. Thus, the full dynamical evolution of nonlocality must be studied to draw conclusions on its fragility.

In this paper we study the evolution of nonlocality, in comparison to that of entanglement, for two-qubit maximally entangled and Greenberger-Horne-Zeilinger (GHZ) [15] states subject to independent depolarization or phase damping [16]. We find that decoherence can lead to unexpected behavior in the nonlocal properties of states [17]. For instance, in the two-qubit case, local decoherence can lead to the natural appearance of anomalies in the orderings of states, such that the less-entangled states have more nonlocality. In turn, in the multipartite case, we identify regimes of noise for which the violation of the Mermin inequality [18] by decohered GHZ states grows exponentially with the number of particles N, despite the fact that both entanglement and nonlocal content [19] decay exponentially. Remarkably, in some cases, this exponentially decaying entanglement is even bound [11]. To get a physical insight of the practical consequences of such anomalous behavior, we study the quantum gains for CCPs. We find that the gain increase with N for small N but decreases exponentially for large N. So an exponentially large visibility coexists with exponentially small entanglement, nonlocal content and usefulness for solving the associated physical problem.

II. NOTATION AND DEFINITIONS

In this section we introduce the definitions, concepts, and tools applied in the derivation of the results.

A. Multiqubit states under noisy channels

The initial states we will consider throughout are the GHZ states [15]

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}), \tag{1}$$

which reduce to maximally entangled Bell states for $N = 2$.

As paradigmatic models of noise we focus on the independent depolarizing (D) and phase-damping (PD) channels [16]. Channel D describes the situation in which a qubit remains untouched with probability $1 - p$ or is mapped to the maximally mixed state (white noise) with probability p:

$$\varepsilon_p^D(\rho) = (1-p)\,\rho + p\tfrac{1}{2}. \tag{2}$$

Channel PD induces the complete loss of quantum coherence with probability p, but without any population exchange,

$$\varepsilon_p^{PD}(\rho) = (1-p)\,\rho + p\sum_i |i\rangle\langle i|\langle i|\rho|i\rangle, \tag{3}$$

where i denotes the energy basis. Probability p can also be interpreted as a convenient parametrization of time, where $p = 0$ refers to the initial time $t = 0$ and $p = 1$ refers to the asymptotic limit $t \to \infty$.

The action of these models of noise on states (1) has been explicitly calculated several times before. Here, we make use of the expressions derived in Ref. [11] and refer the interested reader to Refs. [10,11] for more details. For example, states (1) under independent depolarizing or phase-damping channels can in both cases be expressed as [11]

$$\rho_{(p)} = (1-p)^N|\Phi^+\rangle\langle\Phi^+| + [1-(1-p)^N]\rho_s, \tag{4}$$

where ρ_s is a separable state, diagonal in the computational basis $\{|0\ldots0\rangle, |0\ldots01\rangle, \ldots, |1\ldots1\rangle\}$, that depends on the channel in question.

The simplicity of decomposition (4) allows for exhaustive analytical treatments. For example, the entanglement $E(\rho_{(p)})$—as quantified by any convex entanglement measure E—of decohered states (4) always decays faster than exponentially with N: $E(\rho_{(p)}) \leqslant (1-p)^N E(\rho_{(0)})$ [11]. In particular, for small p and large N, it typically saturates the inequality as $E(\rho_{(p)}) \approx (1-p)^N E(\rho_{(0)})$. Exponentially small physical perturbations are enough to fully disentangle $\rho_{(p)}$. Furthermore, as also mentioned, for the case of channel D and for any noise strength, there exists an N above which the entanglement in states (4) is bound [11]. We will see in what follows how the above-mentioned symmetry can also be exploited to understand the nonlocal properties of the states.

