Analytic approximate expression for the spectral distribution of the emission from a slab of resonant two-level atoms prepared by an ultrashort δ pulse

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Using the slowly varying envelope approximation in space and time and the expression for the cooperative Lamb shift, I obtain an approximate analytic expression for the spectral distribution of the radiation from a slab of two atoms that was initially weakly excited by an ultrashort resonant pulse. The closed-form expression obtained reproduces very accurately the numerical exact results computed from the eigenmode analysis for the emission from the front face of the slab.

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I. INTRODUCTION

In a recent publication [1], I computed numerically, using the eigenmode analysis, the spectral distribution of the radiation emitted from a slab of two-level atoms which was excited by a resonant ultrashort δ pulse. This problem had been the topic of many previous theoretical studies [2–7]. The results numerically obtained in [1] confirm the general features of the Friedberg-Hartmann classic work [4].

The eigenmode analysis formalism [1,7–9] provides the means to compute exactly (by which I mean to arbitrary precision) the time dependence of the forward and backward intensities and their spectral distributions, it gives these quantities in the form of infinite sums over the one-dimensional Lienard-Wiechert eigenmodes. The limitation on the use of this method for slabs with a thickness much larger than the resonance wavelength is only the size of the RAM memory in the computer used. In this Brief Report, I report on a successful effort for obtaining closed-form analytic approximate expressions for the same quantities which reproduce very accurately the exact results for the emission from the front face of the slab. It is to be noted that successfully reproducing the numerical results with the present approximate analytical expression will provide further proof that knowledge of the cooperative decay rate and the cooperative Lamb shift at initial time is necessary and sufficient to predict almost all of the features of the spectral distribution of the forward radiation from the ensemble of two-level atoms.

In Sec. II, I review quickly the results for this problem obtained in [1] using the eigenmode analysis. In Sec. III, I give the closed-form approximate analytic expression for the forward emission spectral distribution obtained using the standard slowly varying envelope approximation (SVEA) and find that, comparing this approximate result with the exact eigenfunction-derived results, the two spectra agree well far from the line center; however, they differ significantly near the line center. Then, I propose a modification to the SVEA which includes the cooperative Lamb shift that gives everywhere an approximate expression for the forward spectral distribution which reproduces everywhere very accurately the eigenfunction analysis results. In the conclusion, I examine the results of this approximation for the spectral distribution of the backward radiation and show that the new approximation, which does not include the contribution of the counterrotating term, fails to give accurate predictions near the line center.

II. SUMMARY OF THE EIGENMODE-DERIVED RESULTS

The integral equation for $b(z,t)$, the excitation amplitude for an initially weakly excited system, in the Markov approximation L/c much less than superradiant lifetime, where $L = 2z_0$ is the thickness of the slab, is given, in the interaction picture, by [8,9]

$$\dot{b}(z,t) - i\omega_L b(z,t) + \gamma_T b(z,t) = -\frac{Ck_0}{2}\int_{-z_0}^{z_0} dz' \exp(ik_0|z-z'|)b(z',t), \tag{1}$$

where $\omega_L = C/3$ is the Lorentz shift, γ_T is the total width $(= \gamma_2 + \gamma_1/2)$, which includes in a gas the single-atom contribution and all the line broadening present (resonant and foreign gas), $C = 4\pi n\wp^2/\hbar$, $\wp$ is the reduced matrix element of the electric dipole operator of the atomic transition, $k_0 = \omega_0/c$ is the wave number of the atomic transition, and n is the number density of the atoms. [The spontaneous decay rate of the excitation probability in the isolated atom, as function of the previous parameters, is $\gamma_1 = \frac{4}{3}(\wp^2 k_0^3/\hbar)$.]

The eigenvalues $\lambda_s^{o,e}$ and eigenfunctions $\phi_s^{o,e}(Z)$ associated with the above kernel are obtained by solving the integral equation

$$\lambda_s^{o,e}\phi_s^{o,e}(z) = \frac{Ck_0}{2}\int_{-z_0}^{z_0} dz' \exp(ik_0|z-z'|)\phi_s^{o,e}(z'). \tag{2}$$

This integral equation admits two families of solutions, where the superscripts o and e refer, respectively, to odd and even parity in space. The form, metric, and normalization of these eigenmodes are given in [1].

