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Three-fluid hydrodynamics of spin-1 Bose-Einstein condensates

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We study excitations of the spin-1 Bose gas at finite temperatures and in the presence of a not-so-strong magnetic field or, equivalently, when the gas sample is partially polarized. Motivated by the success of two-fluid hydrodynamics of scalar superfluids we develop a three-fluid hydrodynamic description to treat the low-frequency and long-wavelength excitations of the spin-1 Bose gas. We derive coupled linear hydrodynamic equations of the three sounds and evaluate them numerically in a self-consistent mean-field approximation valid for dilute gas in the intermediate and critical temperature regions. In this latter region we identify the critical mode.

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I. INTRODUCTION

Multicomponent dilute gases are fascinating systems that can exhibit—at sufficiently low temperatures—phase transitions where the interplay between a superfluid transition and the breaking of some other continuous symmetry can be observed. Maybe the simplest example is the Bose-Einstein condensation of a spin-1 Bose gas [1,2], where the system has a global SU(2) (spin rotation) symmetry as well as the usual U(1). For the weakly interacting gas, Bose-Einstein condensation always infers the breaking of the SU(2) symmetry, too, because the three-component condensate wave function violates the spin rotation symmetry, even when spontaneous magnetization is zero. Depending on the sign and magnitude of the spin-dependent part of the interaction a magnetic phase transition can occur prior to or at the onset of Bose condensation [3–7]. Note that, for relatively large interactions, it has been shown recently that a pairing transition occurs instead of Bose condensation [8].

In the Bose condensed phase, the global U(1) symmetry is always broken and correspondingly a Goldstone mode has to exist among the elementary excitations of the system. Although the spin rotation SU(2) symmetry is also broken by the condensate wave function two possibilities still remain: (1) A U(1) symmetry—the rotation around the z axis—remains a symmetry, at least with the combination of the global U(1) gauge symmetry. Or (2), even this kind of z-axis rotation symmetry is broken and the ground state has no remaining symmetry at all. In this latter case another Goldstone mode must exist. This kind of double symmetry breaking was illustrated in the random phase approximation (RPA) for the polar spin-1 gas earlier, and the Goldstone modes have been identified among the quasiparticle excitations [9].

However, identifying the critical mode and studying its properties can be achieved in the hydrodynamic approximation which can be continued by scale invariance to the critical region. Such a hydrodynamic approximation—the celebrated 2-fluid hydrodynamics of superfluids—is the basis of understanding the critical properties of liquid ^{4}He even around the λ transition and also serve for understanding the behavior of dilute, scalar Bose gases when interparticle collisions take place sufficiently frequently [10–12]. In this paper we will follow the route originally developed for superfluid ^{3}He [13] and derive a three-fluid model to describe the low-energy excitations of the system. We will evaluate the thermodynamic quantities, entering the hydrodynamic equations, in a self-consistent mean-field approximation that treats the noncondensed atoms as an ideal Bose gas above the spinor Bose-Einstein condensate (BEC).

The rest of the paper is organized as follows: In Sec. II, the model is introduced with the help of the grand canonical Hamiltonian, and its symmetry properties are discussed. In Sec. III, the two possible phases and the phase diagram of the system are given in the magnetic-field–temperature plane. Three-fluid hydrodynamics is developed for the partially polarized phase in Sec. IV, with some details of the derivation moved to Appendix A. The thermodynamic quantities are evaluated in the Bogoliubov-Hartree approximation and are presented together with the results for the hydrodynamic excitations also in Sec. IV. The thermodynamic potential is derived in Appendix B in the framework of the model. Finally, we summarize the results in Sec. V.

II. HAMILTONIAN OF SPIN-1 BOSE GAS

We consider a system of homogeneous, weakly interacting, dilute, spin-1 Bose gas at ultralow temperatures and in a homogeneous magnetic field. The interparticle interaction is modelled by s-wave scattering (i.e., we neglect the relatively weak dipolar interaction of the gas). The grand-canonical Hamiltonian of the system takes the following form:

$$\mathcal{H} = \sum_{\substack{\vec{k}\\ r,s}} [(e_{\vec{k}} - \mu)\delta_{rs} - g\mu_{\mathrm{B}} B (F_z)_{rs}] a_r^\dagger(\vec{k}) a_s(\vec{k}) + \frac{1}{2V} \sum_{\substack{\vec{k}_1+\vec{k}_2=\vec{k}_3+\vec{k}_4\\ r,s,r',s'}} a_{r'}^\dagger(\vec{k}_1) a_r^\dagger(\vec{k}_2) V_{rs}^{r's'} a_s(\vec{k}_3) a_{s'}(\vec{k}_4), \quad (1)$$

where $a_r(\vec{k})$ and $a_r^\dagger(\vec{k})$ are the annihilation and creation operators, respectively, of plane-wave states with momentum $\vec{k}$ and spin projection r. The spin index r refers to the eigenvalue of the z component of the spin operator and can take values from $+,0,-$. Correspondingly, $F_z = \mathrm{diag}(1,0,-1)$ is a 3×3 diagonal matrix. In Eq. (1), $e_{\vec{k}} = \hbar^2 k^2/(2M)$ refers to the kinetic energy of an atom, μ refers to the chemical potential, g refers to the gyromagnetic ratio, μ_{B} refers to the Bohr magneton, and B refers to the modulus of the homogeneous magnetic field. V is the volume of the system and $V_{rs}^{r's'}$ the amplitude of the two-particle interaction, given for spin-1 bosons by [1,2,14]

$$V_{rs}^{r's'} = c_n \delta_{rs} \delta_{r's'} + c_s (\vec{F})_{rs} (\vec{F})_{r's'}, \tag{2}$$

with $c_n = 4\pi\hbar^2(a_0 + 2a_2)/(3M)$, and $c_s = 4\pi\hbar^2(a_2 - a_0)/(3M)$. The parameters a_0 and a_2 are the scattering lengths in the total hyperfine spin channel zero and two, respectively. The constant $c_n > 0$, while c_s can be both positive or negative, depending on the relative values of a_0 and a_2. If $c_s < 0$ the interaction tends to align the spins, while for $c_s > 0$ a zero net spin is energetically favorable (in the absence of a magnetic field). For this reason systems with $c_s < 0$ are referred to as ferromagnetic systems, while those with $c_s > 0$ are called polar gases [1]. For example, the ultracold gas of ^{87}Rb atoms in the $f = 1$ hyperfine state is ferromagnetic [15], while the ^{23}Na gas (also in the $f = 1$ hyperfine state) is polar [16].