B. Bell-inequality violations by noisy multiqubit states

We consider throughout the same type of Bell inequalities as considered in Ref. [9]:

$$\mathcal{I}_N \doteq \sum_{x_1,\ldots,x_N=0}^{1} g(x_1,\ldots,x_N)\,C(x_1,\ldots,x_N) \leqslant \mathcal{I}_N^{\mathrm{L}}. \tag{5}$$

Each part i measures randomly in one of two settings, $x_i = 0$ or $x_i = 1$, and obtains 1 or -1 as the outcome. Here, $g(x_1,\ldots,x_N)$ is any real-valued function and $C(x_1,\ldots,x_N)$ denotes the correlation function for the measurements of N separated parties. $\mathcal{I}_N^{\mathrm{L}}$ is in turn the local bound; that is, the maximum possible value of polynomial $\mathcal{I}_N$ attainable by any local-hidden-variable (LHV) model. In the quantum case, $x_i = 0$ or $x_i = 1$ correspond to observables O_{i_0} or O_{i_1}, respectively, each one with eigenvalues ±1. Then, the correlation function is given by $C(x_1,\ldots,x_N) = \mathrm{Tr}[\rho \cdot O_{1_{x_1}} \otimes \ldots \otimes O_{N_{x_N}}]$, where ρ is the state under scrutiny.

C. Nonlocal content of noisy multiqubit states

We will see next that, apart from the entanglement, the nonlocality of $\varrho_{(p)}$ also decays as the state's violation grows exponentially. For this, we consider a measure of nonlocality based on the Elitzur-Popescu-Rohrlich (EPR2) decomposition [19]: Any joint-probability distribution P, which characterizes the correlations of some experiment, can be decomposed into the convex mixture of purely local and purely nonlocal parts as

$$P = (1-p_{\mathrm{NL}})\,P_{\mathrm{L}} + p_{\mathrm{NL}}P_{\mathrm{NL}}. \tag{6}$$

P_{L} and P_{NL} are respectively the corresponding local and nonsignalling joint-probability distributions in the decomposition, and $0 \leqslant p_{\mathrm{NL}} \leqslant 1$. The minimal weight of the nonlocal part over all such possible decompositions provides an unambiguous quantification of the nonlocality in P:

$$\tilde{p}_{\mathrm{NL}} \doteq \min_{P_{\mathrm{L}},P_{\mathrm{NL}}} p_{\mathrm{NL}}. \tag{7}$$

It is also called the *nonlocal content of* P and its counterpart $\tilde{p}_{\mathrm{L}} \equiv 1 - \tilde{p}_{\mathrm{NL}}$ gives the *local content* of P.

The violation of any Bell inequality allows one to obtain a nontrivial lower bound to the nonlocal content. Indeed, for any (linear) Bell inequality $\mathcal{I} \leqslant \mathcal{I}^{\mathrm{L}}$, the optimal decomposition $P = (1-\tilde{p}_{\mathrm{NL}})\tilde{P}_{\mathrm{L}} + \tilde{p}_{\mathrm{NL}}\tilde{P}_{\mathrm{NL}}$ yields $\mathcal{I}(P) \equiv (1-\tilde{p}_{\mathrm{NL}})\mathcal{I}(\tilde{P}_{\mathrm{L}}) + \tilde{p}_{\mathrm{NL}}\mathcal{I}(\tilde{P}_{\mathrm{NL}})$. Now, on the one hand, since it is local, $\tilde{P}_{\rm L}$ cannot violate the inequality: $\mathcal{I}(\tilde{P}_{\rm L}) \leqslant \mathcal{I}^{\rm L}$. On the other hand $\mathcal{I}(\tilde{P}_{\rm NL})$ cannot be larger than the maximal nonsignalling value, $\mathcal{I}^{\rm NL}$, of $\mathcal{I}$. Therefore, it is always $\mathcal{I}(P) \leqslant (1-\tilde{p}_{\rm NL})\mathcal{I}^{\rm L} + \tilde{p}_{\rm NL}\mathcal{I}^{\rm NL}$, from which follows that

$$\tilde{p}_{\rm NL} \geqslant \frac{\mathcal{I}(P) - \mathcal{I}^{\rm L}}{\mathcal{I}^{\rm NL} - \mathcal{I}^{\rm L}}. \quad (8)$$