If $b(z,t)$ is written as

$$b(z,t) = \exp[-(\gamma_2 - i\omega_L)t]\tilde{b}(Z,T), \tag{3}$$

where $T = Ct$, and $Z = z/z_0$, then the initial polarization of the system

$$b(Z,0) = \exp(i\kappa Z) \tag{4}$$

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can be written for arbitrary time as

$$\tilde{b}(Z,T)=\sum_{s=1}^{\infty}\left[c_s^o\phi_s^o(Z)\exp\left(-\Lambda_s^oT\right)+c_s^e\phi_s^e(Z)\exp\left(-\Lambda_s^eT\right)\right], \tag{5}$$

where $\Lambda_s^{o,e}=\lambda_s^{o,e}/C$, and the coefficients $c_s^{o,e}$ in Eq. (5) are obtained by projecting the initial state of the system over the basis functions. The field at each of the exit planes of the slab is equal to

$$E(z_0,t)=A\exp(ik_0z_0)\int_{-z_0}^{z_0}dz'\exp(-ik_0z')b(z',t), \tag{6}$$

$$E(-z_0,t)=A\exp(-ik_0z_0)\int_{-z_0}^{z_0}dz'\exp(ik_0z')b(z',t). \tag{7}$$

Using (5), one can then write

$$\begin{aligned}E_f(T)=&A\exp[-(\Gamma_2-i\Omega_L)T]\\&\times\sum_{s=1}^{\infty}\left[c_s^e(\kappa)c_s^e(u_0)N_s^e\exp\left(-\Lambda_s^eT\right)\right.\\&\left.-c_s^o(\kappa)c_s^o(u_0)N_s^o\exp\left(-\Lambda_s^oT\right)\right],\end{aligned} \tag{8}$$

$$\begin{aligned}E_b(T)=&A\exp[-(\Gamma_2-i\Omega_L)T]\\&\times\sum_{s=1}^{\infty}\left[c_s^e(\kappa)c_s^e(u_0)N_s^e\exp\left(-\Lambda_s^eT\right)\right.\\&\left.+c_s^o(\kappa)c_s^o(u_0)N_s^o\exp\left(-\Lambda_s^oT\right)\right],\end{aligned} \tag{9}$$

where the subscripts f and b refer, respectively, to forward and backward for the exit planes, $\Gamma_2=\gamma_2/C\gg\gamma_1/C$, and $N_s^{e,o}$ are the normalization constants for the basis functions. The spectral distributions for the forward and backward emitted radiation are obtained by taking the magnitude square of the Fourier transforms of the fields $E_f(T)$ and $E_b(T)$, respectively, which give

$$\begin{aligned}|\tilde{E}_f(\tilde{\Omega})|^2=A^2\Bigg|\sum_{s=1}^{\infty}\Bigg(&\frac{c_s^e(\kappa)c_s^e(u_0)N_s^e}{-i\tilde{\Omega}-(\Gamma_2-i\Omega_L)-\Lambda_s^e}\\&-\frac{c_s^o(\kappa)c_s^o(u_0)N_s^o}{-i\tilde{\Omega}-(\Gamma_2-i\Omega_L)-\Lambda_s^o}\Bigg)\Bigg|^2,\end{aligned} \tag{10}$$

$$\begin{aligned}|\tilde{E}_b(\tilde{\Omega})|^2=A^2\Bigg|\sum_{s=1}^{\infty}\Bigg(&\frac{c_s^e(\kappa)c_s^e(u_0)N_s^e}{-i\tilde{\Omega}-(\Gamma_2-i\Omega_L)-\Lambda_s^e}\\&+\frac{c_s^o(\kappa)c_s^o(u_0)N_s^o}{-i\tilde{\Omega}-(\Gamma_2-i\Omega_L)-\Lambda_s^o}\Bigg)\Bigg|^2,\end{aligned} \tag{11}$$

where the different normalized frequencies are the physical quantities normalized to C, Γ_2 is the resonant broadening [8], and $\kappa=u_0$.

