Duality relations in a two-path interferometer with an asymmetric beam splitter

Li Li,[1,*] Nai-Le Liu,[1,†] and Sixia Yu[1,2]

[1]*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

[2]*Centre for Quantum Technologies, National University of Singapore, 2 Science Drive 3, Singapore 117542*

(Received 15 February 2012; published 2 May 2012)

We investigate quantitatively the wave-particle duality in a general Mach-Zehnder interferometer setup with an asymmetric beam splitter. The asymmetric beam splitter introduces additional *a priori* which-path knowledge, and the additional knowledge is different for particles detected at one or the other output port. Accordingly, the interference patterns appearing at the two output ports are also different. Hence, in sharp contrast with the symmetric case, in the present case we should treat the two output ports separately. It turns out that two nonorthogonal unsharp observables are measured jointly in this setup. We apply the condition for joint measurability of these unsharp observables to obtain a trade-off relation between the fringe visibility of the interference pattern and the which-path distinguishability.

DOI: 10.1103/PhysRevA.85.054101 PACS number(s): 03.65.Ta, 03.67.−a

Wave-particle duality is a striking manifestation of Bohr's principle of complementarity [1] which lies at the heart of quantum mechanics. In 1979, Wootters and Zurek [2] first *quantified* the wave-particle duality in an Einstein version of the double-slit experiment. Later, two kinds of duality inequality were established in the standard Mach-Zehnder interferometer (MZI) setup. The first one [3], $\mathcal{P}^2 + \mathcal{V}_0^2 \leqslant 1$, concerns the trade-off between the predictability $\mathcal{P}$ of two possible paths taken by a particle passing through the interferometer and the *a priori* fringe visibility $\mathcal{V}_0$ of the interference pattern emerging at each output port of the interferometer. The second one [4–6], i.e.,

$$\mathcal{D}^2 + \frac{1-\mathcal{P}^2}{\mathcal{V}_0^2}\mathcal{V}^2 \leqslant 1, \tag{1}$$

concerns the trade-off between the which-path distinguishability $\mathcal{D}$ and the fringe visibility $\mathcal{V}$ when each particle is coupled to another physical system which serves as a which-path detector (WPD).

Another celebrated quintessential feature of quantum mechanics is that there exist incompatible observables, i.e., observables which cannot be jointly measured in a single device. However, in some cases, two incompatible sharp observables could still be jointly measured on condition that some imprecision is allowed. Exactly speaking, the unsharp versions of these observables could be marginals of a bivariate joint observable so that measurement of the joint observable offers simultaneously the values of the two unsharp observables [7,8]. The so-called joint measurability problem—given two unsharp observables, are they jointly measurable?—was first brought forward by Busch [9], who solved it in a very special case. Although there have been many partial results concerning this problem in the past few years [10–15], the necessary and sufficient condition for joint measurability of two general unsharp observables of a two-level system was derived only recently by three independent groups [16–18].

The problem of joint measurability has also been studied from other aspects, including the uncertainty relation [19,20], quantum cloning [21,22], Bell inequalities [23], and so on. Recently, we brought to light an intimate relationship between the joint measurability of two unsharp qubit observables and the wave-particle duality illustrated in the standard MZI setup [15]. In fact, the measurement made on the WPD system provides us the which-path information, and meanwhile a counting detection at each output port of the interferometer yields an interference pattern. Since these two measurements are made simultaneously on different systems, the whole setup provides *de facto* a joint measurement of two unsharp observables of the particle. Due to the fact that the beam splitters in the standard MZI setup are *symmetric*, i.e., the proportion of the transmissivity and the reflectivity of each beam splitter is 50:50, the two unsharp observables jointly measured turn out to be *orthogonal*. The condition for their joint measurability leads to a duality inequality which is similar to but has a minor difference from Eq. (1) (see Ref. [15] for details).

The duality inequalities in Refs. [4,5,15] were derived in the standard MZI setup with symmetric beam splitters. In this Brief Report, however, we shall take into account the wave-particle duality illustrated in a more generic scenario, i.e., in a general MZI setup with an *asymmetric* beam splitter (ABS). Unlike in the symmetric case, the *a priori* which-path information is *different* for particles detected at one or the other output port, and the interference patterns appearing at the two output ports are also different. Thus in the asymmetric case the two output ports need to be treated separately. We show that the general MZI setup, when coupled to an additional WPD system, provides a simultaneous measurement of two *nonorthogonal* unsharp observables. Furthermore, the condition for joint measurability of these two observables, which we obtained recently [17], enables us to obtain a duality inequality.