Notice that any correlations P that violate a Bell inequality saturating the maximal nonsignalling value, $\mathcal{I}(P) = \mathcal{I}^{\rm NL}$ are automatically fully nonlocal (i.e., with $\tilde{p}_{\rm NL} = 1$). This is precisely what happens to states (1), which saturate the algebraic violations of an infinite-setting Bell inequality [20] for $N = 2$, and that of the Mermin inequality [18] for an odd number of parties. This is why states (1) are said to be maximally nonlocal. Actually, GHZ states are maximally genuine multipartite nonlocal, as they reach the algebraic violation of a Bell inequality for this form of nonlocality [21].

III. RESULTS

A. Anomalies in noisy dynamics of nonlocality versus entanglement

The first example we consider is the familiar Clauser-Horne-Shimony-Holt (CHSH) inequality [22], $\mathcal{I}_2 \equiv \mathcal{I}_{\rm CHSH}$, defined by Eq. (5) (for $N = 2$) with $g(x_1,x_2) \equiv (-1)^{x_1x_2}$ and $\mathcal{I}_2^{\rm L} \equiv \mathcal{I}_{\rm CHSH}^{\rm L} = 2$. Its maximal quantum violation ($\mathcal{I}_{\rm CHSH} = 2\sqrt{2}$) is realized by Bell state (1) (for $N = 2$) with the observables $O_{1_0} = -X_1$, $O_{1_1} = Z_1$, $O_{2_0} = (X_2 - Z_2)/\sqrt{2}$ and $O_{2_1} = (X_2 + Z_2)/\sqrt{2}$, where X_i and Z_i are respectively the first and third Pauli operators of qubit i. In the noisy scenario, the maximal violation for decohered states (4) is immediately calculated with the criterion of Ref. [23]. In Fig. 1 we compare the evolution of the maximal value of $\mathcal{I}_{\rm CHSH}$ with that of entanglement, as a function of p, for the cases of independent phase damping of noise strength p and independent depolarization of noise strength $p/2$. Entanglement is quantified by the negativity [24], which—

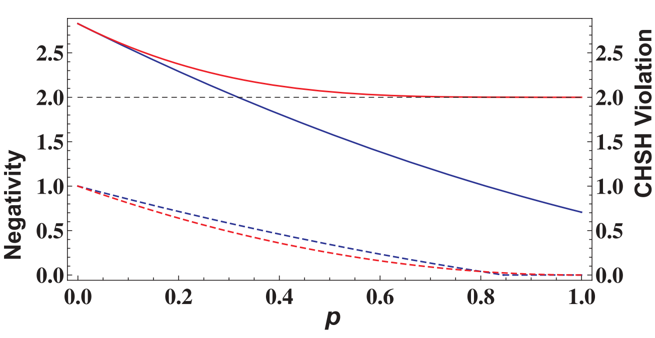

FIG. 1. (Color online) Entanglement (lower dashed curves) and maximal value of the CHSH polynomial $\mathcal{I}_{\rm CHSH}$ (upper solid curves), for maximally entangled two-qubit states under independent phase damping of noise strength p (in red, upper solid and lower dashed curves) and independent depolarization of noise strength $p/2$ (in blue, middle two curves), as a function of p. The horizontal black dashed line represents the local bound $\mathcal{I}_{\rm CHSH}^{\rm L} = 2$, below which there is no more CHSH violation. Decoherence naturally drives the system to dephased states that possess less entanglement than depolarized states but that, at the same time, violate the CHSH inequality more.

since the considered states are Bell diagonal—coincides with the concurrence [25] and can be taken as an unambiguous entanglement quantifier. A curious effect is observed: While the depolarized states display more entanglement than the dephased ones (except for large p), the violation of the CHSH inequality given by the dephased states is always above that of the depolarized states. In fact, as p increases from zero to the point where the depolarized states stop violating the inequality, the gap between the entanglement and the violation grows.