III. THE SVEA AND THE MODIFIED SVEA EXPRESSIONS

Making the following substitution in Eq. (1),

$$b(z,t)=\exp[-(\gamma_2-i\omega_L)t]\exp(ik_0z)B(z,t), \tag{12}$$

the integral equation for $B(z,t)$ becomes

$$\begin{aligned}\dot{B}(z,t)=-\frac{Ck_0}{2}\Bigg[&\int_{-z_0}^{z}B(z',t)dz'+\exp(-2ik_0z)\\&\times\int_{z}^{z_0}B(z',t)\exp(2ik_0z')dz'\Bigg].\end{aligned} \tag{13}$$

The slowly varying envelope approximation consists of neglecting the contribution of the second term in the square brackets in Eq. (13), which amounts to neglecting the local backward wave at each point in the sample. In this approximation the expression of $B(Z,T)$ is found to be [2]

$$B(Z,T)=B_0J_0(\sqrt{2u_0(Z+1)T}). \tag{14}$$

Combining Eq. (6) with Eq. (14), one obtains for the normalized field at the front plane (i.e., $Z=1$) [10–12]

$$\begin{aligned}\frac{E_f^{\mathrm{SVEA}}(T)}{E_f^{\mathrm{SVEA}}(0)}&=\exp[-(\Gamma_2-i\Omega_L)T]\frac{\int_{-1}^{1}dZ'J_0(\sqrt{2u_0(Z'+1)T})}{\int_{-1}^{1}dZ'}\\&=\exp[-(\Gamma_2-i\Omega_L)T]\frac{J_1\sqrt{4u_0T}}{\sqrt{u_0T}}.\end{aligned} \tag{15}$$

The normalized spectral distribution of the radiation from the front plane is the magnitude square of the expression $\tilde{a}_f^{\mathrm{SVEA}}(\tilde{\Omega})=\tilde{E}_f^{\mathrm{SVEA}}(\tilde{\Omega})/E_f^{\mathrm{SVEA}}(0)$, where

$$\begin{aligned}\tilde{a}_f^{\mathrm{SVEA}}(\Omega)&=\frac{1}{2}\int_0^{\infty}dT\exp(-i\tilde{\Omega}T)\exp[-(\Gamma_2-i\Omega_L)T]\\&\quad\times\frac{J_1\sqrt{4u_0T}}{\sqrt{u_0T}}\\&=\frac{1}{u_0}[1-\exp(-iu_0/\Delta^{\mathrm{SVEA}})],\end{aligned} \tag{16}$$

where $\Delta^{\mathrm{SVEA}}=i\Gamma_2+(-\tilde{\Omega}+\Omega_L)$. (Recall that $\tilde{\Omega}$ is the normalized frequency measured from the atomic resonance frequency.)

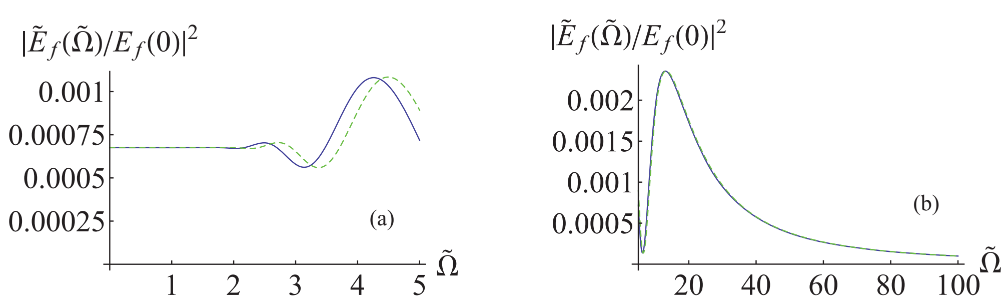

FIG. 1. (Color online) Comparison of the numerical exact results (solid line) for the forward emission spectral distribution with those of the SVEA expressions (dashed line). $u_0=k_0z_0=12.25\pi,\Gamma_2=2.33/4$. (a) $0<\tilde{\Omega}<5$; (b) $5<\tilde{\Omega}<100$.