Specifically, consider the two-path MZI setup as depicted schematically in Fig. 1. For a particle passing through the interferometer, the two distinct paths after the first beam splitter BS_1 define two orthonormal states $|0\rangle$ and $|1\rangle$ which span a two-dimensional Hilbert space. Without loss of generosity

[*]eidos@ustc.edu.cn

[†]nlliu@ustc.edu.cn

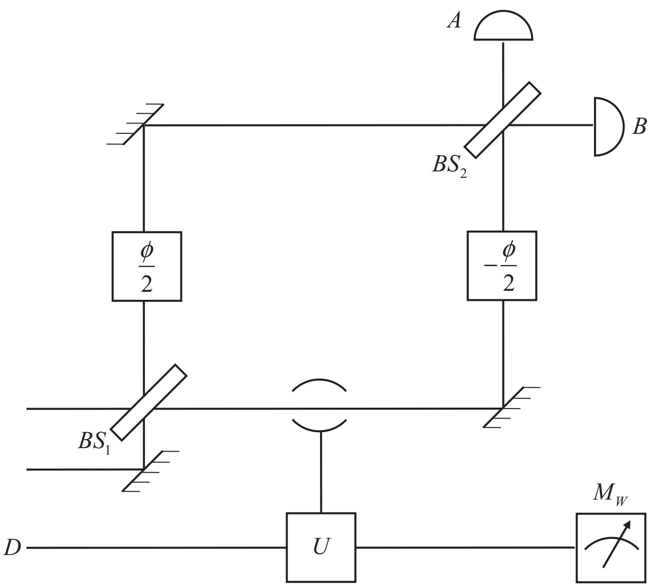

FIG. 1. The Mach-Zehnder interferometer with an asymmetric beam splitter. The particle is prepared in an arbitrary state before entering the interferometer. Two phase shifts $\pm\phi/2$ are introduced into the two distinct paths, respectively. After the second beam splitter BS_2, two counting detectors A and B record two different interference patterns. After the particle has interacted with a WPD system D via a controlled unitary transformation, a special observable W of the WPD system is measured, from which the which-path information of the particle is inferred.

we can take BS_1 as symmetric since the initial state of the particle is taken to be arbitrary. Two phase shifts $\pm\phi/2$ are introduced for the two paths $|0\rangle, |1\rangle$, respectively. The second beam splitter BS_2 is taken to be asymmetric and we denote by r its reflectivity, i.e., the probability of the particle being reflected, and by $t = 1 - r$ its transmissivity. The action of BS_2 on the particle is effectively a unitary transformation

$$B = \begin{pmatrix} \sqrt{r} & \sqrt{t} \\ \sqrt{t} & -\sqrt{r} \end{pmatrix} \tag{2}$$

on the above-mentioned two-dimensional Hilbert space.

Another physical system D, which serves as a WPD, is coupled to the particle; see Fig. 1. The interaction between the particle and the WPD system is effectively a controlled unitary transformation $|0\rangle\langle 0| \otimes I_D + |1\rangle\langle 1| \otimes U$, where I_D and U are the identity and a unitary operator acting on the Hilbert space of the WPD system. Thus the evolution of the particle and the WPD system is governed by the unitary operator

$$U_{QD} = e^{i\phi/2}|\varphi_0\rangle\langle 0| \otimes I_D + e^{-i\phi/2}|\varphi_1\rangle\langle 1| \otimes U, \tag{3}$$

where $|\varphi_0\rangle \equiv B|0\rangle$ and $|\varphi_1\rangle \equiv B|1\rangle$. Let ρ be the state of the particle after it has passed through BS_1 and let ρ_D be the initial state of the WPD system D. Then the final state of the whole system is described by

$$\begin{aligned} \rho_f^{(QD)} &= U_{QD}(\rho \otimes \rho_D)U_{QD}^\dagger \\ &= \sum_{a,b=0}^{1} e^{i(b-a)\phi}|\varphi_a\rangle\langle a|\rho|b\rangle\langle\varphi_b| \otimes U^a \rho_D U^{\dagger b}. \end{aligned} \tag{4}$$