Result 0. Local phase damping can naturally drive two-qubit systems to states with less entanglement but more CHSH violation than those driven by local depolarization.

As N increases this type of anomaly becomes stronger. For $N > 2$ we consider the Mermin inequality [18], defined by inequality (5) with $g(x_1,\ldots,x_N) \equiv \cos[\frac{\pi}{2}(x_1 + \cdots + x_N)]$ and $\mathcal{I}_N^{\rm L} = 2^{N/2}$, for N even, and $\mathcal{I}_N^{\rm L} = 2^{(N-1)/2}$, for N odd. Its maximal quantum violation is $\mathcal{I}_N = 2^{N-1}$ and is attained by GHZ states (1) (for $N > 2$) with observables $O_{i_0} = X_i$ and $O_{i_1} = Y_i$, where X_i and Y_i are, respectively, the first and second Pauli operators acting on qubit i. Since the local bound $\mathcal{I}_N^{\rm L}$ is also an exponentially growing function of N, instead of $\mathcal{I}_N$ one usually quantifies the violation with the ratio $\mathcal{V}_N = \mathcal{I}_N/\mathcal{I}_N^{\rm L}$. $\mathcal{V}_N$ is sometimes called the visibility of the inequality and, in terms of it, inequality (5) reads $\mathcal{V}_N \leqslant 1$. It is immediate to see that the maximal violation of the Mermin inequality for noisy GHZ states (4) takes place with the same observables as for $p = 0$. Thus, their maximal visibility is immediately calculated to be

$$\mathcal{V}_N = (1-p)^N\, 2^{(N-1)/2}, \quad (9)$$

where for simplicity we have taken N odd. Another curious effect appears here. The entanglement in $\rho_{(p)}$ decays always smoothly (exponentially) with N (except for channel D and when $p \gtrsim 0.49$) [11]. Nevertheless, its visibility displays an abrupt transition from exponentially growing to exponentially decreasing at the relatively small value $p_{\rm t} = 1 - 1/\sqrt{2} \approx 0.29$, for the two channels considered. In turn, the critical noise strength beyond which the inequality is not violated any further is $p_{\rm c} = 1 - 1/\sqrt{2}^{(N-1)/N} < p_{\rm t}$. This leads to the following effect: For $p < p_{\rm c}$, and as N increases, $\rho_{(p)}$ gets exponentially close to the separable states while at the same time yielding an exponentially large violation of Eq. (5). Furthermore, we know that, for channel D, the little entanglement remaining in $\varrho_{(p)}$ very rapidly becomes bound [11]. The situation is illustrated in Fig. 2, where the regions with the different regimes are plotted.

Result 1. States with exponentially small and bound entanglement can provide an exponentially large Bell inequality violation.

In addition, from Eqs. (9) and (8), we obtain

$$\tilde{p}_{\rm NL} \geqslant \frac{(1-p)^N\, 2^{(N-1)/2} - 1}{2^{(N-1)/2} - 1}, \quad (10)$$

where again for simplicity we have taken N odd. Additionally, decomposition (4) of $\rho_{(p)}$ immediately yields an upper bound to its local content. This is because the correlations $P(|\Phi^+\rangle)$ in $|\Phi^+\rangle$ are purely nonlocal whereas correlations $P(\rho_s)$ of ρ_s are purely local (because ρ_s is separable). Therefore, $P(\rho_{(p)}) = (1-p)^N P(|\Phi^+\rangle) + [1-(1-p)^N] P(\rho_s)$ realizes a particular EPR2 decomposition of the correlations $P(\rho_{(p)})$