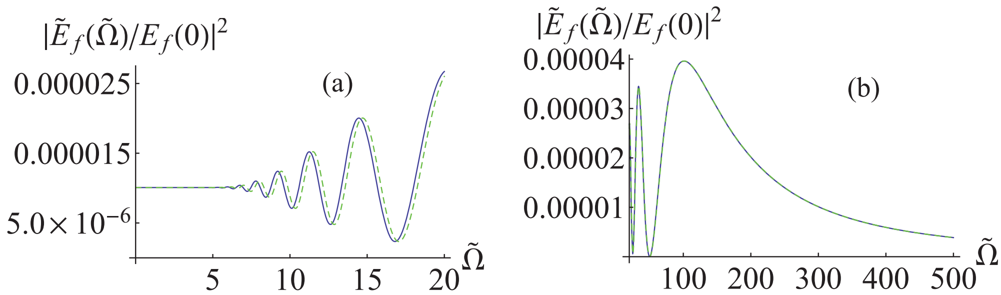

FIG. 2. (Color online) Comparison of the numerical exact results (solid line) for the forward emission spectral distribution with those of the SVEA expressions (dashed line). $u_0 = k_0 z_0 = 100.25\pi, \Gamma_2 = 2.33/4$. (a) $0 < \tilde{\Omega} < 20$; (b) $20 < \tilde{\Omega} < 500$.

I compare in Figs. 1 and 2 the spectra given by Eq. (16) with those from the exact numerical calculations in [1]. For $\tilde{\Omega} \geqslant O(u_0/\pi)$ the agreement between the SVEA results and the exact results is very good, but for lower values of $\tilde{\Omega}$, near the line center, the agreement is not good.

I shall now examine the reason for the above discrepancy and propose a way to fix it. If we consider the integral equation (1), its derivative at $t = 0$ gives

$$\left.\frac{\partial B(z,t)}{\partial t}\right|_{t=0} = (i\omega_L - \gamma_T)B(z,0) - \frac{Ck_0}{2}\int_{-z_0}^{z_0} dz' \times \exp(ik_0|z-z'|)\exp[ik_0(z'-z)]B(z',0). \tag{17}$$

Using the initial condition $B(z,0) = 1$, the above quantity averaged over z gives

$$\overline{\left.\frac{\partial B(z,t)}{\partial t}\right|_{t=0}} = (i\omega_L - \gamma_T) - \frac{Ck_0}{4z_0}\int_{-z_0}^{z_0} dz \times \int_{-z_0}^{z_0} dz' \exp(ik_0|z-z'|)\exp[ik_0(z'-z)]. \tag{18}$$

Now using the value of the double integral

$$\int_{-1}^{1} dZ \int_{-1}^{1} dZ' \exp(iu_0|Z-Z'|)\exp[i(Z'-Z)] = 2 - \frac{\exp(-4iu_0)-1}{4u_0^2} - \frac{i}{u_0}, \tag{19}$$

the right-hand side of Eq. (18) reduces to $i(\omega_L - |\Delta\omega^{\mathrm{CLS}}|) - (\gamma_T + \frac{\gamma_{\mathrm{CDR}}}{2})$, where the normalized cooperative decay rate (CDR) and the normalized cooperative Lamb shift (CLS) slab are [9]

$$\Gamma^{\mathrm{CDR}} = \frac{1}{8u_0}\left[1 - \cos(4u_0) + 8u_0^2\right], \tag{20}$$

$$|\Delta\Omega^{\mathrm{CLS}}| = \frac{1}{4}\left[1 - \frac{\sin(4u_0)}{4u_0}\right] = \left|\frac{3}{4}\Omega_L\left[1 - \frac{\sin(4u_0)}{4u_0}\right]\right|, \tag{21}$$

where the sign of the CLS is opposite to that of the Lorentz shift $\Omega_L = \frac{1}{3}$.

Now let us examine what the standard SVEA gives for these quantities at $t = 0$. The rate of change of the SVEA expression of the intensity there is

$$\left.\frac{d}{dT}\left(\frac{J_1^2(\sqrt{4u_0T})}{u_0T}\right)\right|_{T=0} = u_0, \tag{22}$$

This result for the width agrees with the large-u_0 limit of Eq. (20); however, for the shift, the SVEA value is zero because the nonexponential part of the right-hand side of Eq. (15) is real, whereas Eq. (21) gives Ω^{CLS}.