The probability that a particle is detected at the output port A is given by

$$\begin{aligned} p(\phi) &= \mathrm{tr}_{QD}\big[|0\rangle\langle 0|\rho_f^{(QD)}\big] = rw_+ + tw_- \\ &\quad + 2\sqrt{rt}|\langle 0|\rho|1\rangle \mathrm{tr}_D(\rho_D U)|\cos(\phi + \alpha + \delta), \end{aligned} \tag{5}$$

where α and δ are defined by

$$\langle 0|\rho|1\rangle = |\langle 0|\rho|1\rangle|e^{i\alpha} \tag{6}$$

and

$$\mathrm{tr}_D(\rho_D U) = |\mathrm{tr}_D(\rho_D U)|e^{-i\delta}. \tag{7}$$

One should notice that the ABS BS_2 can be regarded as a kind of which-path detector. Let $w_+ = \langle 0|\rho|0\rangle$ and $w_- = \langle 1|\rho|1\rangle$ be the probabilities for taking the two paths between the two beam splitters. Consider a simple case where $w_+ = w_- = 1/2$ so that the intermediate stage of the interferometer provides no *a priori* which-path knowledge. If the reflectivity r of BS_2 is larger than $1/2$ and the particle is detected at the counting detector A, immediately one can infer that the particle passes more likely through the path $|0\rangle$. So the ABS introduces additional *a priori* which-path knowledge, and meanwhile the fringe visibility of the interference pattern observed at the counting detector A will decrease. In a general case where $w_+ \neq w_-$, the *a priori* which-path knowledge provided by BS_2 is different for particles detected at one output port of the interferometer and for particles detected at the other output port, and accordingly the interference patterns emerging at the two output ports are also different.

Hence, in sharp contrast with the symmetric case, when we explore the duality relation between the which-path information and the fringe visibility of the interference pattern in the general MZI setup, *the two output ports should be treated separately*. In what follows we shall consider only particles detected at the output port A; the case for the output port B is similar.

First we consider the case where the unitary transformation U is turned off, i.e., $U = I_D$. For all particles detected at the output port A, an interference pattern can be observed when ϕ is varied, and the *a priori* fringe visibility reads [24]

$$\mathcal{V}_0 = \frac{p_{\max} - p_{\min}}{p_{\max} + p_{\min}} = \frac{2\sqrt{rt}|\langle 0|\rho|1\rangle|}{rw_+ + tw_-}, \tag{8}$$

where p is defined in Eq. (5), and the maximum and minimum values are calculated with respect to ϕ. At the same time, for all particles detected at the output port A, the probabilities of taking the two paths $|0\rangle$ and $|1\rangle$ are respectively

$$w_0 = \frac{rw_+}{rw_+ + tw_-}, \quad w_1 = \frac{tw_-}{rw_+ + tw_-}, \tag{9}$$

and the predictability of the two paths is then [25]

$$\mathcal{P} = |w_0 - w_1| = \frac{|rw_+ - tw_-|}{rw_+ + tw_-}. \tag{10}$$

From the positivity of the density operator ρ, i.e., $|\langle 0|\rho|1\rangle|^2 \leqslant w_+ w_-$, it is easy to check that

$$\mathcal{P}^2 + \mathcal{V}_0^2 = \frac{4rt|\langle 0|\rho|1\rangle|^2 + (rw_+ - tw_-)^2}{(rw_+ + tw_-)^2} \leqslant \frac{4rtw_+w_- + (rw_+ - tw_-)^2}{(rw_+ + tw_-)^2} = 1. \quad (11)$$

That is, the well-known duality relation between the predictability and the *a priori* fringe visibility previously derived in the standard MZI setup [3,5] still holds in the present case. In other words, the two quantities $\mathcal{P}$ and $\mathcal{V}_0$ belonging to the same ouput port are mutually complementary quantities.

At this point, we have some remarks.

(1) One might wonder whether the path predictability for *all* particles

$$\mathcal{P}_{\text{all}} = |w_+ - w_-| \quad (12)$$

is also complementary to $\mathcal{V}_0$. As is well known, the answer is yes for the standard MZI case where $\mathcal{P}$ reduces to $\mathcal{P}_{\text{all}}$. However, for the general MZI setup with an ABS, the answer is no. In fact, there exist cases where

$$\mathcal{P}_{\text{all}}^2 + \mathcal{V}_0^2 > 1. \quad (13)$$

We will prove this fact in the Appendix.