The Hamiltonian (1) is invariant under the global U(1) symmetry transformation, which leads to particle number conservation. In this case, when the external magnetic field is zero, the Hamiltonian is invariant also under the global spin rotation SU(2), which leads to the conservation of the total magnetization (spin) of the system. When a magnetic field is present (breaking explicitly the rotational invariance), only another U(1) symmetry remains; namely, the rotation around the z axis. These two U(1) symmetries can be described either by $a_\pm \to a_\pm e^{i\varphi_\pm}$ together with $a_0 \to a_0 e^{i\frac{1}{2}(\varphi_+ + \varphi_-)}$, or equivalently by $a_r \to a_r e^{i(\phi + r\theta)}$. It is easy to see that $\phi = \frac{1}{2}(\varphi_+ + \varphi_-)$ corresponding to the symmetry generated by the particle number, and $\theta = \frac{1}{2}(\varphi_+ - \varphi_-)$ corresponding to the symmetry responsible for the conservation of the z component of the total spin. The constraint due to the conservation of the total magnetization can be resolved similarly as that for the conservation of particle number [i.e., with the introduction of a Lagrange multiplier in the Hamiltonian (1)]. This multiplier shows up in the same way as the magnetic field does, therefore an effective magnetic field can be introduced as a sum of the external magnetic field plus the Lagrange multiplier. In the following, B will mean this kind of effective magnetic field even when the physical magnetic field is zero. In the following, we assume that either such is the case, or the external magnetic field is so small that the quadratic Zeeman effect can be neglected.

III. PHASES OF SPIN-1 POLAR BOSE GAS

The weakly interacting spin-1 Bose gas has two types of Bose-Einstein condensed phases. At zero temperature these two phases are obtained from minimizing the following energy functional [1,2,17]:

$$E[\zeta] = -b f_z[\zeta] + c_s \mathbf{f}^2[\zeta], \tag{3}$$

where $b = g\mu_B B$, $f_i[\zeta] = \zeta^* F_i \zeta$ is the magnetization of the condensate calculated with the help of the condensate spinor $\zeta = (\zeta_+,\zeta_0,\zeta_-)$. One of the phases is the fully polarized Bose condensed phase that is realized by the thermodynamic ground state of a gas with $c_s < 0$ (ferromagnetic coupling), or even when $c_s > 0$ but the gas sample is originally polarized [or equivalently when the (effective) magnetic field is sufficiently large]. This phase is characterized by $f_z = 1$, $f_x = f_y = 0$. The condensate wave function is single component [$\zeta = (1,0,0)$], and correspondingly only one of the two global symmetries are broken; namely, the symmetry parametrized by the φ_+ angle. More generally, and especially at finite temperature, we call the P1 phase the phase where the condensate is fully polarized and the transformation characterized by the φ_- angle remains a symmetry while the one characterized by the φ_+ angle is broken.

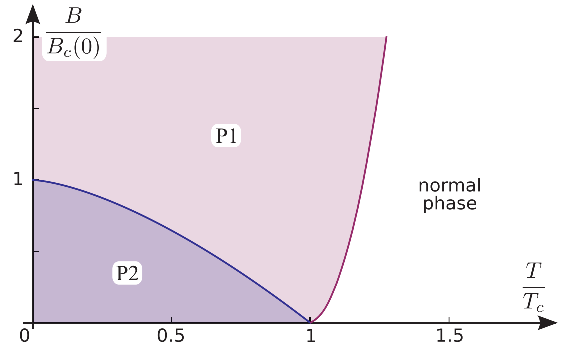

FIG. 1. (Color online) The phase diagram of the polar spin-1 Bose gas in the Bogoliubov-Hartree approximation. $B_c(0)$ is the critical magnetic field at zero temperature, above which the condensate is fully polarized. Here we have chosen the coupling constants as $c_n = 0.2 M k_B T_c/\rho$ and $c_s = 0.4 c_n$.

In the case of a *polar gas* and starting from the P1 phase, when the effective magnetic field is lowered, the system undergoes a "second" Bose-Einstein condensation. The condensate wave function is no longer single component and the other remaining global symmetry becomes broken, too [2]. At zero temperature, by minimizing the energy functional (3), one finds that $f_z = b/c_s$ and $f_x = f_y = 0$. From the latter equations it follows directly that $\zeta_0 = 0$ for $b > 0$, and the condensate spinor is $\zeta = (\zeta_+,0,\zeta_-)$. Generally, and at finite temperature the phase, where the condensate is no longer single component and both of the two global gauge symmetries are broken will be called P2.

We illustrate the phase diagram of the polar spin-1 gas in Fig. 1 in the self-consistent Bogoliubov-Hartree approximation [9], described in Appendix B. Within the framework of this theory one can see that the condensate wave function in the P2 phase remains with only two nonzero components even at finite temperature. The case when $\zeta_0 = 0$ in equilibrium might be interesting in general since it is very intuitive. In the presence of a magnetic field the magnetization of the system points in the direction of the magnetic field. By assuming that the magnetization of the condensate itself points in the direction of the magnetic field, the x and y components of the condensate magnetization have to vanish and therefore $\zeta_0 = 0$. Note that the local equilibrium states occurring in the hydrodynamic description of fluctuations might contain condensates in the $m = 0$ Zeeman level, too, even when $\zeta_0 = 0$ in equilibrium.

Also note that the point $(T = T_c, B = 0)$ is a special one, where two critical lines cross each other, and a renormalization group analysis around $4 - \epsilon$ dimensions to leading order in ϵ shows that there is no stable fixed point [18].

IV. THREE-FLUID HYDRODYNAMICS IN P2 PHASE

A. Equations of motion

The system has two conserved global charges associated with the symmetry of the internal states; namely, the total number of particles and the z component of the total magnetization, therefore two continuity equations arise:

$$\partial_t \rho + \vec{\nabla} \cdot \vec{g}_\rho = 0, \tag{4a}$$

$$\partial_t \sigma + \vec{\nabla} \cdot \vec{g}_\sigma = 0, \tag{4b}$$

where ρ is the mass density and σ is the spin density (multiplied by the mass of the particles M). Correspondingly, $\vec{g}_\rho$ is the total current (momentum) density and $\vec{g}_\sigma$ is the spin current density.

