# Optimal measurement for quantum discord of two-qubit states

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We present a complete treatment for the quantum discord of two-qubit $X$ states, by developing a geometric picture of a quantum steering ellipsoid. It is shown that either a von Neumann measurement or a three-element positive-operator-valued measurement is optimal. The condition for the latter is obtained and expressed in geometric language. We show, by using analytical as well as numerical results, that there is a systematic structure in the optimal decomposition which exists in a class of states including the $X$ states. More significantly, we establish the relation to the quantum channel by identifying the steering ellipsoid with the quantum channel ellipsoid. Thus the quantum discord and classical correlation are closely related to the concept of the entanglement entropy of the quantum channel.

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*Introduction.* Quantum discord (QD) is regarded as a measure of the quantumness of correlation, even in the absence of quantum entanglement [1]. Many works have been devoted to the significance and application of QD [2]. However, very few analytic results for QD are known [3–6]. Although much effort has been made [7–10], there is no exact expression for QD for general two-qubit states, even for the $X$ states (i.e., states such that the nonzero elements of the density matrix lie only along the diagonal or antidiagonal in the product basis). The main obstacle lies in the complicated optimization procedure involving not only projective but also generalized measurements.

We present in this work a complete treatment of the QD of two-qubit $X$ states. We focus on the induced decomposition of the reduced state of qubit $B$ (say) rather than the measurements on qubit $A$. Using the geometric picture of the quantum steering ellipsoid (QSE) [11,12], we show that the optimal decomposition contains at most three states, and determine the conditions under which the optimal decomposition has two or three components. The QD of two-qubit $X$ states is then obtained without resorting to any numerical optimization. However, the condition for the optimal three-state decomposition involves a transcendental equation, which prevents us from getting an analytical result.

What is interesting is the systematic structure existing in the optimal decomposition not only for $X$ states but also for a slightly more general class of states. We introduce the concept of the *invariant set of optimal components* (ISOC) to characterize the fact that the set of optimal components for a specific state remains optimal for a class of states. Using analytical as well as numerical results, we demonstrate the "phase" diagrams illustrating the optimal decomposition. More importantly, we recognize that the QSE is identical to the quantum channel ellipsoid. Then the relation to the entanglement entropy of the quantum channel is established. This result provides a deeper insight into the nature of the QD as well as classical correlation.

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*Quantum discord and steering ellipsoid.* Suppose that a positive-operator-valued measure (POVM) measurement $\mathcal{M}$ is performed on particle $A$ of a bipartite system in the state $\rho$, where $\mathcal{M} = \{M_i\}$ with $M_i \geqslant 0$ and $\sum_i M_i = \mathbb{1}$. The measurement $\mathcal{M}$ induces the ensemble $\{p_i, \rho_{B|M_i}\}$ for $\rho_B$, where $p_i = \mathrm{Tr}[(M_i \otimes \mathbb{1})\rho]$, $\rho_{B|M_i} = \mathrm{Tr}_A[(M_i \otimes \mathbb{1})\rho]/p_i$, and $\rho_{A(B)} = \mathrm{Tr}_{B(A)}\,\rho$. The QD of $\rho$ is defined as $\mathcal{Q} = S(\rho_A) + \min_{\mathcal{M}} S(\rho_B|\mathcal{M}) - S(\rho)$, where $S$ is the von Neumann entropy, $S(\rho_B|\mathcal{M}) = \sum_i p_i S(\rho_{B|M_i})$, and the minimization is performed over all POVMs.

For the two-qubit state $\rho$, the QSE is a useful tool. Denoted by $\mathfrak{E}$, QSE is the three-dimensional ellipsoidal region in which the Bloch vectors of $\rho_{B|M_i}$ are distributed. The ellipsoid $\mathfrak{E}$ is determined by $\boldsymbol{x}^T R^{-1} \eta R^{-T} \boldsymbol{x} \geqslant 0$, where $\eta = \mathrm{diag}(1,-1,-1,-1)$, $\boldsymbol{x}^T = (1,x,y,z)$ is a four-vector in row form (the superscript $T$ denotes matrix transposition), and $R$ is a matrix with entries $R_{\mu\nu} = \mathrm{Tr}[\rho(\sigma_\mu \otimes \sigma_\nu)]$ for $\mu,\nu = 0,1,2,3$. Here $\sigma_0$ is the identity matrix and $\sigma_i$ ($i = 1,2,3$) the Pauli matrices. When $\mathcal{M}$ is of rank 1, all states $\rho_{B|M_i}$ lie on the surface of $\mathfrak{E}$, denoted by $\partial\mathfrak{E}$ [12]. It is shown in Refs. [13,14] that the optimal $\mathcal{M}$ must be of rank 1. So, in geometric language, the optimal decomposition must be attained on $\partial\mathfrak{E}$.

