Critical frequency control for arbitrarily slow decoherence of a qubit

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(Received 2 February 2012; published 8 June 2012)

The decoherence of a qubit is studied for a general class of spectral densities with an arbitrary band gap. The exact dynamics reveals the existence of critical values of the qubit transition frequency, such that relaxations of coherence show discontinuities, jumping from decays with powers arbitrarily larger than unity to powers arbitrarily larger than 0. The critical frequency is the sum of the band-gap frequency and the first negative moment of the spectral density. In the superohmic case, coherence is not entirely lost and the ratio of the trapped population approaches unity as the second negative moment of the spectral density becomes negligible with respect to the scale frequency. Arbitrarily slow decoherence processes are obtained in the critical configuration by approaching the boundary between subohmic and superohmic regimes. The setup of critical configurations constitutes a different way to control qubit decoherence.

DOI: 10.1103/PhysRevA.85.062107 PACS number(s): 03.65.Yz, 03.65.Ta

I. INTRODUCTION

In open quantum systems, the external environment is usually described by a discrete or continuous distribution of bosonic or fermionic modes [1,2]. The Jaynes-Cummings model (JCM) mimics the simplest quantum interaction between radiation and matter: a two-level system linearly coupled to a single frequency mode [3]. An excitation of the system corresponds to a loss of a photon from the radiation field, and, viceversa, the transition to the ground level provokes an emission of a photon [4].

The extension of the JCM to a discrete distribution of bosonic modes has been widely used in quantum optics, magnetic resonance, circuit QED, and quantum computing, to name few [5]. As matter of fact, oscillatory behaviors and dressed states emerge in the spontaneous emission of a two-level atom coupled to the radiation field in a three-dimensional periodic dielectric [6]. The further extension to a continuous distribution of field modes (reservoir) provides an efficient theoretical model for a two-level atom placed inside a cavity [7]. The internal atomic dynamics can be described through the theoretical construct of pseudomodes, finding exponential-like relaxation and trapping [8]. The spontaneous and induced decay of a two-level atom weakly coupled to a continuous distribution of frequency modes with a band-gap structure has been analyzed in detail [9].

Another standard model for a two-level system interacting with a bath of frequency modes is the spin-boson model, whose dynamics is usually described via the spectral density [1,10]. Usually, continuous spectral densities are shaped with power laws at low frequencies, subohmic, for positive powers less than unity, and superohmic for powers larger than unity. Commonly, the cutoff at high frequencies is chosen as an exponential or an abrupt vanishing decay. In the latter case, the adoption of the nonperturbative numerical renormalization group shows interesting properties of the dynamics, like a continuous quantum phase transition emerging in the subohmic condition with the absence of a single energy scale [11].

The search for feasible and efficient ways to control qubit decoherence induced by the interaction with the external environment remains a leading argument of research in quantum information processing [12]. Various methods have been proposed: quantum error-correction and error-prevention codes, ancillary qubits, and decoherence-free subspaces [13]. An alternative approach consists in the dynamical control of the qubit-environment interaction via external fields [14]. Recently, a theory for the optimal dynamical control of decoherence reduction, induced by an arbitrary thermal bath, has been proposed [15].

Open systems dynamics can also be controlled by manipulating the dissipative environment. Inhibition or enhancement of the decay of a two-level atom is obtained in the rotating wave approximation by the fast chirping of the reservoir frequency modes, keeping the gross reservoir structure unchanged. Similarly, moderate chirping can create partial trapping of the level population [16].

The aim of the present paper is to introduce a further scenario for the control of qubit decoherence, providing arbitrarily slow relaxations by setting the system in critical configurations. Such conditions are obtained by properly shaping the spectral density near an arbitrary band gap and considering qubit transition frequencies equal to the corresponding critical value.

II. THE MODEL

We consider a qubit (two-state system) interacting with a reservoir of bosons, at zero temperature, in the rotating wave approximation [2,7,16,17]. By choosing $\hbar = 1$, the Hamiltonian of the whole system is $H_S + H_E + H_{SE}$, where

$$H_S = \omega_0\, \sigma_+ \sigma_-, \quad H_E = \sum_{k=1}^{\infty} \omega_k\, a_k^\dagger a_k,$$

$$H_{SE} = \sum_{k=1}^{\infty} (g_k\, \sigma_+ \otimes a_k + g_k^*\, \sigma_- \otimes a_k^\dagger).$$

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The rising and lowering operators, σ_+ and σ_-, respectively, act on the Hilbert space of the qubit and are defined through the equalities $\sigma_+ = \sigma_-^\dagger = |1\rangle\langle 0|$, while $a_k^\dagger$ and a_k are the creation and annihilation operators, acting on the Hilbert space of the kth boson.