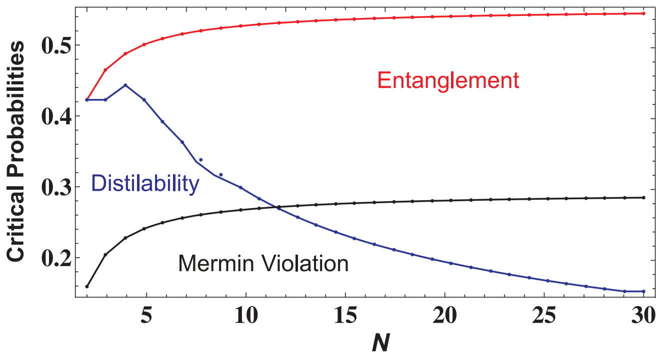

FIG. 2. (Color online) Critical probabilities for full separability (in red), distillability (in blue), and violation of the Mermin inequality (p_c, in black), for N-qubit GHZ states under independent depolarization of strength p. Below the red curve the states are entangled; they are distillable though only below the blue curve [11]. Between the red and the blue curves the states are thus bound entangled. The total entanglement (not plotted) decreases exponentially with N. Below the black curve in turn the states violate the Mermin inequality—and they do it exponentially with N. Hence, for $N > 11$, and between the black and the blue curves, decoherence naturally drives the system to states with exponentially small and bound entanglement that yet violate the inequality exponentially.

in $\rho_{(p)}$, and the optimal one must thus necessarily satisfy

$$\tilde{p}_{\rm NL} \leqslant (1-p)^N , \tag{11}$$

in a way reminiscent to GHZ entanglement decay [11].

Result 2. The nonlocal content in locally depolarized, or dephased, states (4) cannot decay slower than exponentially with N.

Notice further that, for $p < p_{\rm t} = 1 - 1/\sqrt{2}$, lower bound (10) converges, as N grows, to the upper bound (11). Then, in the limit $N \to \infty$, the exact value for the nonlocal content of $\rho_{(p)}$ is $\tilde{p}_{\rm NL} = (1-p)^N$. So, exponentially large Mermin visibility (9) coexists not only with an exponentially small (and in some cases bound) entanglement but also with an exponentially small nonlocal content. This indicates that the visibility of a Bell inequality may not always constitute an unambiguous quantitative figure of merit for the nonlocal resources of quantum states.

B. Efficiency gain in communication complexity problems with noisy multiqubit states

As discussed, the previous results may be a manifestation of the fact that the visibility of a Bell test does not necessarily quantify a state's usefulness for a practical (nonlocal) problem. In particular, in view of both the entanglement and nonlocal content of $\rho_{(p)}$ decaying at slowest exponentially with N, it is interesting to explore what implication the observed exponentially growing Mermin-inequality visibility has on some concrete physical task. Here, we focus on the gain in efficiency—with respect to all protocols assisted by classical correlations—for solving probabilistic distributed computations given by communication complexity problems (CCPs) [8]. In the pure-state case of $p = 0$, an exponentially growing Mermin visibility is responsible for an exponentially growing quantum gain [9]. However, for $p > 0$, we find that the visibility in question yields a gain that, after a short transient period of growth with N, converges to the universal exponential-decay law of $(1-p)^N$.