We now turn to the $X$ states. An $X$ state is expressed as $\mathcal{X} = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + a\,\sigma_3\otimes\mathbb{1} + b\,\mathbb{1}\otimes\sigma_3 + \sum_{i=1}^{3} t_i\,\sigma_i\otimes\sigma_i)$. Both reduced states $\mathcal{X}_A$ and $\mathcal{X}_B$ take the diagonal form, so all coherence applies jointly to the two qubits, and not to either separately. The class of $X$ states is actually not unusual and arises naturally in a wide variety of physical situations, e.g., entanglement sudden death [15], and the relation between quantum correlations and quantum phase transitions [16].

The QSE of the state $\mathcal{X}$ is given by $x^2/\lambda_1^2 + y^2/\lambda_2^2 + (z-\kappa)^2/\lambda_3^2 = 1$, where $\kappa = (b - at_3)/(1-a^2)$, $\lambda_1 = t_1/\sqrt{1-a^2}$, $\lambda_2 = -t_2/\sqrt{1-a^2}$, and $\lambda_3 = (t_3 - ab)/(1-a^2)$. The center of $\mathfrak{E}$ is the point $(0,0,\kappa)$. Let $P$ and $Q$ denote the upper and the lower vertices of $\mathfrak{E}$. Their $z$ coordinates are $z_P = \kappa + |\lambda_3|$ and $z_Q = \kappa - |\lambda_3|$, respectively.

For the sake of convenience of analysis, we let $|\lambda_1| \geqslant |\lambda_2|$, $\kappa \geqslant 0$. These conditions can be satisfied by performing local unitary operations. Henceforth these conditions are assumed. In this setting, the center of $\mathfrak{E}$ is on the upper half of the $z$ axis (containing the origin), and the largest vertical intersection is

 

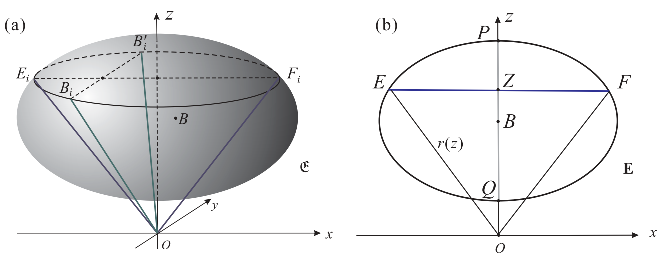


FIG. 1. (Color online) QSE $\mathfrak{E}$ and the largest vertical intersection **E**. (a) is used to prove Lemma 1. See [17] for details. In (b), the chord $EF$ (in blue) and $PQ$ (in gray) denote, respectively, the horizontal and vertical decompositions for point $Z$.

in the $x$-$z$ plane. Let **E** denote the boundary of the intersection. The equation of **E** is $x^2/\lambda_1^2 + (z-\kappa)^2/\lambda_3^2 = 1$. Let point $B = (0,0,b)$ denote the reduced state $\mathcal{X}_B = \frac{1}{2}(\mathbb{1} + b\sigma_3)$. Point $B$ is inside **E**. See Fig. 1. We are concerned with the decomposition of the point $B$ into the convex combination of the points $B_i \in \partial\mathfrak{E}$. The following lemma allows us to greatly simplify the optimization procedure. The proof is given in Ref. [17].

*Lemma 1*. For any point inside **E**, the optimal decomposition is attained on **E**, and has at most three components.

*Properties of* **E**. The ellipse **E** is then our focus of attention. We will analyze its properties by contracting **E** horizontally while keeping the upper and lower vertices unchanged. That is, we fix $z_P$ and $z_Q$, and decrease $|\lambda_1|$ from the maximal value $\lambda_{\max}$ to zero [18].