The coupling between the transition $|0\rangle \leftrightarrow |1\rangle$ and the kth bosonic mode is represented by the parameters g_k, while ω_0 is the qubit transition frequency. Initially, the qubit is unentangled from the vacuum state $|0\rangle_E$ of the reservoir,

$$|\Psi(0)\rangle = [c_0|0\rangle + c_1(0)|1\rangle] \otimes |0\rangle_E. \quad (1)$$

In this way, the exact time evolution reads

$$|\Psi(t)\rangle = c_0|0\rangle \otimes |0\rangle_E + c_1(t)|1\rangle \otimes |0\rangle_E + \sum_{k=1}^{\infty} b_k(t)|0\rangle \otimes |k\rangle_E,$$
$$|k\rangle_E = a_k^\dagger|0\rangle_E, \quad k = 0,1,2,\ldots,$$

due to the conservation of the number of excitations [16]. The dynamics is easily studied in the interaction picture,

$$|\Psi(t)\rangle_I = e^{i(H_S+H_E)t}|\Psi(t)\rangle = c_0|0\rangle \otimes |0\rangle_E + C_1(t)|1\rangle \otimes |0\rangle_E + \sum_{k=1}^{\infty} B_k(t)|0\rangle \otimes |k\rangle_E,$$

where i is the imaginary unity, $C_1(t) = e^{i\omega_0 t}\, c_1(t)$, and $B_k(t) = e^{i\omega_k t}\, b_k(t)$ for every $k = 1,2,\ldots$.

The Schrödinger equation for the total system leads to the system of equations

$$\dot{C}_1(t) = -i \sum_{k=1}^{\infty} g_k\, B_k(t)\, e^{-i(\omega_k-\omega_0)t},$$
$$\dot{B}_k(t) = -i\, g_k^*\, C_1(t)\, e^{i(\omega_k-\omega_0)t},$$

and the amplitude $\langle 1| \otimes {}_E\langle 0||\Psi(t)\rangle_I$, labeled as $C_1(t)$, fulfills the convolution equation

$$\dot{C}_1(t) = -(f * C_1)(t), \quad (2)$$

$$f(\tau) = \sum_{k=1}^{\infty} |g_k|^2 e^{-i(\omega_k-\omega_0)\tau}, \quad (3)$$

where the symbol $(*)$ denotes the convolution product, while f is the correlation function of the reservoir [2, 17]. A continuous distribution of modes, described by the density $\eta(\omega)$, gives $J(\omega) = \eta(\omega)\,|g(\omega)|^2$ as the spectral density, while the corresponding correlation function results in

$$f(\tau) = \int_0^\infty J(\omega)\, e^{-i(\omega-\omega_0)\tau} d\omega, \quad (4)$$

where $g(\omega)$ is the frequency dependent coupling.

The exact dynamics of the qubit is described by the time evolution of the reduced density matrix obtained by tracing over the Hilbert space of the bosons,

$$\rho_{1,1}(t) = 1 - \rho_{0,0}(t) = \rho_{1,1}(0)\, |G(t)|^2, \quad (5)$$

$$\rho_{1,0}(t) = \rho_{0,1}^*(t) = \rho_{1,0}(0)\, e^{-i\omega_0 t} G(t). \quad (6)$$

The function $G(t)$, fulfilling the convolution equation

$$\dot{G}(t) = -(f * G)(t), \qquad G(0) = 1, \quad (7)$$

drives the dynamics of the levels populations, the decoherence term, and the amplitude $c_1(t) = \langle 1| \otimes {}_E\langle 0||\Psi(t)\rangle$,

$$c_1(t) = c_1(0)\, e^{-i\omega_0 t} G(t). \quad (8)$$

III. THE CRITICAL FREQUENCY

We analyze the dynamics induced by reservoirs described by spectral densities with a general band-gap condition:

$$J(\omega) = \theta((\omega - \Omega_g)/\omega_s)\Lambda((\omega - \Omega_g)/\omega_s), \quad (9)$$