This suggests multiplying $B^{\mathrm{SVEA}}(Z,T)$ by $\exp(-i|\Delta\Omega^{\mathrm{CLS}}|T)$ in the value of the derivative of $B^{\mathrm{SVEA}}(Z,T)$ in order to reproduce, in the neighborhood of $T = 0$, the exact value of the derivative of $\overline{B(Z,T)}$. This ansatz is exact at $T = 0$. If this functional form is extrapolated to all values of T (an approximation), this will mean replacing

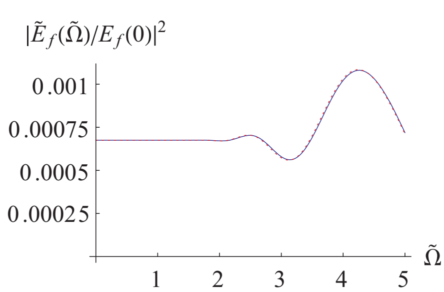

FIG. 3. (Color online) Comparison of the numerical exact results (solid line) for the forward emission spectral distribution with those of the MSVEA expressions (dotted line). $u_0 = k_0 z_0 = 12.25\pi, \Gamma_2 = 2.33/4$. The dotted line is difficult to see as it overlaps with the solid line.

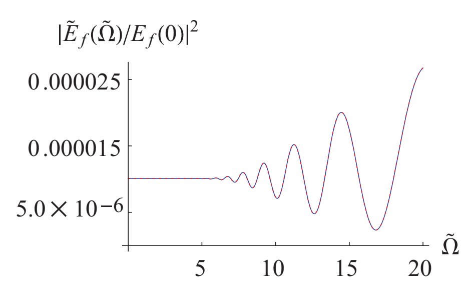

FIG. 4. (Color online) Comparison of the numerical exact results (solid line) for the forward emission spectral distribution with those of the MSVEA expressions (dotted line). $u_0 = k_0 z_0 = 100.25\pi, \Gamma_2 = 2.33/4$. The dotted line is difficult to see as it overlaps with the solid line.

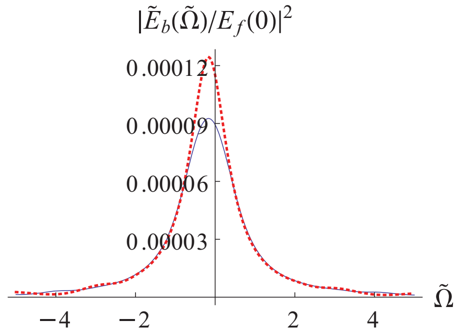

FIG. 5. (Color online) Comparison of the numerical exact results (solid line) for the backward emission spectral distribution with those of the MSVEA expressions (dotted line). $u_0 = k_0 z_0 = 12.25\pi, \Gamma_2 = 2.33/4$.

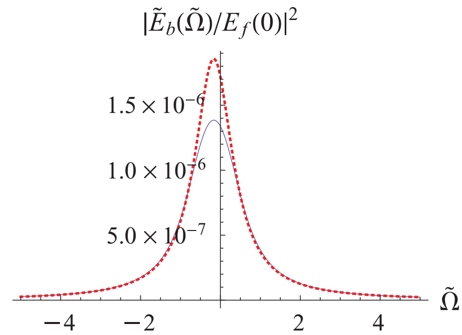

FIG. 6. (Color online) Comparison of the numerical exact results (solid line) for the backward emission spectral distribution with those of the MSVEA expressions (dotted). $u_0 = k_0 z_0 = 100.25\pi, \Gamma_2 = 2.33/4$.

Ω_L by Ω_M in Eq. (16), where

$$\Omega_M = \Omega_L - |\Delta\Omega^{\text{CLS}}| = \Omega_L\left[1 - \frac{3}{4}\left(1 - \frac{\sin(4u_0)}{4u_0}\right)\right] = \Omega_L\left(\frac{1}{4} + \frac{3}{16}\frac{\sin(4u_0)}{u_0}\right), \tag{23}$$

thus giving for the modified SVEA expression

$$\tilde{a}_f^{\text{MSVEA}}(\Omega) = \frac{1}{u_0}[1 - \exp(-iu_0/\Delta^{\text{MSVEA}})], \tag{24}$$

where $\Delta^{\text{MSVEA}} = i\Gamma_2 + (-\tilde{\Omega} + \Omega_M)$.