To proceed, we define some functions. Let $Z = (0,0,z)$ be some point on the $z$ axis and $EF$ the horizontal chord of **E** passing through $Z$ [see Fig. 1(b)]. The distance from the origin $O$ to the point $E$ or $F$ is $r(z) = [\lambda_1^2 - (z-\kappa)^2\lambda_1^2/\lambda_3^2 + z^2]^{1/2}$ for $z \in [z_Q, z_P]$. Define

$$H(z) = \xi(r(z)), \tag{1}$$

$$V(z) = \frac{z - z_Q}{z_P - z_Q}\xi(z_P) + \frac{z_P - z}{z_P - z_Q}\xi(z_Q), \tag{2}$$

where $\xi(x) = -\frac{1+x}{2}\log_2\frac{1+x}{2} - \frac{1-x}{2}\log_2\frac{1-x}{2}$ for $x \in [-1,1]$. The functions $H(z)$ and $V(z)$ are in fact the average entropy over the horizontal and vertical decompositions of point $Z$, i.e., $Z \to \{E,F\}$ and $Z \to \{P,Q\}$, respectively. When $\mathfrak{E}$ is contracted, some specific values of $|\lambda_1|$ are important, which we define as follows.

When $\lambda_1^2 = \lambda_P^2$, the second derivative of $H(z)$ with respect to $z$ vanishes at $z = z_P$. We similarly define $\lambda_Q^2$ at $z = z_Q$. When $\lambda_1^2 = \lambda_T^2$, the tangent line of $H(z)$ at $z = z_Q$ is just the line that connects the two end points $(z_P, H(z_P))$ and $(z_Q, H(z_Q))$. The expressions for $\lambda_P^2$, $\lambda_Q^2$, and $\lambda_T^2$ are given at the end of the paper. We also define $z^\star$ as the solution of the equation

$$\frac{dH(z)}{dz} = \frac{H(z) - H(z_P)}{z - z_P}. \tag{3}$$

Let $E^\star F^\star$ denote the horizontal chord of **E** passing through the point $(0,0,z^\star)$.

The function $H(z)$ has the following properties. (i) $\lambda_Q^2 < \lambda_T^2 < \lambda_P^2$. (ii) $H(z)$ is concave with respect to $z$ for $\lambda_1^2 \in [0,\lambda_Q^2]$, and convex for $\lambda_1^2 \in [\lambda_P^2, \lambda_{\max}^2]$. (iii) $H(z)$ has a unique inflection point for $\lambda_1^2 \in [\lambda_Q^2, \lambda_P^2]$. (iv) $H(z) \geqslant V(z)$ for $\lambda_1^2 \in [0,\lambda_T^2]$. (v) $z^\star$ exists for $\lambda_1^2 \in [\lambda_T^2, \lambda_P^2]$. These results can be verified by directly analyzing $H(z)$ and its derivatives. But the proof is too technical to be described here. A schematic plot of $H(z)$ is shown in Fig. 2(a).

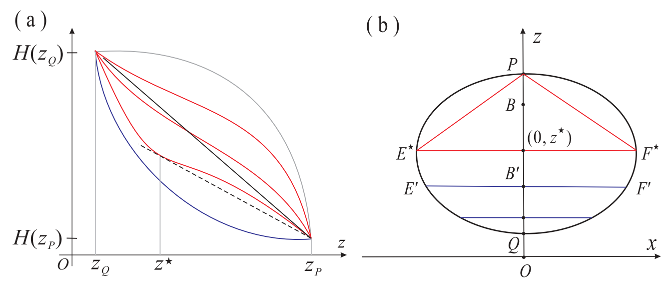


FIG. 2. (Color online) (a) Schematic plot of the function $H(z)$. The five curves (from bottom to top) correspond to five different values of $|\lambda_1|$ in decreasing order. The bottom (blue) curve is convex, and the top (gray) is concave. Each of the three middle (red) curves has only one inflection point. The solid line denotes the function $V(z)$. (b) In $\mathbf{E}_\triangle$, the decomposition $B \to \{P,E^\star,F^\star\}$ is optimal for any point $B$ with $b \in [z^\star, z_P]$, while for $b \in [z_Q, z^\star]$ the horizontal decomposition is optimal.