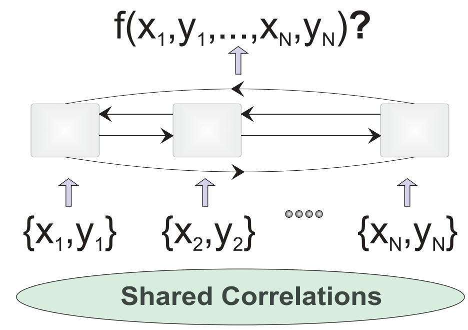

FIG. 3. (Color online) Distributed computing scenario [8]. N distant users receive each a two-bit input string $\{x_i, y_i\}$, with $1 \leqslant i \leqslant N$. Distributed bits y_i are chosen randomly between 1 and -1, and each x_i is chosen as 0 or 1 depending on a joint probability distribution $Q(x_1, \ldots, x_N)$. The users are endowed with some preestablished shared correlations, but they can only exchange a restricted amount of public communication. The problem is, for each and all of them, to compute the value $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of a given function f with some probability. The minimum number of bits that they must broadcast to do so defines the communicational complexity of the problem. We consider here a specific subclass of such problems with $Q(x_1, \ldots, x_N) \equiv |g(x_1, \ldots, x_N)| / \sum_{x_1, \ldots, x_N = 0}^{1} |g(x_1, \ldots, x_N)|$, for $g(x_1, \ldots, x_N)$ some real-valued function, f a boolean function of the form $f = y_1 \cdots \; y_n S[g(x_1, \ldots, x_n)]$, with $f = \pm 1$ and $S[g] \equiv g/|g| = \pm 1$ being the sign function, and where each user is allowed to broadcast only a single bit. For this subclass, it was shown in Ref. [9], for a broad family of protocols, that if the shared correlations violate the associated Bell inequality (5) the users can solve the problem with a higher probability than with any classical protocol (assisted by LHV correlations).

The family of CCPs we consider is described in Fig. 3. For these, the maximal probability of success, p^s, achievable through a broad class of protocols [9] with preestablished correlations P as the resource is

$$p^s = \frac{1}{2}\left(1 + \frac{\mathcal{I}_N(P)}{\sum_{x_1, \ldots, x_N = 0}^{1} |g(x_1, \ldots, x_N)|}\right). \tag{12}$$

Therefore, no such strategy can do as well when using classical resources as when based on correlations that violate inequality (5). For the Mermin inequality one has $\mathcal{I}_N^{\rm L} = 2^{N/2}$ for N even, $\mathcal{I}_N^{\rm L} = 2^{(N-1)/2}$ for N odd, and $\sum_{x_1, \ldots, x_N = 0}^{1} |g(x_1, \ldots, x_N)| = 2^{N-1}$. Therefore, the best such protocol with classical correlations solves the associated CCP with a probability $p_{\rm L}^s = \frac{1}{2}(1 + 1/\sqrt{2^{N-1}})$ for N odd, and $p_{\rm L}^s = \frac{1}{2}(1 + 1/\sqrt{2^{N-2}})$ for N even. If states (4) are used as the resource in contrast the protocol succeeds with $p_Q^s = \frac{1}{2}[1 + (1-p)^N]$.

The quantum gain in the protocol is defined as $G_Q \doteq p_Q^s - p_{\rm L}^s$. For N odd, for instance, it reads

$$G_Q \equiv G_Q(p,N) = \tfrac{1}{2}((1-p)^N - 1/\sqrt{2^{N-1}}). \tag{13}$$

From this, we can see that, for $p = 0$, G_Q grows monotonically and exponentially with N, converging to the maximal value 1/2 in the limit of $N \to \infty$. For $p > 0$, however, both terms

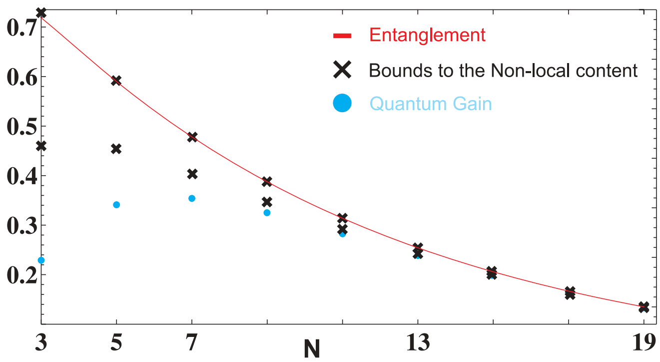

FIG. 4. (Color online) Quantum gain $G_Q(p,N)$—for odd N—(blue circles) in the efficiency to solve CCPs with $\varrho(p)$, lower and upper bounds—also for odd N—to the nonlocal content $\tilde{p}_{\rm NL}$ in $\varrho(p)$ (black crosses) and negativity (solid red) in the most robust bipartitions of $\varrho(p)$, for $p = 0.1$ and as a function of N. For $3 \leqslant N \leqslant 7$ the quantum gain displays a region of growth. However, for large N, it very rapidly converges to the universal upper bound $(1-p)^N$, and so do all other curves.