Directly following from Galilean invariance generalized from the treatment of the scalar problem [19] we have two independent superfluid components with their velocities obeying

$$M \partial_t \vec{v}_{s,+} + \vec{\nabla}(\mu + b) = 0, \tag{5a}$$

$$M \partial_t \vec{v}_{s,-} + \vec{\nabla}(\mu - b) = 0. \tag{5b}$$

Note that $\vec{v}_{s,0}$ is not independent; that is, $\vec{v}_{s,0} = (\vec{v}_{s,+} + \vec{v}_{s,-})/2$, in harmony with the gauge transformation discussed in Sec. II. The effect of inhomogeneities in the chemical potential μ and in the magnetic energy b is nicely exhibited in an alternative representation as follows: Instead of $\vec{v}_{s,+}$ and $\vec{v}_{s,-}$ one can introduce their linear combinations

$$\vec{v}_\rho = \frac{\vec{v}_{s,+} + \vec{v}_{s,-}}{2}, \tag{6a}$$

$$\vec{v}_\sigma = \frac{\vec{v}_{s,+} - \vec{v}_{s,-}}{2}. \tag{6b}$$

Clearly, $\vec{v}_\rho$ is the superfluid velocity for the density current and $\vec{v}_\sigma$ is the superfluid velocity for the spin current. They obey the following equations of motion:

$$M \partial_t \vec{v}_\rho + \vec{\nabla} \mu = 0, \tag{7a}$$

$$M \partial_t \vec{v}_\sigma + \vec{\nabla} b = 0. \tag{7b}$$

Momentum conservation yields the well-known Euler equation

$$\partial_t \vec{g}_\rho + \vec{\nabla} p = 0, \tag{8}$$

with p being the pressure field. Another equation of motion is provided by the entropy transport of the normal component of the fluid:

$$\partial_t s + \vec{\nabla} \cdot (s \vec{v}_n) = 0, \tag{9}$$

with s being the entropy density of the gas.

To close the set of hydrodynamic equations one has to relate the complete density and spin currents to the 3 components of the superfluid (i.e., to the normal component and to the two distinct superfluid components). The only linear relation consistent with both the Maxwell relations detailed in Appendix A and the zero-temperature results one has to choose them to be

$$\vec{g}_\rho = \rho_n \vec{v}_n + \rho_s \vec{v}_\rho + \sigma_s \vec{v}_\sigma, \tag{10a}$$

$$\vec{g}_\sigma = \sigma_n \vec{v}_n + \sigma_s \vec{v}_\rho + \rho_s \vec{v}_\sigma, \tag{10b}$$

with $\rho = \rho_n + \rho_s$ and $\sigma = \sigma_n + \sigma_s$. In the special case when $\zeta_0 = 0$ one can also split ρ_s and σ_s according to

$$\rho_s = \rho_{s,+} + \rho_{s,-}, \tag{11a}$$

$$\sigma_s = \rho_{s,+} - \rho_{s,-}, \tag{11b}$$

where $\rho_{s,+}$ and $\rho_{s,-}$ can be interpreted as the superfluid densities in spin components $+$ and $-$.

We turn now to the discussion of the hydrodynamic waves.

B. Wave equations

With the help of the equation of continuity for the density (4a) and with the Euler equation (8), one can derive the first of the three coupled wave equations:

$$\partial_t^2 \rho - \nabla^2 p = 0. \tag{12}$$

The second equation follows from the continuity equation for the spin density (4b) by combining it with the Euler equation (8) and the equations of motion for the superfluid velocities (7) and the Gibbs-Duham relation. It reads

$$\begin{aligned} 0 = \; & \partial_t^2 \sigma - \frac{\sigma}{\rho} \nabla^2 p - \tilde{s} \frac{\sigma_n \rho_s - \rho_n \sigma_s}{\rho_n} \nabla^2 T \\ & - \frac{\sigma(\sigma_n \rho_s - \rho_n \sigma_s) - \rho(\sigma_n \sigma_s - \rho_n \rho_s)}{\rho \rho_n M} \nabla^2 b, \end{aligned} \tag{13}$$

where we have introduced $\tilde{s} = s/\rho$, which is the entropy per unit mass. The final equation is obtained from the entropy continuity equation (9) in combination with the Euler equation (8) and the equations of motion of the superfluid velocities (7):

$$\partial_t^2 \tilde{s} - \tilde{s}^2 \frac{\rho_s}{\rho_n} \nabla^2 T - \tilde{s} \frac{\sigma \rho_s - \rho \sigma_s}{\rho \rho_n M} \nabla^2 b = 0. \tag{14}$$

Close to equilibrium the fluctuation of the three densities can be expressed with the help of the fluctuations of the intensive parameters

$$\partial_t^2 \rho = \left(\frac{\partial \rho}{\partial p} \right) \partial_t^2 p + \left(\frac{\partial \rho}{\partial T} \right) \partial_t^2 T + \left(\frac{\partial \rho}{\partial b} \right) \partial_t^2 b, \tag{15a}$$

$$\partial_t^2 \sigma = \left(\frac{\partial \sigma}{\partial p} \right) \partial_t^2 p + \left(\frac{\partial \sigma}{\partial T} \right) \partial_t^2 T + \left(\frac{\partial \sigma}{\partial b} \right) \partial_t^2 b, \tag{15b}$$

$$\partial_t^2 \tilde{s} = \left(\frac{\partial \tilde{s}}{\partial p} \right) \partial_t^2 p + \left(\frac{\partial \tilde{s}}{\partial T} \right) \partial_t^2 T + \left(\frac{\partial \tilde{s}}{\partial b} \right) \partial_t^2 b. \tag{15c}$$