(2) Even in the special case where $w_+ = w_- = 1/2$ we still have $\mathcal{P} \neq 0$, that is, we still have *a priori* which-path knowledge, which does not arise from the discrepancy of the two paths but is provided by the ABS. In fact, this is the case considered in the experiment scheme in [26]. The ABS in the experiment plays also the role of a which-path detector so that the which-path information and the interference pattern are obtained via *the same* counting detector. What the setup in [26] tested simply is the trade-off relation between $\mathcal{P}$ and $\mathcal{V}_0$ in this special case.

Now we consider the case where the unitary transformation U is turned on. From Eqs. (5) and (8) it follows that the fringe visibility reads

$$\mathcal{V} = \mathcal{V}_0 |\text{tr}_D(\rho_D U)|. \quad (14)$$

A general strategy $\mathcal{S}$ to guess the path taken by the particle is to divide the outcomes W of the measurement of an observable $\hat{W}$ performed on the WPD system D into two disjoint sets S and $\bar{S}$. If $W \in S$, then one guesses the path to be 1; if $W \in \bar{S}$, then one guesses the path to be 2. The probability of guessing the right path is given by

$$\mathcal{L}_{\hat{W},S} = w_0 \sum_{W\in S} \langle W|\rho_D|W\rangle + w_1 \sum_{W\in\bar{S}} \langle W|U\rho_D U^\dagger|W\rangle. \quad (15)$$

Let us denote

$$\eta_S \equiv \sum_{W\in S} \langle W|\rho_D|W\rangle, \quad \eta_S^U \equiv \sum_{W\in S} \langle W|U\rho_D U^\dagger|W\rangle, \quad (16)$$

together with $\eta_{\bar{S}} = 1 - \eta_S$ and $\eta_{\bar{S}}^U = 1 - \eta_S^U$. The which-path *distinguishability* for the given strategy $\mathcal{S}$ is then

$$\mathcal{D}_S = 2\mathcal{L}_{\hat{W},S} - 1 = 2w_0\eta_S + 2w_1\eta_S^U - 1. \quad (17)$$

Recently, we showed that the duality relation in a standard MZI setup is intimately related to the joint measurement of two unsharp observables [15]. Generally, for a two-level system an unsharp observable is nothing else than a two-outcome positive-operator-valued measure. Two general unsharp observables $\{O_\pm\}$ and $\{O'_\pm\}$ of a qubit take the forms

$$O_\pm = \frac{I \pm (xI + \boldsymbol{m}\cdot\boldsymbol{\sigma})}{2}, \quad O'_\pm = \frac{I \pm (yI + \boldsymbol{n}\cdot\boldsymbol{\sigma})}{2}, \quad (18)$$

where I is the identity operator acting on the particle, and $\boldsymbol{\sigma}$ is the Pauli operator. The non-negativity imposes $|x| + |\boldsymbol{m}| \leqslant 1$ and so on. When $x = 0$ and $|\boldsymbol{m}| = 1$, $O_\pm$ are projectors of eigenstates of a sharp observable $\boldsymbol{m}\cdot\boldsymbol{\sigma}$. So generally an unsharp observable is the smeared version of a sharp observable. The above two unsharp observables are jointly measurable if and only if there exists a bivariate joint unsharp observable $\{M_{\mu\nu}\}$ whose outcomes can be so grouped that the marginals correspond exactly to the two given unsharp observables, i.e.,

$$O_\mu = \sum_{\nu=\pm} M_{\mu\nu}, \quad O'_\nu = \sum_{\mu=\pm} M_{\mu\nu}. \quad (19)$$

If $y = 0$, then the necessary and sufficient condition for their joint measurability reads [17]

$$\sqrt{(1+x)^2 - |\boldsymbol{m}|^2} + \sqrt{(1-x)^2 - |\boldsymbol{m}|^2} \geqslant \frac{2|\boldsymbol{m}\times\boldsymbol{n}|}{\sqrt{|\boldsymbol{m}|^2 - (\boldsymbol{m}\cdot\boldsymbol{n})^2}}. \quad (20)$$