*Optimal decomposition for X states*. We have the following proposition.

*Proposition 1*. There are three cases.

(a) If $\lambda_1^2 \in [\lambda_P^2, \lambda_{\max}^2]$, then $\min S(\mathcal{X}_B|\mathcal{M}) = H(b)$. Horizontal decomposition is optimal; it is induced by performing on qubit $A$ the measurement $\mathcal{M} = \{|x_+\rangle\langle x_+|, |x_-\rangle\langle x_-|\}$ with $\sigma_x|x_\pm\rangle = \pm|x_\pm\rangle$.

(b) If $\lambda_1^2 \in [0,\lambda_T^2]$, then $\min S(\mathcal{X}_B|\mathcal{M}) = V(b)$. Vertical decomposition is optimal; it comes from the measurement $\mathcal{M} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

(c1) If $\lambda_1^2 \in [\lambda_T^2, \lambda_P^2]$ and $b \in [z_Q, z^\star]$, the horizontal decomposition is optimal and $\min S(\mathcal{X}_B|\mathcal{M}) = H(b)$.

(c2) If $\lambda_1^2 \in [\lambda_T^2, \lambda_P^2]$ and $b \in [z^\star, z_P]$, then $\min S(\mathcal{X}_B|\mathcal{M}) = p^\star H(z^\star) + (1-p^\star)H(z_P)$ with $p^\star = (z_P - b)/(z_P - z^\star)$. The optimal decomposition is $B \to \{P, E^\star, F^\star\}$ for which a three-element POVM is needed.

We denote by $\mathbf{E}_\leftrightarrow$, $\mathbf{E}_\updownarrow$, and $\mathbf{E}_\triangle$ the ellipses corresponding to cases (a), (b), and (c), respectively. The QD of $X$ states is obtained by examining the type of $\mathfrak{E}$, in particular the type of **E**, as well as Eq. (3). Unfortunately Eq. (3) can only be solved numerically.

Let us prove Proposition 1. Consider the decomposition $B \to \{p_i, B_i\}$ with $B_i \in \mathbf{E}$. The average entropy is given by $\sum_i p_i H(z_i)$ with $z_i$ the $z$ coordinate of point $B_i$. If $H(z)$ is convex, we have $H(b) \leqslant \sum_i p_i H(z_i)$ for $b = \sum_i p_i z_i$, meaning that the horizontal decomposition is optimal. If $H(z)$ is larger than $V(z)$, then for any combination $\sum_i p_i z_i = b$, we have $\sum_i p_i H(z_i) \geqslant \sum_i p_i V(z_i) = V(b)$. Thus the vertical decomposition is optimal. Cases (a) and (b) are proved.

For case (c), we consider the tangent line of $H(z)$ passing through the end point $(z_P, H(z_P))$, i.e., the dashed line in Fig. 2(a). The tangent point is determined by Eq. (3). According to the property (v), this tangent line exists when $\lambda_1^2 \in [\lambda_T^2, \lambda_P^2]$. In this case, if $b \in [z^\star, z_P]$, the tangent line determines the vale of $\min S(\mathcal{X}_B|\mathcal{M})$, that is, $p^\star H(z^\star) + (1-p^\star)H(z_P)$. The minimum is attained on the three-state decomposition $B \to \{P, E^\star, F^\star\}$. If $b \in [z_Q, z^\star]$, the optimal decomposition of $B$ is still the horizontal one, as the function $H(z)$ remains convex in the interval $[z_Q, z^\star]$. Case (c) is proved. We emphasize that the three-state decomposition is optimal for any point $B$ with $b \in (z^\star, z_P)$. The set $\{P, E^\star, F^\star\}$ is the ISOC for any such point.

Function $H(z)$ has another property: If $\kappa = 0$ then $H(z)$ is either convex (when $|\lambda_1| > |\lambda_3|$) or concave (when $|\lambda_1| < |\lambda_3|$). Then we have the following.

*Corollary 1*. If the center of $\mathbf{E}$ is at the origin, the optimal decomposition is horizontal for an oblate ellipse (i.e., $|\lambda_1| > |\lambda_3|$), and vertical for a prolate ellipse (i.e., $|\lambda_1| < |\lambda_3|$).