in Eq. (13) decay exponentially and their difference displays a nonmonotonic behavior, with a transition from growing to decaying with N at

$$N \approx \frac{\log_2[\sqrt{2}\ln(1/\sqrt{2})/\ln(1-p)]}{\log_2[\sqrt{2}(1-p)]}. \qquad (14)$$

Quantum gain (13) is plotted in Fig. 4 as a function of N for $p = 0.1$. Together with it, in the figure, also the lower [26] and upper bounds—for N odd—to the nonlocal content in $\varrho(p)$, as well as the negativity in its half-versus-half bipartitions, are plotted. These bipartitions are the most robust ones and the disappearance (red upper curve of Fig. 2) of their negativity characterizes the full separability of $\varrho(p)$ [11]. Therefore, the plotted negativity can be taken as a valid quantitative figure of merit for the total entanglement of $\varrho(p)$. In the figure, we can see how, after a short region of growth, from $N = 3$ to $N = 7$, G_Q becomes a decreasing function of N. Furthermore, we can see how it rapidly converges to the universal upper bound $(1-p)^N$, as well as all other curves in the figure for large N. This constitutes our final result.

Result 3. The usefulness for the nonlocal task associated with the Mermin inequality of locally depolarized, or dephased, states (4) decays, for large N, exponentially with N.

So an exponentially large visibility renders, apart from coexisting already with small entanglement and nonlocal content, an exponentially small usefulness for solving the associated physical problem. We emphasize that this remarkable anomaly can happen only in a noisy scenario, as for pure states a monotonically growing visibility implies a monotonically growing gain. Nevertheless, it constitutes a further confirmation of the fact that, as discussed below Result 2, at present day we still do not possess fully satisfactory tools for the unambiguous quantification of the entanglement or nonlocal resources of quantum states.

IV. CONCLUSION

In this work we have analyzed how local noisy environments affect the nonlocal properties of N-qubit states. Interestingly, the derived picture is more complex than initially expected, as there are regimes where, although entanglement and nonlocal content show an exponential decay with the number of parties, the violation of some Bell inequalities exponentially increase with N. This improvement in the Bell violation may in fact be the reason for the existence of a regime of N in which decaying entanglement and nonlocal content coexists with an increasing efficiency to solve probabilistic distributed computing tasks. Our results, then, provide a manifestation of the subtle relation between entanglement and nonlocality in quantum states. We hope that our work opens new perspectives in the study of nonlocality decay under decoherence. In particular, it would be interesting to study how the recent proposals for robust encoding of N-qubit entanglement introduced in Refs. [27,28] apply to nonlocality.