where $\theta(\nu)$ represents the Heavside step function, Ω_g is the band-gap frequency, and ω_s is a scale frequency. The nonnegative and summable function $\Lambda(\nu)$ describes the spectral density with the origin in the band gap. The class of spectral densities under study, labeled as $\Lambda_\alpha(\nu)$, is defined by requiring that the nonnegative and summable functions fulfill the constraints

$$\Lambda_\alpha(\nu) = O(\nu^{-1-\epsilon}), \quad \text{for } \nu \to +\infty, \quad \epsilon > 0, \quad (10)$$

$$\Lambda_\alpha(\nu) \sim \Omega_0\, \nu^\alpha + \sum_{n=1}^{\infty} \Omega_n\, \nu^{\alpha_n}, \quad \text{for } \nu \to 0^+, \quad (11)$$

$$-1 < \alpha < \alpha_1 < \alpha_2 < \alpha_3, \ldots, \quad \alpha \neq \lfloor \alpha \rfloor, \quad (12)$$

$$\alpha_n \neq \lfloor \alpha_n \rfloor, \quad \text{for } n = 1,2,\ldots, \quad \alpha_n \to +\infty, \quad \text{for } n \to +\infty, \quad (13)$$

where the symbol $\lfloor \cdot \rfloor$ denotes the integer part. The condition (10) means that at high frequencies $\Lambda_\alpha(\nu) \leqslant K\nu^{-1-\epsilon}$, where $K > 0$ and ϵ is a positive arbitrarily small parameter, and guarantees that the function $\Lambda(\nu)$ is summable. The constraint (11) describes various power law behaviors of the spectral density near the band-gap edge: subohmic for $0 < \alpha < 1$, and superohmic for $\alpha > 1$, while the case $0 > \alpha > -1$ shows divergencies near the band-gap frequency referring to structures describing photonic band-gap materials [1,10]. The additional conditions (12) and (13) state that the power series expansion in the band gap contains no integer powers and play a fundamental role in the determination of the asymptotic dynamics; see Appendix below for details.

The above class includes spectral densities that are the result of the product of ν^α and every nonnegative function, admitting an infinite terms Taylor series expansion in the origin, due to the request for divergency of the powers α_n, and fulfilling the nonnegativity and summability conditions. Examples of such spectral densities are given by the expression $[(\omega - \Omega_g)/\omega_s]^\alpha f((\omega - \Omega_g)/\omega_s)$, where the function $f(\nu)$ takes the forms $a^{n_0\,\nu^{n_1}}$, $(\bar{\Omega}/\omega_s + \nu)^\beta$, and $[\ln(1+\nu)]^{n_0}$, where $\bar{\Omega}$ is a frequency, and β and a are nonvanishing and positive constants, respectively, while n_0 and n_1 are nonvanishing natural numbers. Notice that for spectral densities with a band-gap frequency coinciding with the qubit transition frequency, $\omega_0 = \Omega_g$, corresponding to forms $\Lambda_\alpha(\nu) = \nu^\alpha/(a + \nu^2)$ obtained for $0 < \nu < 1$, the exact dynamics can be described through either incomplete Gamma functions [18] or Fox H functions [19] for either $\alpha = 1/2$ or $0 < \alpha < 1$, respectively.

The time evolution depends on a general spectral density via the following relationship:

$$\tilde{G}(u) = \frac{1}{u - i\mathcal{S}_\Lambda((\Omega_g - \omega_0 - iu)/\omega_s)}, \qquad (14)$$

where $\tilde{G}(u)$ is the Laplace transform of the function $G(t)$, driving the dynamics of the qubit through Eqs. (5), (6), and (8), while $\mathcal{S}_\Lambda$ represents the Stieltjes transform of the function Λ [20,21]. Details necessary to derive Eq. (14), are given in the Appendix. The inverse problem, finding the spectral density corresponding to a general dynamics, assigned through the function $G(t)$, is solved from Eq. (14) by performing the inverse Stieltjes transform [22],

$$\Lambda(\nu) = \lim_{\epsilon\to 0} \frac{1}{\pi} \,\mathrm{Re}\left\{\frac{1}{\tilde{G}(\omega_s(\epsilon - i\nu) + i(\omega_0 - \Omega_g))}\right\}. \qquad (15)$$

The constraints of real value, nonnegativity, and summability that the function $\Lambda(\nu)$ has to fulfill deny the possibility to design a general dynamics, via the function $G(t)$, through the spectral density. For example, the spectral density inducing an exponential decay, $G(t) = e^{-\lambda t}$, where $\lambda > 0$, results in $J(\omega) = \lambda\Theta((\omega - \Omega_g)/\omega_s)/\pi$, which is not summable. The exponential-like decays can be obtained, for example, by adding negative values of the frequencies [7].