We conclude the discussion of $X$ states with some remarks regarding case (c2). First, this class of states has a particular significance in the dynamics of the QD. Suppose that the initial state is such that the ellipse $\mathbf{E}$ is the largest one, i.e., $|\lambda_1| = \lambda_{\max}$. Let both qubits experience the identical local phase-damping noise. Under the influence of the noise, the QSE is contracted towards the $z$ axis, while the upper and lower vertices remain fixed. The ellipse $\mathbf{E}$ is of the type $\mathbf{E}_\leftrightarrow$ at first and of the type $\mathbf{E}_\updownarrow$ in the end. If $\kappa = 0$, then from Corollary 1 an ellipse of the type $\mathbf{E}_\triangle$ does not appear in the whole evolution. The change from $\mathbf{E}_\leftrightarrow$ to $\mathbf{E}_\updownarrow$ occurs abruptly, which leads to a sudden transition between classical and quantum decoherence [19]. If $\kappa \neq 0$, an ellipse of the type $\mathbf{E}_\triangle$ will appear in the decoherence process. Consequently, QD and classical correlation change smoothly and there is no sudden transition.

Second, consider a typical class of states belonging to case (c2): the $X$ states such that $H(b) = V(b)$. It can be verified that this class of states appears on the boundary curves illustrating the relationships between QD and other quantities, such as classical correlation [20], linear entropy [21], and von Neumann entropy [22].

*Phase diagrams of optimal decomposition*. Now we extend the results for $X$ states to a more general case: Point $B$ may be moved to any position inside $\mathbf{E}$, in which case the corresponding state may not take the $X$ form.

*Proposition 2*. For any point inside $\mathbf{E}_\leftrightarrow$, the horizontal decomposition is optimal. For any point inside $\mathbf{E}_\triangle$ and below the chord $E^\star F^\star$, the horizontal decomposition is optimal. For any point inside the triangle $\triangle P E^\star F^\star$ in $\mathbf{E}_\triangle$, the optimal decomposition is $\{P, E^\star, F^\star\}$.

This proposition reveals the systematic structure of the optimal decomposition. To prove it, we first consider an ellipse of the type $\mathbf{E}_\triangle$. Suppose that a point $B$ inside $\triangle P E^\star F^\star$ is decomposed into $\{p_i, B_i\}$ with $B_i \in \mathbf{E}$. By rotating the points $B$ and $B_i$ around the $z$ axis by 180°, we get the symmetric points $B'$ and $B'_i$. The following three ensembles yield the same average entropy: $\{p_i, B_i\}$, $\{p_i, B'_i\}$, and $\{\frac{p_i}{2}, B_i; \frac{p_i}{2}, B'_i\}$. Note that the average over the last ensemble gives the midpoint of the line segment $BB'$, for which the optimal decomposition is $\{P, E^\star, F^\star\}$ according to Proposition 1. Hence the optimal decomposition of $B$ is also given by $\{P, E^\star, F^\star\}$. Similar reasoning leads to the other results stated in Proposition 2.

The cases which are not covered by Proposition 2 concern the points inside the ellipse $\mathbf{E}_\updownarrow$ and the points in the region above $\triangle P E^\star F^\star$ in $\mathbf{E}_\triangle$. These cases are illustrated by gray regions in Fig. 3. We cannot provide a rigorous treatment for such points. By examining numerically about $10^5$ states we find that for such a point $B''$ the two-component decomposition $B'' \to \{B''_1, B''_2\}$, is optimal. And the set $\{B''_1, B''_2\}$ is the ISOC for any point on the chord $B''_1 B''_2$. Hence we obtain the phase diagram depicted in Fig. 3.