Combining Eqs. (12)–(15), one gets a closed set of wave equations for the quantities p, T, and b describing density waves, temperature waves, and magnetization waves. When looking for plane-wave solutions one uses

$$\begin{pmatrix} p \\ T \\ b \end{pmatrix} = \begin{pmatrix} p_0 \\ T_0 \\ b_0 \end{pmatrix} e^{i(\vec{k} \cdot \vec{r} - ckt)}, \tag{16}$$

and the plane-wave amplitudes and dispersions are obtained from the eigenvector problem:

$$\mathbf{M}\begin{pmatrix} p_0 \\ T_0 \\ b_0 \end{pmatrix} = 0, \tag{17}$$

where

$$\mathbf{M} = \begin{pmatrix} \left(\frac{\partial\rho}{\partial p}\right)c^2 - 1 & \left(\frac{\partial\rho}{\partial T}\right)c^2 & \left(\frac{\partial\rho}{\partial b}\right)c^2 \\ \left(\frac{\partial\sigma}{\partial p}\right)c^2 - \frac{\sigma}{\rho} & \left(\frac{\partial\sigma}{\partial T}\right)c^2 - \tilde{s}\frac{\sigma_n\rho_s - \rho_n\sigma_s}{\rho_n} & \left(\frac{\partial\sigma}{\partial b}\right)c^2 - \frac{\sigma(\sigma_n\rho_s - \rho_n\sigma_s) - \rho(\sigma_n\sigma_s - \rho_n\rho_s)}{\rho\rho_n M} \\ \left(\frac{\partial\tilde{s}}{\partial p}\right)c^2 & \left(\frac{\partial\tilde{s}}{\partial T}\right)c^2 - \tilde{s}^2\frac{\rho_s}{\rho_n} & \left(\frac{\partial\tilde{s}}{\partial b}\right)c^2 - \tilde{s}\frac{\sigma\rho_s - \rho\sigma_s}{\rho\rho_n M} \end{pmatrix}. \tag{18}$$

The dispersion follows from $\det\mathbf{M} = 0$ which is a cubic equation for c^2 and gives the velocities of the first sound, the quadrupolar mode, and the second sound.

C. Evaluation of thermodynamic quantities

In experiments made with ^{23}Na atoms the typical ratio of the coupling constants is $\epsilon \equiv c_s/c_n \approx 3 \times 10^{-2} \ll 1$. In this limit the cubic equation for the speed of sound resulting from the vanishing of the determinant of Eq. (18) factorizes into a quadratic equation: the first- and second-sound modes, and to a separate linear equation with a solution on the order of ϵ: the quadrupolar spin-wave mode. This kind of separation is a generic feature of the dilute Bose gas and results from the behavior of the thermodynamic derivatives appearing in Eq. (18). We are going to illustrate this feature in the Bogoliubov-Hartree approximation but keep the value of ϵ general in order to study the hybridization of the different sound waves.

The advantage of the Bogoliubov-Hartree approximation is that it is an extension of the Bogoliubov approximation to finite temperatures with accounting interaction effects with noncondensed atoms in a way which leads to a conserving and gapless approximation together with a continuous phase transition [9]. Here we give only the results and sketch the derivation in Appendix B, and also refer to earlier papers where the backbone of the perturbation theory is worked out [3,9].

In this approximation the chemical potential and the magnetic field are expressed as

$$\mu = c_n\frac{\rho}{M}, \tag{19a}$$

$$b = c_s\frac{\sigma}{M}. \tag{19b}$$

The normal density of spin component r reads as

$$\rho_{n,r} = \frac{M\zeta(3/2)}{\Lambda^3}, \tag{20}$$

where $\Lambda = \sqrt{2\pi\hbar^2/(Mk_BT)}$ is the de Broglie wavelength, and $\zeta(s)$ is the Riemann zeta function. The total densities then are

$$\rho_n = \rho_{n,+} + \rho_{n,0} + \rho_{n,-} = \frac{3M\zeta(3/2)}{\Lambda^3} = \rho t^{3/2}, \tag{21a}$$

$$\rho_s = \rho_{s,+} + \rho_{s,-} = \rho(1 - t^{3/2}), \tag{21b}$$

$$\sigma_n = \rho_{n,+} - \rho_{n,-} = 0, \tag{21c}$$

$$\sigma_s = \rho_{s,+} - \rho_{s,-} = \sigma = \frac{Mb}{c_s}, \tag{21d}$$

where $t = T/T_c$ is the reduced temperature. T_c is the temperature of Bose-Einstein condensation of the spin-1 Bose gas in zero magnetic field (or with zero magnetization)

$$T_c = \frac{2\pi\hbar^2}{k_BM}\left(\frac{\rho}{3\zeta(3/2)M}\right)^{2/3}. \tag{22}$$

The grand canonical thermodynamic potential in this approximation is

$$\Phi(T,V,\mu) = -\frac{k_BTV}{\Lambda^3}3\zeta(5/2) - \frac{V}{2M^2}(c_n\rho^2 + c_s\sigma^2). \tag{23}$$

The pressure $p = -\Phi/V$, and its thermodynamic derivatives are

$$\left(\frac{\partial p}{\partial T}\right)_{\rho,b} = s = \frac{5}{2}\frac{k_B}{\Lambda^3}3\zeta(5/2), \tag{24a}$$

$$\left(\frac{\partial p}{\partial\rho}\right)_{T,b} = c_n\frac{\rho}{M}, \tag{24b}$$

$$\left(\frac{\partial p}{\partial b}\right)_{T,\rho} = \frac{\sigma}{M}, \tag{24c}$$

$$\left(\frac{\partial\sigma}{\partial T}\right)_{\rho,b} = 0, \tag{24d}$$

$$\left(\frac{\partial\sigma}{\partial\rho}\right)_{T,b} = 0, \tag{24e}$$

$$\left(\frac{\partial\sigma}{\partial b}\right)_{T,\rho} = \frac{M}{c_s}. \tag{24f}$$

The specific heat under constant volume is

$$c_V = T\left(\frac{\partial s}{\partial T}\right)_{\rho,b} = \frac{3}{2}s. \tag{25}$$

D. Results and discussion

The three solutions of $\det\mathbf{M} = 0$ provides the three speeds of sound of the hydrodynamic modes, which we will call here the first, second, and third sounds. To identify which speed corresponds to which physical excitation mode one has to look for the eigenvectors [i.e., the solution of Eq. (17)] corresponding to the specific speed of sound substituted into the matrix $\mathbf{M}$. In general, the hydrodynamic modes hybridize,