By analyzing the long time scale dynamics induced by spectral densities described by Eqs. (10) and (11), a critical condition emerges if the qubit transition frequency equals the band-gap frequency plus an effective frequency resulting to be the first negative moment of the spectral density of the reservoir,

$$\omega_\alpha^{(c)} = \Omega_g + \int_0^\infty \frac{\Lambda_\alpha(\nu)}{\nu}\, d\nu. \qquad (16)$$

Such critical behavior derives from the function $G_\alpha(t)$, driving the time evolution through Eqs. (5), (6), and (8); details are given in the Appendix below. For times $t \gg 1/\omega_s$, the following asymptotic forms hold true in the subohmic regime, $0 < \alpha < 1$,

$$G_\alpha(t) \sim \gamma_\alpha\, e^{i(\omega_0 - \Omega_g)t}\, (\omega_s\, t)^{-1-\alpha}, \quad \omega_0 \neq \omega_\alpha^{(c)}, \qquad (17)$$

$$G_\alpha(t) \sim \gamma_\alpha^{(c)}\, e^{i(\omega_0 - \Omega_g)t} (\omega_s\, t)^{-1+\alpha}, \quad \omega_0 = \omega_\alpha^{(c)}, \qquad (18)$$

where

$$\gamma_\alpha = \frac{\pi\,\omega_s \Omega_0\, e^{i\pi(1-\alpha)/2}}{\left[\omega_0 - \Omega_g - \int_0^\infty \Lambda_\alpha(\nu)/\nu\, d\nu\right]^2 \sin(\pi\alpha)\Gamma(-\alpha)},$$

$$\gamma_\alpha^{(c)} = \frac{\omega_s\, \sin(\pi\alpha)\, e^{i\pi(-1+\alpha)/2}}{\pi\,\Omega_0\Gamma(\alpha)}.$$

In the superohmic case, for noninteger values $\alpha > 1$, outside the critical condition, $\omega_0 \neq \omega_\alpha^{(c)}$, the long time-scale dynamics is driven by

$$G_\alpha(t) \sim \frac{\gamma_\alpha\, \sin(\pi\alpha) e^{i(\omega_0 - \Omega_g)t}\, (\omega_s\, t)^{-1-\alpha}}{(-1)^n\, \sin(\pi(\alpha - n))}, \qquad (19)$$

where $n = \lfloor \alpha \rfloor + 1$, while for $\omega_0 = \omega_\alpha^{(c)}$, the resulting form

$$G_\alpha(t) \sim \frac{\omega_s (1 + \eta_\alpha^{(c)}\, (\omega_s\, t)^{1-\alpha})\, e^{i(\omega_0 - \Omega_g)t}}{\omega_s + \int_0^\infty \Lambda_\alpha(\nu)/\nu^2\, d\nu}, \qquad (20)$$

where

$$\eta_\alpha^{(c)} = \frac{(-1)^n \pi\,\Omega_0 \csc(\pi(\alpha - n)) e^{-i\pi(1+\alpha)/2}}{\Gamma(2-\alpha)\left[\omega_s + \int_0^\infty \Lambda_\alpha(\nu)/\nu^2\, d\nu\right]},$$

gives the following limit

$$|G_\alpha(\infty)| = \frac{\omega_s}{\omega_s + \int_0^\infty \Lambda_\alpha(\nu)/\nu^2\, d\nu}. \qquad (21)$$

Thus, according to Eq. (6), the oscillating coherent term does not vanish, and, according to Eq. (5), trapping of the initial population appears with the ratio given by Eq. (21), approaching unity for $\omega_s \gg \int_0^\infty \Lambda_\alpha(\nu)/\nu^2\, d\nu$.

For $0 > \alpha > -1$, the resulting function $G_\alpha(t)$ is the same as the one given by Eq. (18) and shows oscillations enveloped in inverse power law relaxations arbitrarily faster than $1/t$ and slower than $1/t^2$. No critical behavior emerges from the long time scale dynamics induced by the analyzed spectral densities, which are divergent near the band-gap frequency. Details on the demonstrations concerning the superohmic, subohmic, and divergent spectral densities are given in the Appendix.