In our general MZI setup, the unsharp observable $\mathcal{N} = \{N_0, N_1 = I - N_0\}$ corresponding to the interference pattern registered in the counting detector A is given by $p(\phi) = \text{tr}_Q(\rho N_0)$ for an arbitrary ρ. Thus we obtain

$$N_0 = \text{tr}_D[U_{QD}^\dagger(|0\rangle\langle 0| \otimes I_D)U_{QD}(I \otimes \rho_D)] = \frac{I + \boldsymbol{n}\cdot\boldsymbol{\sigma}}{2} \quad (21)$$

with

$$\boldsymbol{n} = \left[2\frac{\mathcal{V}}{\mathcal{V}_0}\sqrt{rt}\cos(\phi+\delta), 2\frac{\mathcal{V}}{\mathcal{V}_0}\sqrt{rt}\sin(\phi+\delta), 2r-1\right]. \quad (22)$$

For a given strategy $\mathcal{S}$, the probability of finding the WPD system D in one of the eigenstates in S is given by $\text{tr}_Q(M_S\rho)$ for an arbitrary ρ where

$$M_S = \sum_{W\in S} \text{tr}_D[U_{QD}^\dagger(I \otimes |W\rangle\langle W|)U_{QD}(I \otimes \rho_D)]. \quad (23)$$

Thus the unsharp observable corresponding to the observable $\hat{W}$ and the strategy $\mathcal{S}$ is $\mathcal{M} = \{M_S, I - M_S\}$ with

$$M_S = \frac{1}{2}\left[(\eta_S + \eta_S^U)I + (\eta_S - \eta_S^U)\sigma_z\right] = \frac{I + xI + \boldsymbol{m}\cdot\boldsymbol{\sigma}}{2}, \quad (24)$$

in which notations in Eq. (16) have been used and

$$x = \eta_S + \eta_S^U - 1, \quad \boldsymbol{m} = (0,0,\eta_S - \eta_S^U). \quad (25)$$

It is clear that as long as $r \neq 1/2$ we have $\boldsymbol{n}\cdot\boldsymbol{m} \neq 0$, i.e., the two unsharp observables jointly measured in the general MZI setup are nonorthogonal, in contrast with a standard MZI setup [15]. From the joint measurement condition Eq. (20) it follows that

$$\sqrt{\eta_S \eta_S^U} + \sqrt{\eta_{\bar{S}} \eta_{\bar{S}}^U} \geqslant \frac{\mathcal{V}}{\mathcal{V}_0}. \tag{26}$$

Using Eq. (17) and similarly to the derivation in [15], we obtain

$$\mathcal{D}_S^2 + \frac{1-\mathcal{P}^2}{\mathcal{V}_0^2}\mathcal{V}^2 \leqslant 1 - \gamma_S^2, \tag{27}$$

where $\gamma_S = 2|w_0\sqrt{\eta_S \eta_{\bar{S}}} - w_1\sqrt{\eta_S^U \eta_{\bar{S}}^U}|$. By maximizing over all possible strategies we obtain a duality inequality in the same form as Eq. (1).

To conclude, we have shown how to illustrate quantitatively the wave-particle duality in a general MZI scenario with an ABS. It turns out that the nonorthogonality of two unsharp observables involved is caused by the ABS. We have employed the condition for joint measurability of the two unsharp observables to obtain a duality inequality.

It would be interesting to determine the experimental setup necessary to measure jointly a pair of most general unsharp observables of a two-level system, although the condition for their joint measurability has been established [17]. Conversely, does the most general joint measurability condition imply a "thorough" complementarity relation with realizable and observable effects not limited by the known duality inequalities? These questions remain open for further research. The answers may lead to a device-independent duality inequality.

This work was supported by the NNSF of China, the CAS, and the National Fundamental Research Program (under Grant No. 2011CB921300).

APPENDIX

In this appendix, we present the proof that there exist cases where the inequality in Eq. (13) holds, and therefore $\mathcal{P}_{\text{all}}$ and $\mathcal{V}_0$ are not mutually complementary quantities.