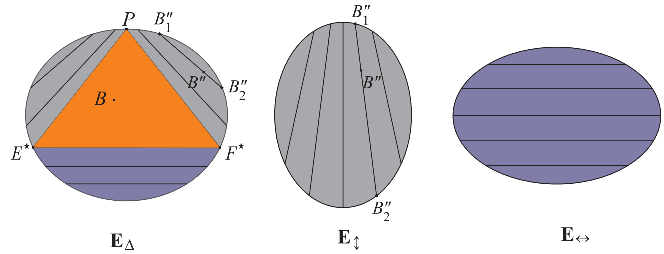


FIG. 3. (Color online) Phase diagram of the optimal decomposition. The triangle $\triangle P E^\star F^\star$ is in red, the region below $\triangle P E^\star F^\star$ and the region inside $\mathbf{E}_\leftrightarrow$ are in blue, and the region above $\triangle P E^\star F^\star$ and the region inside $\mathbf{E}_\updownarrow$ are in gray. For any point in the red, blue, or gray regions, the optimal decomposition is, respectively, three-state, horizontal, or tilted decomposition.

*Relation to quantum channel*. We will show below the fundamental significance of the QSE: It is identical to the quantum channel ellipsoid. For the details of the quantum channel the reader is referred to Refs. [23,24].

Given a two-qubit state $\rho$, the reduced state $\rho_A$ is in general invertible. Let $\tilde{\rho} = [(2\rho_A)^{-1/2} \otimes \mathbb{1}]\rho[(2\rho_A)^{-1/2} \otimes \mathbb{1}]$. There is an isomorphism between $\tilde{\rho}$ and a one-qubit quantum channel $\Phi$, i.e., $\tilde{\rho} = (\mathrm{id} \otimes \Phi)\beta$, where id denotes the identity transformation and $\beta = |\beta\rangle\langle\beta|$ with $|\beta\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. In terms of the Bloch vectors, the effect of $\Phi$ can be expressed as a $4 \times 4$ matrix $L_\Phi$ [25]. We have shown in Ref. [5] that the QSE can be expressed equivalently as $\boldsymbol{x}^T L_\Phi^{-T} \eta L_\Phi^{-1} \boldsymbol{x} \geqslant 0$. On the other hand, the channel $\Phi$ transforms the Bloch ball to an ellipsoid, which we call the *quantum channel ellipsoid*. In terms of $L_\Phi$, a point on the Bloch sphere, denoted by the unit vector $\vec{u}$, is mapped to the point $\vec{x}$ such that $\boldsymbol{x} = L_\Phi \boldsymbol{u}$ with $\boldsymbol{x} = (1, \vec{x})$ and $\boldsymbol{u} = (1, \vec{u})$. The surface of the channel ellipsoid is determined by $1 - \vec{u} \cdot \vec{u} = \boldsymbol{u}^T \eta \boldsymbol{u} = 0$, that is, $\boldsymbol{x}^T L_\Phi^{-T} \eta L_\Phi^{-1} \boldsymbol{x} = 0$. Thus we see that the QSE is identical to the channel ellipsoid.

Define the entanglement entropy of a one-qubit quantum channel $\Phi$ with the input state $\varrho$ as $E_\Phi(\varrho) = \min \sum p_j S(\Phi(\pi_j))$, where the minimum is taken over all convex decompositions of the state $\varrho$ into pure states, i.e., $\varrho = \sum p_j \pi_j$ with $\pi_j$ pure [26]. We have the following.

*Proposition 3*. Given a two-qubit state $\rho$, let $\Phi$ be the quantum channel isomorphic to $\tilde{\rho}$. We have that $\min_{\mathcal{M}} S(\rho_B|\mathcal{M}) = E_\Phi(\rho_A^T)$.

We first note that $\Phi(\rho_A^T) = \rho_B$, which can be proved by direct calculation. Also note that $E_\Phi(\rho_A^T) = \min \sum p_j S(\Phi(\pi_j^T)) = \min \sum p_j S(\rho_{B_j})$, where $\sum p_j \pi_j = \rho_A$, $\rho_{B_j} = \Phi(\pi_j^T)$, and $\sum p_j \rho_{B_j} = \rho_B$. All $\rho_{B_j}$ lie on the surface of the channel ellipsoid (or QSE). Then the minimum of $\sum p_j S(\rho_{B_j})$ is equal to the minimum of $S(\rho_B|\mathcal{M})$, and the proof is completed.

The results obtained about the optimal decomposition can be directly applied to find the entanglement entropy of the quantum channel. Considering the axially symmetric channel discussed in Ref. [26], a more rigorous treatment is available in our approach: The optimal output ensemble is determined by the properties of **E**, which we have characterized in Proposition 1 by an analytical rather than a numerical method.