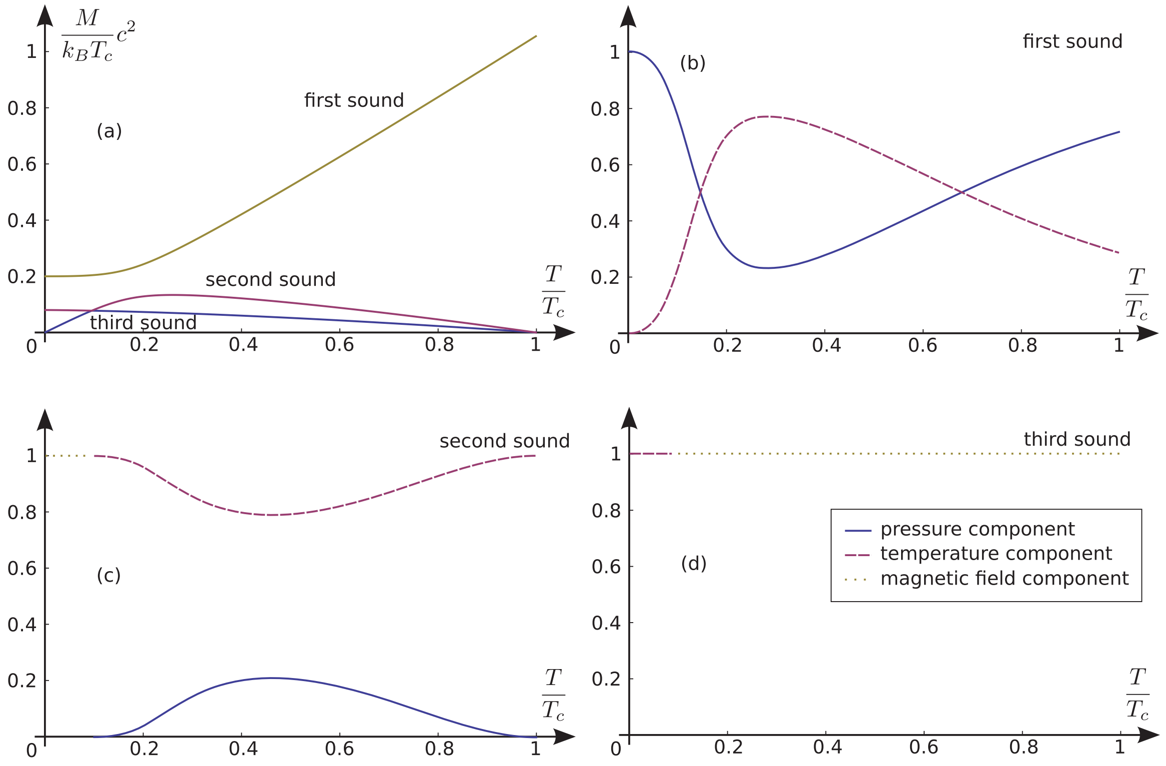

FIG. 2. (Color online) Speeds of sound versus temperature at $B = 0$ (a). The relative values of the three components of the eigenvectors corresponding to the given hydrodynamic mode: the first sound (b), the second sound (c), and the third sound (d). The coupling constants are chosen to be the same as in Fig. 1.

and their roles exchange. In Fig. 2(a) we plot the three speeds of sound at the whole temperature range for zero magnetization (or equivalently at zero magnetic field). At this specific choice of zero magnetization the phase transition happens at T_c and the P2 phase goes directly to the normal phase (compare with Fig. 1 at $B = 0$). This transition is characterized by the simultaneous vanishing of the condensates both in $r = +$ and in $r = -$. Correspondingly, two Goldstone modes are present, which are the second and third sounds. The speed of the first sound stays finite at this transition. In Figs. 2(b)–2(d) we demonstrate the amplitudes of the fluctuations of the sound modes. It can be seen that the first sound is basically a pressure wave when we are far away from the point of the hybridization of the first and second sounds. The second sound is mainly a temperature wave after the point of crossing with the third sound and not too close to the hybridization with the first sound. The third sound is a typical magnetic field wave after the crossing with the second sound. It is also very nicely demonstrated that the nature of the second and third sound changes at the crossing point.

In Fig. 3 the same quantities are plotted but for finite magnetic field (or finite magnetization). It is visible from the figure that it is the quadrupolar mode that goes soft at the P1-P2 transition point; thus it plays the role of the Goldstone mode. The second sound stays finite, since it only softens when the BEC gives way to the normal phase; that is, only at the P1-normal transition point, which is not discussed in this paper. The first sound remains unmodified by the change of the magnetic field, since it is basically a pressure wave. The second and third sounds are hybridizing in this case and the crossing is resolved to an avoided crossing in the presence of the magnetic field; the explicit breaking of the rotational symmetry.

In the limit when $\epsilon = c_s/c_n \ll 1$, the pressure and temperature waves separate from the quadrupolar spin wave. This limit can be treated by looking for solutions of $\det \mathbf{M} = 0$ at the order of ϵ. To leading order this gives the following solution:

$$c^2 = \frac{k_B T_c \epsilon \gamma_c}{M} \frac{1 - \omega^2 + t^3 - 2t^{3/2}}{1 - t^{3/2}}, \tag{26a}$$

which is the dispersion of the quadrupolar spin wave. For the other solutions one looks for $c \sim O(1)$ and to leading order gets

$$c^4 - \frac{k_B T_c}{M}\left(\gamma_c + \frac{2}{3\sqrt{t}} \frac{sM}{k_B \rho}\right) c^2 + \frac{2}{3}\left(\frac{k_B T_c}{M}\right)^2 \gamma_c \left(\frac{1}{\sqrt{t}} - t\right) = 0, \tag{26b}$$

which is quadratic in c^2 and its solution gives the first and second sounds which formally agrees with that of the scalar Bose gas [11]. Note that in the $t \to 0$ limit we get back the solutions $c_1^2 = k_B T_c \gamma_c / M$ and $c_2^2 = 0$. Thus at low temperature the lower of the two solutions crosses the quadrupolar mode of Eq. (26a), which stays finite at zero temperature. We have introduced the following dimensionless parameters: $\gamma_c = c_n \rho / (M k_B T_c)$, $\omega = Mb/(c_s \rho) = \sigma/\rho$.