In summary, over long time scales, by continuously varying the qubit transition frequency, the oscillating function $G_\alpha(t)$ jumps from decays proportional to $t^{-1-\alpha}$ for $\omega_0 \neq \omega_\alpha^{(c)}$ to either vanishing forms proportional to $t^{-1+\alpha}$ in the subohmic case, $0 < \alpha < 1$, or nonvanishing terms partially decaying proportionally to $t^{1-\alpha}$ in the superohmic case, for noninteger $\alpha > 1$, for $\omega_0 = \omega_\alpha^{(c)}$. Thus, discontinuities appear in the long time scale dynamics of the populations and the coherence term of the reduced density matrix, described by Eqs. (5) and (6), respectively, and in the wave function, through Eq. (8). The critical configuration cannot be realized if the qubit transition frequency is smaller than either the band-gap frequency, if it exists, or the first negative moment of the reservoir.

The long time scale dynamics exhibits features relevant for the control of decoherence in the limiting cases of α tending to 1^- and 1^+. In an occurrence of the critical frequency, $\omega_0 = \omega_\alpha^{(c)}$, arbitrarily slow decoherence processes are obtained in the subohmic regime for $\alpha \to 1^-$, Eq. (18), while, in the superohmic case, the oscillating coherence term relaxes arbitrarily slowly, Eq. (20), to a nonvanishing value, showing partial trapping of the initial population with the ratio $\left|\omega_s/[\omega_s + \int_0^\infty \Lambda_\alpha(\nu)/\nu^2\, d\nu]\right|^2$, approaching unity as the second negative moment of the spectral density becomes negligible if compared to the scale frequency. The above analysis fails for natural values of the parameter α. By considering different examples of ohmic spectral densities near the band-gap frequency, $\alpha = 1$, the asymptotic behavior does not depend uniquely on the power, equal to unity, and, differently from the previous cases, a collective behavior does not emerge.

Qualitatively, by setting $\omega_s \gg 1$, the inverse power law regime appears at early times and the continuous function $G(t)$ approaches unity, since $G(0) = 1$. Thus, arbitrarily slow decoherence processes of a qubit are obtained in the critical configuration, by approaching the boundary between sub- and superohmic regimes.

A. Particular cases

We consider reservoirs described by two relevant forms of spectral densities, with either stretched exponential or inverse power law cutoff at high frequencies, and we recover the general results in these particular cases.

The first spectral density analyzed corresponds to

$$\Lambda^{(e)}_{\alpha}(\nu) = \Omega_e \nu^{\alpha} e^{-\lambda\,\nu^{1/2}}, \quad \alpha > -1, \quad \alpha \neq \lfloor \alpha \rfloor, \quad \lambda > 0, \tag{22}$$

subohmic for $0 < \alpha < 1$, and superohmic for $\alpha > 1$ near the band-gap frequency Ω_g. The superscript (e) indicates the stretched exponential cutoff at high frequencies. The critical frequency reads

$$\omega^{(e)}_{\alpha} = \Omega_g + 2\Omega_e \lambda^{-2\alpha}\Gamma\,(2\alpha)\,, \tag{23}$$

and, over long time scales, $t \gg 1/\omega_s$, the corresponding function $G^{(e)}_{\alpha}(t)$ is described by the following asymptotic form, holding true for $t \to +\infty$,

$$G^{(e)}_{\alpha}(t) \sim \frac{2\Omega_e \omega_s \Gamma(-2\alpha)\Gamma(2\alpha+1)\cos(\pi\alpha)}{\Gamma(-\alpha)[\omega_0 - \Omega_g - 2\Omega_e\lambda^{-2\alpha}\Gamma(2\alpha)]^2} \times e^{i[-\pi(1+\alpha)/2+(\omega_0-\Omega_g)t]}\,(\omega_s\,t)^{-1-\alpha}, \quad \omega_0 \neq \omega^{(e)}_{\alpha}, \tag{24}$$

both for $0 < \alpha < 1$ and noninteger $\alpha > 1$. If the qubit transition frequency equals the critical value, at the subohmic condition, $0 < \alpha < 1$, it results in

$$G^{(e)}_{\alpha}(t) \sim \frac{\omega_s\, e^{i[\pi(1+\alpha)/2+(\omega_0-\Omega_g)t]}}{2\Omega_e \Gamma(\alpha)\Gamma(-2\alpha)\Gamma(2\alpha+1)\cos(\pi\alpha)} \times e^{i(\omega_0-\Omega_g)t}(\omega_s\,t)^{-1+\alpha}, \quad \omega_0 = \omega^{(e)}_{c}. \tag{25}$$