*Summary*. We provide a complete treatment of the QD of two-qubit $X$ states. The results are generalized to a slightly more general class of two-qubit states. Using analytical and numerical methods, we find the systematic structure of the optimal decomposition, i.e., the existence of the ISOC. Moreover, we identify the QSE with the quantum channel ellipsoid, and show that the entanglement entropy of the quantum channel is the core of the QD.

*Expressions for $\lambda_P^2$, $\lambda_Q^2$, and $\lambda_T^2$*. By solving the equations $d^2H(z)/dz^2\big|_{z=z_P} = 0$ and $dH(z)/dz\big|_{z=z_Q} = [H(z_P) - H(z_Q)]/(z_P - z_Q)$ with respect to $\lambda_1^2$, we have, respectively,

$$\lambda_P^2 = \frac{z_P}{2\left[z_P + \left(z_P^2 - 1\right)\operatorname{arctanh}(z_P)\right]}\Big[z_P^2 - z_P z_Q + \left(1 - z_P^2\right) z_Q \operatorname{arctanh}(z_P) - \sqrt{\left(1 - z_P^2\right)\operatorname{arctanh}(z_P)\left[z_P^3 - z_P z_Q^2 + \left(1 - z_P^2\right) z_Q^2 \operatorname{arctanh}(z_P)\right]}\Big], \tag{4}$$

$$\lambda_T^2 = z_Q \frac{(1+z_P)\log_2(1+z_P) + (1-z_P)\log_2(1-z_P) - (1+z_P)\log_2(1+z_Q) - (1-z_P)\log_2(1-z_Q)}{2[\log_2(1+z_Q) - \log_2(1-z_Q)]}. \tag{5}$$

The expression for $\lambda_Q^2$ can be obtained by interchanging $z_P$ and $z_Q$ in Eq. (4). Also note that (4) and (5) were derived in Ref. [26] in an alternative way.

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[17] Suppose that a point $B$ inside **E** is decomposed into the ensemble $\{p_i, B_i\}$ with all $B_i \in \partial\mathfrak{E}$. Mirror reflection about the $x$-$z$ plane gives the image points $B_i'$. See Fig. 1(a). The three ensembles $\{p_i, B_i\}$, $\{p_i, B_i'\}$, and $\{p_i/2, B_i; p_i/2, B_i'\}$ yield the same average entropy. Imagine that $\mathfrak{E}$ is cut by a horizontal plane containing the points $B_i$ and $B_i'$. The intersection is an ellipse and its major axis $E_i F_i$ lies in the $x$-$z$ plane. A new ensemble for $B$ can be constructed as $\{p_i q_i, E_i; p_i(1-q_i), F_i\}$ with $q_i \in (0,1)$. The average entropy over the new ensemble is no greater than that over $\{p_i, B_i\}$, as the length of $OE_i$ (or $OF_i$) is no smaller than that of $OB_i$. So we need only consider the points on **E**. It follows from the Carathéodory theorem [27] that at most three points on **E** are needed to realize the optimal decomposition.

[18] This process can be realized by applying a phase-damping channel on both qubits (e.g., see [12]). It can be seen that $\lambda_{\max} = \frac{1}{2}[\sqrt{(1+z_P)(1-z_Q)} + \sqrt{(1-z_P)(1+z_Q)}]$, which corresponds to the ellipse touching the Bloch sphere .

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[25] Let $\vec{r}$ and $\vec{r}'$ be the Bloch vectors of the one-qubit state $\varrho$ and the image $\varrho' = \Phi(\varrho)$, respectively. Define the $4 \times 4$ matrix $L_\Phi = \left(\begin{smallmatrix} 1 & \vec{0} \\ \vec{\kappa} & \Lambda \end{smallmatrix}\right)$, with $\Lambda$ being a real $3 \times 3$ matrix (and $\vec{0}$ and $\vec{\kappa}$ the row and column vectors, respectively). The effect of $\Phi$ can be expressed equivalently as $\boldsymbol{r} \to \boldsymbol{r}' = L_\Phi \boldsymbol{r}$ with $\boldsymbol{r} = (1,\vec{r})$, $\boldsymbol{r}' = (1,\vec{r}')$. See [24] for details.
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