We have evaluated the thermodynamic derivatives in the Bogoliubov-Hartree approximation, which gives even quantitatively good interpolation between zero temperature and the

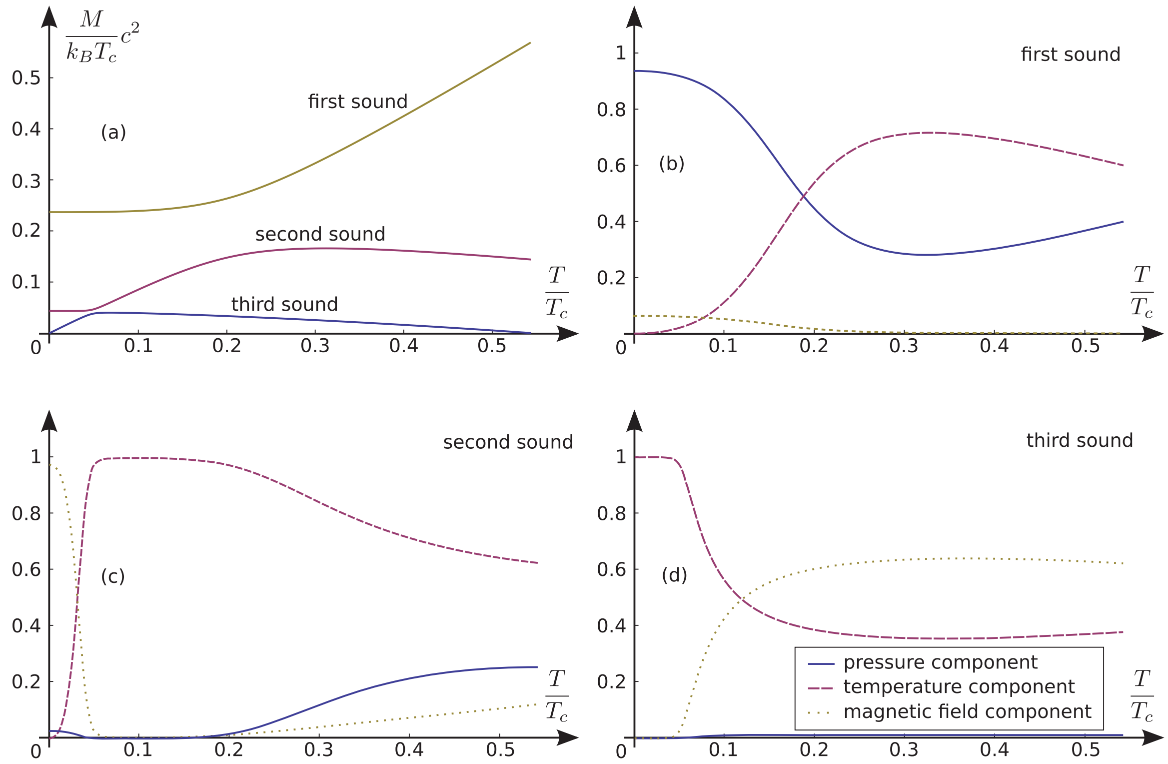

FIG. 3. (Color online) Speeds of sound versus temperature at $B = 0.6B_c(0)$ (a). The relative values of the three components of the eigenvectors corresponding to the given hydrodynamic mode: the first sound (b), the second sound (c), and the third sound (d). The coupling constants are chosen to be the same as in Fig. 1.

critical point as far as equilibrium properties are concerned. As a matter of fact, at intermediate temperatures also for the scalar gas we have evaluated the two fluid hydrodynamic equations in the Bogoliubov-Hartree approximation and found the speeds of sounds very close to (few percent) the Popov approximation discussed in detail in Ref. [11]. In the very-low-temperature region, however, where $k_BT < \mu$, the thermodynamical quantities are dominated by phonon excitations and show a qualitatively different behavior. The big difference is that the speed of the temperature wave does not go to zero but rather to $c_2^2 = c_1^2/3$. Therefore, neither of the three sounds vanishes at zero temperature. In the limit of $\epsilon \ll 1$ the quadrupolar mode is always situated below the temperature wave; thus the third sound is always a quadrupolar spin wave, while the second sound is a temperature wave.

One can assume that the susceptibility given in Eq. (24f) remains singular when $c_s \to 0$ beyond the Bogoliubov-Hartree approximation in the P2 phase when $\zeta_0 = 0$ in equilibrium. Then the term $c^2(\partial\sigma/\partial b)$ can remain finite if c^2 is of $O(c_s)$, and in the matrix (18) all terms containing c^2 can be neglected in this limit except the one multiplied by the susceptibility (first term of $M_{2,3}$). As a result one arrives at the velocity of the quadrupolar spin mode in this limit as

$$c^2 = 4\frac{\rho_{s,-}\rho_{s,+}}{M\rho_s}\left(\frac{\partial\sigma}{\partial b}\right)^{-1}, \tag{27}$$

where Eqs. (11) have been used. One can easily convince oneself that approximating the superfluid densities by the corresponding condensate densities (27) agrees with the expression (26a). However, Eq. (27) is valid to any orders in c_n. By the same token one can write the generalization of Eq. (26b) as the solution of the equation $\det \mathbf{M}_{2,3} = 0$, where $\mathbf{M}_{2,3}$ denotes the submatrix obtained from (18) by removing the second row and the third column.

Concerning the dynamical critical properties, the P2→P1 transition on the $T,B > 0$ plane belongs to the same universality class as the λ transition of liquid Helium [20,21]. Consequently, the quadrupolar spin wave suffers a damping Dk^2, where $D \propto \Delta^{-1/3}$ when approaching the transition line between the P2 and P1 phases along the thermodynamic path on which Δ measures the distance from criticality.

V. SUMMARY

In this paper we have developed a three-fluid hydrodynamical approach for studying the low-energy excitations of the spin-1 Bose gas in the partially polarized Bose-Einstein condensed phase (P2). Such a low-energy and low-momentum approximation of the excitation spectrum is considered to be valid when interparticle scattering events happen sufficiently frequently in order to have large enough regions in the gas sample where thermodynamic equilibrium can be assumed. This kind of approximation is a generalization of the two fluid hydrodynamics, which was successfully applied to scalar Bose-Einstein condensates [11,22] and is in sharp contrast to quantum hydrodynamic approximations, already studied for spinor Bose-Einstein condensates, too [23–25]. These latter approaches are nonlinear equations equivalent to the Gross-Pitaevskii equation and are valid only at (or very close to) zero temperature and where quasiparticle interaction can be assumed to occur very rarely.

In the partially polarized Bose-Einstein condensed phase of polar spin-1 Bose gases (P2 phase) the fluid mass density can be decomposed into three components: a normal mass density and two superfluid densities. The relative motion of these three fluids determine the low-energy excitations of the system. We have calculated the three sound velocities governing these fluctuations. We have found that the speed of sound of the magnetization wave is always below that of the temperature and pressure waves for sufficiently weak spin-spin interaction ($c_s \ll c_n$). However, if c_s is in the same order as c_n, successive hybridizations occur, and the nature of the excitation branches exchange their character. One can hope that, with the help of magnetic and optical Feshbach resonances, such hybridizations can be achieved experimentally, and the three-fluid nature of polar spin-1 Bose condensates and its implications on the phase transition can be examined.