For noninteger $\alpha > 1$ and $\omega_0 = \omega^{(e)}_{\alpha}$, partially damped coherence emerges,

$$G^{(e)}_{\alpha}(t) \sim \frac{\omega_s e^{i[-\pi/2+(\omega_0-\Omega_g)t]}}{\omega_s + 2\Omega_e\lambda^{2(1-\alpha)}\Gamma(2(\alpha-1))} \times \left(1 + \frac{2\Omega_e\Gamma(2\alpha+1)\Gamma(-2\alpha)\cos(\pi\alpha)\,(\omega_s\,t)^{1-\alpha}}{\Gamma(2-\alpha)(\omega_s + 2\Omega_e\lambda^{2(1-\alpha)})\Gamma(2(\alpha-1))}\right), \tag{26}$$

and trapping of the populationis revealed, with the ratio

$$|G^{(e)}_{\alpha}(\infty)| = \frac{\omega_s}{\omega_s + 2\Omega_e\lambda^{2(1-\alpha)}\Gamma(2(\alpha-1))}. \tag{27}$$

For $0 > \alpha > -1$, the function $G^{(e)}_{\alpha}(t)$ is given by the one reported in Eq. (25), for every value of the frequency ω_0, showing relaxation arbitrarily faster than $1/t$ and no critical behavior. Notice that arbitrarily slow decoherence processes are obtained for $\omega_0 = \omega^{(e)}_{\alpha}$ and α tending to either 1^- or 1^+.

As a second case, we consider

$$\Lambda^{(p)}_{\alpha}(\nu) = \frac{\chi\,\nu^{\alpha}}{\mu^2+\nu^2}, \quad -1 < \alpha < 2, \quad \alpha \neq 0{,}1, \tag{28}$$

describing reservoirs as either subohmic for $0 < \alpha < 1$, or superohmic for $1 < \alpha < 2$, near the band-gap frequency, including the nonanalytical behavior of photonic band-gap materials for $0 > \alpha > -1$. The superscript (p) indicates an inverse power law cutoff at high frequencies. Again, a critical frequency reveals

$$\omega^{(p)}_{\alpha} = \Omega_g + \pi\,\chi\,\mu^{\alpha-2}\csc\,(\pi\alpha/2)\,/2, \tag{29}$$

such that, over long time scales, $t \gg 1/\omega_s$, the dynamics is driven by the function $G^{(p)}_{\alpha}(t)$, described by the following asymptotic forms, holding true in the limit $t \to +\infty$,

$$G^{(p)}_{\alpha}(t) \sim \frac{\pi\,\chi\,\omega_s e^{i\pi(1-\alpha)/2}}{\mu^2[\omega_0 - \Omega_g - \pi\,\chi\,\mu^{\alpha-2}\csc(\pi\alpha/2)/2]^2} \times \frac{e^{i(\omega_0-\Omega_g)t}(\omega_s t)^{-1-\alpha}}{\sin(\pi\alpha)\Gamma(-\alpha)} \tag{30}$$

for either $0 < \alpha < 1$ or $1 < \alpha < 2$ and $\omega_0 \neq \omega^{(p)}_{c}$,

$$G^{(p)}_{\alpha}(t) \sim \frac{\mu^2\omega_s \sin(\pi\alpha)e^{i\pi(-1+\alpha)/2}}{\pi\,\chi\,\Gamma(\alpha)} \times e^{i(\omega_0-\Omega_g)t}(\omega_s t)^{-1+\alpha} \tag{31}$$

for either $\omega_0 = \omega^{(p)}_{\alpha}$ and $0 < \alpha < 1$, or $0 > \alpha > -1$ and every $\omega_0 > 0$. The conditions $\omega_0 = \omega^{(p)}_{\alpha}$ and $1 < \alpha < 2$ lead to an oscillating partially damped coherence term

$$G^{(p)}_{\alpha}(t) \sim \frac{\omega_s\, e^{i[-\pi/2+(\omega_0-\Omega_g)t]}}{\omega_s - \pi\,\chi\,\mu^{\alpha-3}\sec(\pi\alpha/2)/2} \times \left(1 - \frac{\pi\,\chi\,\mu^{-2}\csc(\pi\alpha)(\omega_s t)^{1-\alpha}}{\Gamma(2-\alpha)[\omega_s - \pi\,\chi\,\mu^{\alpha-3}\sec(\pi\alpha/2)/2]}\right), \tag{32}$$

and to trapping of the population, with the ratio

$$|G^{(p)}_{\alpha}(\infty)| = \frac{\omega_s}{\omega_s - \pi\,\chi\,\mu^{\alpha-3}\sec(\pi\alpha/2)/2}. \tag{33}$$