It is easy to check that there exist many cases, e.g.,

$$w_- > 1/2, \quad r \in \left(\frac{1}{2}, \frac{w_-^2}{2w_-^2 - 2w_- + 1}\right), \tag{A1}$$

or

$$w_- < 1/2, \quad r \in \left(\frac{w_-^2}{2w_-^2 - 2w_- + 1}, \frac{1}{2}\right), \tag{A2}$$

where the following inequality holds:

$$\frac{\sqrt{rt}}{rw_+ + tw_-} > 1. \tag{A3}$$

In these cases, by recalling the form of $\mathcal{V}_0$ in Eq. (8), we have

$$\mathcal{V}_0 > 2|\langle 0|\rho|1\rangle|. \tag{A4}$$

Suppose ρ is a pure state. We have

$$\mathcal{P}_{\text{all}}^2 + (2|\langle 0|\rho|1\rangle|)^2 = 1. \tag{A5}$$

Thus, in the cases mentioned above we have

$$\mathcal{P}_{\text{all}}^2 + \mathcal{V}_0^2 > \mathcal{P}_{\text{all}}^2 + (2|\langle 0|\rho|1\rangle|)^2 = 1. \tag{A6}$$

[1] N. Bohr, Nature (London) **121**, 580 (1928).
[2] W. K. Wootters and W. H. Zurek, Phys. Rev. D **19**, 473 (1979).
[3] D. M. Greenberger and A. Yasin, Phys. Lett. A **128**, 391 (1988).
[4] G. Jaeger, A. Shimony, and L. Vaidman, Phys. Rev. A **51**, 54 (1995).
[5] B.-G. Englert, Phys. Rev. Lett. **77**, 2154 (1996).
[6] B.-G. Englert and J. A. Bergou, Opt. Commun. **179**, 337 (2000).
[7] H. Martens and W. M. de Muynck, Found. Phys. **20**, 255 (1990).
[8] W. M. de Muynck, *Foundations of Quantum Mechanics: An Empiricist Approach* (Kluwer Academic, Dordrecht, 2002).
[9] P. Busch, Phys. Rev. D **33**, 2253 (1986).
[10] P. Busch and C. Shilladay, Phys. Rep. **435**, 1 (2006).
[11] Y. Kurotani, T. Sagawa, and M. Ueda, Phys. Rev. A **76**, 022325 (2007).
[12] T. Sagawa and M. Ueda, Phys. Rev. A **77**, 012313 (2008).
[13] P. Busch and T. Heinosaari, Quantum Inf. Comput. **8**, 0797 (2008).
[14] T. Heinosaari, P. Stano, and D. Reitzner, Int. J. Quantum Inf. **6**, 975 (2008).
[15] N.-L. Liu, L. Li, S. Yu, and Z.-B. Chen, Phys. Rev. A **79**, 052108 (2009).
[16] P. Stano, D. Reitzner, and T. Heinosaari, Phys. Rev. A **78**, 012315 (2008).
[17] S. Yu, N.-L. Liu, L. Li, and C. H. Oh, Phys. Rev. A **81**, 062116 (2010).
[18] P. Busch and H.-J. Schmidt, Quantum Inf. Process. **9**, 143 (2010).
[19] E. Andersson, S. M. Barnett, and A. Aspect, Phys. Rev. A **72**, 042104 (2005).
[20] T. Brougham, E. Andersson, and S. M. Barnett, Phys. Rev. A **80**, 042106 (2009).
[21] T. Brougham, E. Andersson, and S. M. Barnett, Phys. Rev. A **73**, 062319 (2006).
[22] A. Ferraro and M. G. A. Paris, Open Syst. Inf. Dyn. **14**, 149 (2007).
[23] M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Phys. Rev. Lett. **103**, 230402 (2009).
[24] Similarly, for all particles detected at the output port B, the *a priori* fringe visibility is $\mathcal{V}_0' = 2\sqrt{rt}|\langle 0|\rho|1\rangle|/(tw_+ + rw_-)$. Obviously, $\mathcal{V}_0' = \mathcal{V}_0 = 2|\langle 0|\rho|1\rangle|$ when and only when BS_2 is symmetric.
[25] Similarly, the path predictability for particles detected at the output port B is $\mathcal{P}' = |tw_+ - rw_-|/(tw_+ + rw_-)$. Obviously, $\mathcal{P}' = \mathcal{P} = |w_+ - w_-|$ when and only when BS_2 is symmetric.
[26] V. Jacques, E. Wu, F. Grosshans, F. Treussart, P. Grangier, A. Aspect, and J. F. Roch, Phys. Rev. Lett. **100**, 220402 (2008).