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APPENDIX A: THERMODYNAMIC QUANTITIES

The internal energy per unit volume of the superfluid spin-1 system in the P2 phase should look like

$$de = MT ds + \mu d\rho + b d\sigma + \vec{v}_n \cdot d\vec{g}_\rho + \vec{j}_+ \cdot d\vec{v}_+ + \vec{j}_- \cdot d\vec{v}_-. \tag{A1}$$

The quantities $\vec{v}_+ = (\hbar/M)\vec{\nabla}\varphi_+$, $\vec{v}_- = (\hbar/M)\vec{\nabla}\varphi_-$ are the superfluid velocities of the $+$ and $-$ spin components, respectively, while $\vec{j}_+$ and $\vec{j}_-$ are their conjugate fields. In the P1 phase the last term is omitted from Eq. (A1). In an alternative treatment ϕ and θ can be introduced instead of ϕ_+ and ϕ_-. It has an advantage of some physical insight. In this formulation the internal energy can be given by

$$de = MT ds + \mu d\rho + b d\sigma + \vec{v}_n \cdot d\vec{g}_\rho + \vec{j}_\rho \cdot d\vec{v}_\rho + \vec{j}_\sigma \cdot d\vec{v}_\sigma. \tag{A2}$$

To obtain further thermodynamic equations one has to consider the transformation properties regarding an infinitesimal Galilean transformation, $d\vec{w}$. The noninvariant quantities transform as follows:

$$de = \vec{g}_\rho \cdot d\vec{w}, \tag{A3a}$$

$$d\vec{g}_\rho = \rho d\vec{w}, \tag{A3b}$$

$$d\vec{v}_\pm = d\vec{w}. \tag{A3c}$$

With their help, the momentum density can be identified as

$$\vec{g}_\rho = \rho\vec{v}_n + \vec{j}_+ + \vec{j}_-. \tag{A4}$$

In the alternative representation of the superfluid velocities, according to Eqs. (6), it is clear that one of the superfluid velocities introduced in Eq. (A2) transform like a velocity: $d\vec{v}_\rho = d\vec{w}$; however, the other one is Galilean invariant: $d\vec{v}_\sigma = 0$. Therefore, the momentum density is as follows:

$$\vec{g}_\rho = \rho\vec{v}_n + \vec{j}_\rho. \tag{A5}$$

Close to thermodynamic equilibrium the currents can be expanded to linear order in the velocities:

$$\vec{g}_\rho = \frac{\partial\vec{g}_\rho}{\partial\vec{v}_n} \cdot \vec{v}_n + \frac{\partial\vec{g}_\rho}{\partial\vec{v}_\rho} \cdot \vec{v}_\rho + \frac{\partial\vec{g}_\rho}{\partial\vec{v}_\sigma} \cdot \vec{v}_\sigma, \tag{A6a}$$

$$\vec{j}_\rho = \frac{\partial\vec{j}_\rho}{\partial\vec{v}_n} \cdot \vec{v}_n + \frac{\partial\vec{j}_\rho}{\partial\vec{v}_\rho} \cdot \vec{v}_\rho + \frac{\partial\vec{j}_\rho}{\partial\vec{v}_\sigma} \cdot \vec{v}_\sigma, \tag{A6b}$$

$$\vec{j}_\sigma = \frac{\partial\vec{j}_\sigma}{\partial\vec{v}_n} \cdot \vec{v}_n + \frac{\partial\vec{j}_\sigma}{\partial\vec{v}_\rho} \cdot \vec{v}_\rho + \frac{\partial\vec{j}_\sigma}{\partial\vec{v}_\sigma} \cdot \vec{v}_\sigma. \tag{A6c}$$

Equation (A2) yields the following Maxwell relations:

$$\frac{\partial\vec{g}_\rho}{\partial\vec{v}_\rho} = -\frac{\partial\vec{j}_\rho}{\partial\vec{v}_n}, \tag{A7a}$$

$$\frac{\partial\vec{g}_\rho}{\partial\vec{v}_\sigma} = -\frac{\partial\vec{j}_\sigma}{\partial\vec{v}_n}, \tag{A7b}$$

$$\frac{\partial\vec{j}_\rho}{\partial\vec{v}_\sigma} = \frac{\partial\vec{j}_\sigma}{\partial\vec{v}_\rho}. \tag{A7c}$$

The nine coefficients of the linear expansion (A6) are not independent. With the help of the Maxwell relations (A7) and Eq. (A5),

$$\vec{j}_\rho = A(\vec{v}_\rho - \vec{v}_n) + B\vec{v}_\sigma, \tag{A8a}$$

$$\vec{j}_\sigma = B(\vec{v}_\rho - \vec{v}_n) + C\vec{v}_\sigma, \tag{A8b}$$

hold for linear order. Choosing A, B, C to be consistent also with the zero temperature linearized hydrodynamic equations

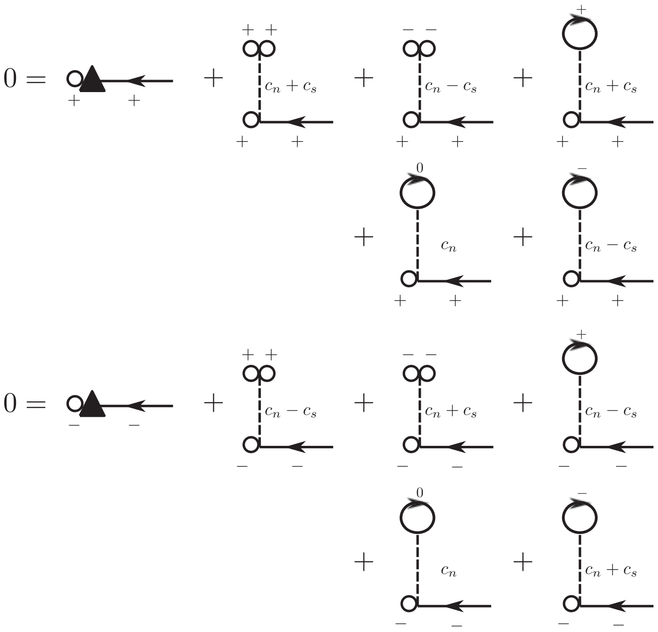

FIG. 4. Feynman diagrams contributing to tadpole graphs in the Bogoliubov-Hartree approximation. The solid lines with an arrow represent single particle propagators, the circles represent condensate contributions, the dashed lines stand for the contact interaction, and the triangle is the contribution from the chemical potential and magnetic field.