As in the previous case, for $0 > \alpha > -1$, the function $G^{(p)}_{\alpha}(t)$ is given by the one reported in Eq. (31), for every positive value of the frequency ω_0, showing relaxation arbitrarily faster than $1/t$ and no critical behavior. Again, arbitrarily slow decoherence processes are obtained for $\omega_0 = \omega^{(p)}_{\alpha}$ and $\alpha \to 1^-$.

IV. CONCLUSIONS

The control of decoherence through the engineering of the external environment has been studied in details, especially for boson-boson models [23,24] and atoms interacting with structured reservoirs [25]. Spectral densities with photonic band gaps provide coherent control of the spontaneous emissions if one resonant frequency is near a photonic band gap [9,26]. Modulations through external fields [15] and manipulations of reservoir modes through time-dependent dissipative environments [16] are relevant examples of dynamical control of decoherence.

We propose a static control of qubit decoherence, obtained by setting special critical conditions involving the reservoir. For special spectral densities either subohmic or superohmic near an arbitrary band gap, the critical conditions are obtained if the band-gap frequency plus the first negative moment of the spectral density equals the qubit transition frequency. By considering scale frequencies $\omega_s \ll 1$, the regime of inverse power law decay starts at early times, and arbitrarily slow decoherence processes are obtained in the critical configuration by approaching the boundary between the sub- and superohmic conditions. This method is based on the discontinuities appearing in the long time inverse power law relaxations describing the dynamics of the qubit. A condition sufficient to make the effect disappear is to consider qubit transition frequencies less than either the band-gap frequency, if it exists, or the first negative moment of the spectral density of the reservoir. In this way, the long time scale dynamics is stabilized.

The proposed effect might be observed experimentally by letting a two-level atom interact weakly [rotating wave approximation (RWA)] with the field modes of a cavity [25,27] or photonic band-gap materials [28] described by the spectral densities designed above. Since an N-period one-dimensional (1D) lattice may reproduce a band gap through a proper sequence of dielectric unit cells, an arbitrarily shaped density of frequency modes can be modeled through the sequence of the transmission coefficients of each unit cell [29]. Also, a diffractive grating and photonic crystals engineer 1D photonic band-gap (PBG) microcavities [30]. Notice that the spectral density $J_e(\omega) = j_0\,\omega_c\,(\omega/\omega_c)^\alpha e^{-\omega/\omega_c}$ gives a simple expression of the corresponding critical frequency, $\omega_\alpha^{(e)} = j_0\,\omega_c\,\Gamma(1+\alpha)$, where ω_c is the cutoff frequency, j_0 is a dimensionless coupling constant, and the superscript e refers to the exponential cutoff at high frequencies. Anyway, a detailed analysis of an apparatus reproducing the above scenario is not one of the purposes of the present paper.

The present results suggest that, under certain conditions, arbitrarily slow decoherence might be derived in the general open quantum system dynamics, described by the quantum dynamical semigroups through the Gorini-Kossakowski-Sudarshan-Lindblad form [31]. This hypothesis will be the argument of further research.

ACKNOWLEDGMENTS

This work is based upon research supported by the South African Research Chairs Initiative of the Department of Science and Technology and National Research Foundation. F.G. is deeply grateful to Prof. Yu-Qiu Zhao and Prof. R. Wong for the suggestions on the asymptotic approximation of the Stieltjes transform. The authors thankfully acknowledge N. Lo Gullo for the discussion on the feasibility of the special reservoirs through dielectric gas mixtures.