In Figs. 3 and 4, I compare the results of the spectral distribution for the modified SVEA as given by Eq. (24) with those of the numerical summation of the analytical terms of the eigenmode expansion and note that the two curves are indeed indistinguishable, a remarkable result given the large time oscillations and large frequency shifts. Having thus established the accuracy of the closed-form expression given by Eq. (24) for the spectral distribution, I have established that knowing the initial values of the cooperative decay rate and of the cooperative Lamb shift is key to predicting the emission from the ensemble. The most important features of the forward emission spectral distribution are (a) the height of the middle plateau is $\cong\frac{1}{u_0^2}$ and its extent is $\approx\sqrt{u_0\Gamma_2}$, (b) the height of the outermost peaks is $\cong \frac{4}{u_0^2}$ and their locations are at $\tilde{\Omega} \cong |\frac{u_0}{\pi}|$, and (c) the locations of the outermost minima are at $\tilde{\Omega} \cong |\frac{u_0}{2\pi}|$.

IV. CONCLUSION

In this Brief Report, I have shown that the spectral distribution for the forward emission from a slab of resonant two-level atoms that was initially weakly excited by a δ pulse can be very well approximated by considering the expression obtained from the SVEA modified to incorporate the cooperative Lamb shift. So far, I have left out any results on the emission from the back face of the slab. As the SVEA essentially neglects locally the contribution of the backward wave, one should not expect the modified approximation which corrects only for the phase of $B(Z,T)$ to also improve on the SVEA expression for the amplitude of the emission spectrum from the back of the slab near line center.

Next, I explore this issue a little further: The expression corresponding to Eq. (24) for the emission from the back face is given in the modified SVEA by

$$\tilde{a}_b^{\text{MSVEA}}(\Omega) = \frac{1}{u_0(1 - 4\Delta^{\text{MSVEA}})}[\exp(-2iu_0) - \exp(2iu_0 - iu_0/\Delta^{\text{MSVEA}})]. \tag{25}$$

In Figs. 5 and 6, I compare the exact numerically obtained results for the backward emission spectral distribution with those obtained from Eq. (25) and find, as expected, that the approximation is poor for the magnitude of the spectral distribution around its peak. However, it is interesting to note that the spectral shift in the location of the peak and the magnitude of the spectral intensity in the wings of the line as predicted by the modified SVEA results are not too far off from the exact results. While the issue of the backward emission is of theoretical interest and the present approximation did not fully address it, practically, it is of little impact as it represents less than 1% of the total radiation emitted by the system.

[1] J. T. Manassah, Laser Phys. **22**, 559 (2012).
[2] D. C. Burnham and R. Y. Chiao, Phys. Rev. **188**, 667 (1967).
[3] M. D. Crisp, Phys. Rev. A **1**, 1604 (1970).
[4] R. Friedberg and S. R. Hartmann, Opt. Commun. **2**, 301 (1970).
[5] V. V. Zheleznyakov, V. V. Kocharovskii, and Vl. V. Kocharovskii, Sov. Phys. JETP **60**, 897 (1984).
[6] V. V. Zheleznyakov, V. V. Kocharovskii, and Vl. V. Kocharovskii, Sov. Phys. Usp. **32**, 835 (1989).
[7] N. S. Ginzburg and A. S. Sergeev, Opt. Commun. **91**, 140 (1992).
[8] S. Prasad and R. J. Glauber, Phys. Rev. A **61**, 063814 (2000).
[9] R. Friedberg and J. T. Manassah, Phys. Lett. A **372**, 2787 (2008).
[10] R. Friedberg and J. T. Manassah, Opt. Commun. **281**, 3755 (2008).
[11] R. Friedberg, S. R. Hartmann, and J. T. Manassah Phys. Rep. **7**, 101 (1972).
[12] R. Friedberg and J. T. Manassah, Phys. Lett. A **373**, 3423 (2009).