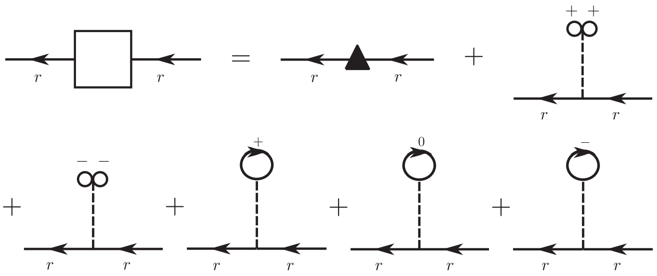

FIG. 5. Feynman diagrams contributing to the normal self-energy Σ^{rr} in the Bogoliubov-Hartree approximation. The contribution of the chemical potential and the magnetic field exactly cancels that of the interaction.

gives

$$A=\frac{\partial \vec{j}_\rho}{\partial \vec{v}_\rho}=-\frac{\partial \vec{j}_\rho}{\partial \vec{v}_n}=\rho-\frac{\partial \vec{g}_\rho}{\partial \vec{v}_n}=\frac{\partial \vec{g}_\rho}{\partial \vec{v}_\rho}\equiv\rho_s, \tag{A8c}$$

$$B=\frac{\partial \vec{j}_\rho}{\partial \vec{v}_\sigma}=\frac{\partial \vec{j}_\sigma}{\partial \vec{v}_\rho}=-\frac{\partial \vec{j}_\sigma}{\partial \vec{v}_n}=\frac{\partial \vec{g}_\rho}{\partial \vec{v}_\sigma}\equiv\sigma_s, \tag{A8d}$$

$$C=\frac{\partial \vec{j}_\sigma}{\partial \vec{v}_\sigma}=\rho_s. \tag{A8e}$$

APPENDIX B: DERIVATION OF BOGOLIUBOV-HARTREE APPROXIMATION

Here we briefly derive the thermodynamic quantities of the polar, spin-1 Bose gas in the presence of a magnetic field in the Bogoliubov-Hartree approximation. We follow the route of the general finite temperature perturbation theory worked out for the spin-1 Bose gas without magnetic field [3] and for the polar gas in the presence of a magnetic field [9].

First one has to take into account that the system is Bose-Einstein condensed; the field operators have nonzero mean values. It is done by introducing new annihilation (and creation) operators with the help of a canonical transformation

$$b_r(\vec{k})=a_r(\vec{k})-\sqrt{N_c}\zeta_r\delta_{\vec{k},0}, \tag{B1}$$

where N_c is the condensate atom number and ζ_r is the condensate spinor introduced earlier in Sec. III. Here we consider only the P2 phase and take $\zeta=(\zeta_+,0,\zeta_-)$. We note that here we can assume a condensate in the $m=0$ Zeeman level too, but after some algebra it turns out that $\zeta_0=0$ in this approximation. The relation between the density, the magnetization, the chemical potential, and the magnetic field can be obtained by the requirement that $\langle b_+(0)\rangle=\langle b_-(0)\rangle=0$. These equations, in the Bogoliubov-Hartree approximation, are depicted graphically in Fig. 4, and read as

$$\mu+b=c_n\frac{\rho}{M}+c_s\frac{\sigma}{M}, \tag{B2a}$$

$$\mu-b=c_n\frac{\rho}{M}-c_s\frac{\sigma}{M}. \tag{B2b}$$

Equations (19) follow directly from Eqs. (B2).

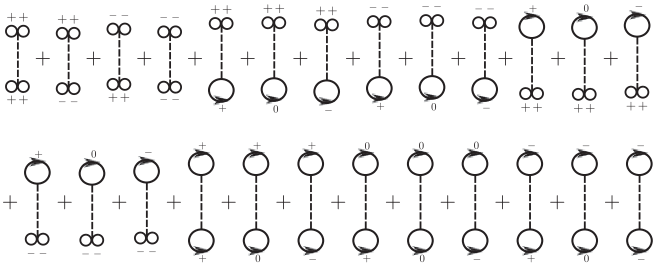

FIG. 6. Feynman diagrams contributing to grand-canonical potential.

The finite temperature (normal) Green's function [26] is used to derive one-particle densities, such as the normal particle density ρ_n and the normal magnetization density σ_n. It is defined as

$$\mathcal{G}^{r,s}(\vec{k},\tau)=-\langle T_\tau b_r(\vec{k},\tau)b_s^\dagger(\vec{k},0)\rangle, \tag{B3}$$

where T_τ is the imaginary time ordering operator and the average is performed over the grand-canonical ensemble defined by the grand-canonical Hamiltonian (1). In the present approximation the self-energy diagrams are zero: the interaction contributions cancel those of the chemical potential and the magnetic field, as depicted in Fig. 5. Consequently, all three Zeeman levels of the thermal atoms have the same free particle occupations, as given by Eq. (20). Note that, starting with the assumption that there is a condensate in the $m=0$ Zeeman level, this statement remains true for all three components. As a result the magnetization is only due to the condensate, which then should point to the direction of the magnetic field, as already discussed in Sec. III. It implies that $\zeta_0=0$ is the only consistent assumption and indeed we are in the P2 phase. The grand-canonical potential is given by

$$\Phi(T,V,\mu,b)=\Phi_0(T,V,\mu,b)+\Phi_1(T,V,\mu,b), \tag{B4a}$$

where Φ_0 is the free contribution [19]

$$\Phi_0(T,V,\mu)=-\frac{k_BTV}{\Lambda^3}3\zeta(5/2), \tag{B4b}$$

and Φ_1 is the contribution of the vacuum diagrams depicted in Fig. 6 and reads as

$$\Phi_1(T,V,\mu)=-\frac{V}{2M^2}(c_n\rho^2+c_s\sigma^2). \tag{B4c}$$

The combination of Eqs. (B4) provides the thermodynamic potential (23).

Finally, we note that a gapless approximation for the quasiparticles is obtained by a hybridization of certain modes via the condensate leading to a kind of RPA [9]. This hybridization, however, does not show up in the equilibrium quantities we are interested in here.

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