ACKNOWLEDGMENTS

This work is supported by the the EU Q-Essence project, the ERC Starting Grant PERCENT, the Chist-Era DIQIP project, the Spanish FIS2010-14830 project and Juan de la Cierva foundation, the Brazilian agency FAPERJ, and the National Institute of Science and Technology for Quantum Information, the National Research Foundation and the Ministry of Education of Singapore.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 2 (2009).
[2] J. S. Bell, Physics **1**, 195 (1964).
[3] N. Gisin, Phys. Lett. A **154**, 201 (1991); S. Popescu and D. Rohrlich, *ibid.* **166**, 293 (1992).
[4] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[5] J. Barrett, Phys. Rev. A **65**, 042302 (2002); A. Acín, N. Gisin, and B. Toner, *ibid.* **73**, 062105 (2006).
[6] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Phys. Rev. Lett. **98**, 230501 (2007); J. Barrett, L. Hardy, and A. Kent, *ibid.* **95**, 010503 (2005); A. Acín, N. Gisin, and L. Masanes, *ibid.* **97**, 120405 (2006).
[7] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Nature (London) **464**, 1021 (2010); R. Colbeck, Ph.D. thesis, University of Cambridge, 2006; R. Colbeck and A. Kent, J. Phys. A **44**, 095305 (2011).
[8] See H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. **82**, 665 (2010), and references therein.
[9] C. Brukner, M. Zukowski, J. W. Pan, and A. Zeilinger, Phys. Rev. Lett. **92**, 127901 (2004).
[10] C. Simon and J. Kempe, Phys. Rev. A **65**, 052327 (2002); A. R. R. Carvalho, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. **93**, 230501 (2004); W. Dür and H.-J. Briegel, *ibid.* **92**, 180403

(2004); M. Hein, W. Dür, and H.-J. Briegel, Phys. Rev. A **71**, 032350 (2005); O. Gühne, F. Bodoky, and M. Blaauboer, *ibid.* **78**, 060301 (2008); H. Wunderlich, S. Virmani, and M. B. Plenio, New J. Phys. **12**, 083026 (2010); R. Chaves and L. Davidovich, Phys. Rev. A **82**, 052308 (2010).

[11] L. Aolita, R. Chaves, D. Cavalcanti, A. Acín, and L. Davidovich, Phys. Rev. Lett. **100**, 080501 (2008); L. Aolita, D. Cavalcanti, A. Acín, A. Salles, Markus Tiersch, A. Buchleitner, and F. de Melo, Phys. Rev. A **79**, 032322 (2009).

[12] D. Cavalcanti, R. Chaves, L. Aolita, L. Davidovich, and A. Acín, Phys. Rev. Lett. **103**, 030502 (2009); L. Aolita, D. Cavalcanti, R. Chaves, C. Dhara, L. Davidovich, and A. Acín, Phys. Rev. A **82**, 032317 (2010).

[13] D. Cavalcanti, L. Aolita, A. Ferraro, A. García-Saez, and A. Acín, New J. Phys. **12**, 025011 (2010).

[14] W. Laskowski, T. Paterek, C. Brukner, and M. Zukowski, Phys. Rev. A **81**, 042101 (2010).

[15] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989).

[16] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[17] A. Acín, T. Durt, N. Gisin, and J. I. Latorre, Phys. Rev. A **65**, 052325 (2002); A. A. Méthot and V. Scarani, Quantum Inf. Comput. **7**, 157 (2007).

[18] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).

[19] A. C. Elitzur, S. Popescu, and D. Rohrlich, Phys. Lett. A **162**, 25 (1992).

[20] J. Barrett, A. Kent, and S. Pironio, Phys. Rev. Lett. **97**, 170409 (2006).

[21] L. Aolita, R. Gallego, A. Cabello and A. Acín, Phys. Rev. Lett. **108**, 100401 (2012).

[22] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[23] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **200**, 340 (1995).

[24] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).

[25] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[26] In general, the exact value of $\tilde{p}_{\mathrm{NL}}$ for a quantum state is the weight of the nonlocal parts of the optimal EPR2 decompositions maximized over all possible measurements on the state. While for a given setting, the optimal nonlocal weight can be efficiently obtained via linear programming, the measurement optimization is a hard computational task. For states(4) of $N = 3$ and 5, we maximized the nonlocal content restricted to the correlations over all projective measurements with two settings and two outcomes, and assuming the same measurement settings for each part. A perfect coincidence, up to numerical precision, with the lower bound was observed.

[27] F. Fröwis and W. Dür, Phys. Rev. Lett. **106**, 110402 (2011).

[28] R. Chaves, L. Aolita, and A. Acín, arXiv:1112.2645.