APPENDIX: DETAILS

We give details about the long but straightforward calculations leading to the results shown in the paper. The time evolution of the qubit is driven by the function $G(t)$, solution of Eq. (7), through Eqs. (5), (6), and (8), related to the spectral density through Eqs. (9) and (12). For spectral densities described by the functions $\Lambda_\alpha(\nu)$, fulfilling the conditions (10) and (11), the asymptotic behavior of the function $G_\alpha(t)$ is obtained through the identity

$$\frac{1}{\nu+z} = \sum_{k=0}^{n-1}(-1)^k \frac{z^k}{\nu^{k+1}} + \frac{(-1)^n z^n}{\nu^n(\nu+z)}. \qquad \text{(A1)}$$

For noninteger values of the powers appearing in the power series expansion of the function $\Lambda_\alpha(\nu)$ around the origin, a series expansion of the Stiltjes transform $\mathcal{S}_{\Lambda_\alpha}(z)$ can be performed. In fact, by considering $\Lambda_\alpha(\nu) = \Omega_0\nu^\alpha + \Lambda_{\alpha,n}(\nu)$, where $\Lambda_{\alpha,n}(\nu) = o(\nu^\alpha)$ and $n = \lfloor\alpha\rfloor + 1$, it results in

$$\mathcal{S}_{\Lambda_\alpha}(z) \sim \sum_{k=0}^{n-1}(-1)^k z^k \int_0^\infty \frac{\Lambda_\alpha(\nu)}{\nu^{k+1}}\,d\nu + \frac{(-1)^{n+1}\pi\,\Omega_0 z^\alpha}{\sin(\pi(\alpha-n))} + o(z^\alpha), \quad z \to 0, \qquad \text{(A2)}$$

holding true in the sector $|\arg z| \leqslant \pi - \delta$, where δ is an arbitrarily small positive parameter. The above treatment holds true for any noninteger $\alpha > 1$, and $0 > \alpha > -1$, due to the divergency of the noninteger sequence α_n. In this way, the function $\tilde{G}_\alpha(u)$, given by Eq. (12), result in

$$\tilde{G}(u) = 1\bigg/\left(i\left(\omega_0 - \Omega_g - \int_0^\infty \Lambda_\alpha(\nu)\,d\nu\right) - i\sum_{k=0}^{n-1}\frac{z^k}{(-1)^k} \times \int_0^\infty \frac{\Lambda_\alpha(\nu)}{\nu^{k+1}}\,d\nu + \frac{i\pi\,\Omega_0(-1)^n z^\alpha}{\sin(\pi(\alpha-n))} + o(\nu^\alpha)\right), \qquad \text{(A3)}$$

where $z = -i[u - i(\omega_0 - \Omega_g)]/\omega_s$. The series expansion of Eq. (A3), around $z = 0$, equivalent to $u = i(\omega_0 - \Omega_g)$, converges in the sector $|\arg z| \leqslant \pi - \delta$, where δ is an arbitrarily small positive parameter. Thus, the long time behavior can be found through the asymptotic series obtained via the term by term Laplace inversion. For $0 < \alpha < 1$, such asymptotic series leads to either Eq. (15) or Eq. (16), whether the constant $[\omega_0 - \Omega_g - \int_0^\infty \Lambda(\nu)/\nu\,d\nu]$ vanishes or not. The latter condition provokes the appearance of the critical frequency $\omega_\alpha^{(c)}$, given by Eq. (14). The above procedure works for noninteger $\alpha > 1$ and leads to either Eq. (17) or Eq. (18), whether the qubit transition frequency equals the critical value or not. The case $0 > \alpha > -1$ is analyzed in the same way and the resulting function $G_\alpha(t)$ is the same as the one reported in Eq. (16), for every $\omega_0 > 0$. The definition of the long time scale, $1/\omega_s$, descends from the convergence criteria of the related power series expansions. This concludes the details necessary to recover the results shown in the paper. The dynamics induced by reservoirs corresponding to the spectral densities $\Lambda_\alpha^{(e)}(\nu)$ and $\Lambda_\alpha^{(p)}(\nu)$, given by Eqs. (22) and (28), is described by the functions $\tilde{G}_\alpha^{(e)}(t)$ and $\tilde{G}_\alpha^{(p)}(t)$, inverse Laplace transforms of the functions $\tilde{G}_\alpha^{(e)}(u)$ and $\tilde{G}_\alpha^{(p)}(u)$, obtained through Eq. (12) and the corresponding Stieltjes transforms [16]. The asymptotic series obtained through the term by term inverse Laplace transforms of the series expansions of the form $\tilde{G}_\alpha^{(e)}(u)$ lead to Eqs. (24), (25), (26), and (27), describing the long time scale dynamics, whether the qubit transition frequency equals the corresponding critical value (23), or not. In similar way, starting from the form $\tilde{G}_\alpha^{(p)}(u)$, Eqs. (30), (31), (32), and (33) are obtained giving the time evolution over long time scales, whether the frequency ω_0 equals the critical value (29) or not. This concludes the details necessary to recover the results concerning the particular cases.

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