Description and measurement of observables in the optical tomographic probability representation of quantum mechanics

G. G. Amosov,[1] Ya. A. Korennoy,[2] and V. I. Man'ko[2]
[1]*Steklov Mathematical Institute, ulica Gubkina 8, Moscow 119991, Russia*
[2]*P. N. Lebedev Physics Institute, Leninsky prospect 53, Moscow 119991, Russia*
(Received 1 October 2011; published 22 May 2012)

Explicit expressions for most interesting quantum observable operators in optical tomography representation are found. A general formalism of the symbols of the operators is presented in the optical tomography representation. The symbols of the operators in the form of singular and regular generalized functions are found, and suggestions for their use in experimental data processing in quantum tomography are given.

DOI: 10.1103/PhysRevA.85.052119 PACS number(s): 03.65.Ca, 03.65.Ta, 42.50.Ar, 42.50.Xa

I. INTRODUCTION

The transition from quantum to classical mechanics has been an important research subject since the beginning of quantum mechanics (see [1] for a review). A suitable setting for this problem is represented by the Wigner-Weyl-Moyal formalism where the operators corresponding to observables and the states, considered linear functionals on the space of observables, are mapped onto functions on a suitable manifold. Such a representation for quantum mechanics has been generalized to yield the deformation quantization program [2]. In this quantization program the operator noncommutativity is implemented by a noncommutative (star) product which is a generalization of the Moyal product [3–5]. Since then, most attention to the star-product quantization scheme has been devoted to the case where the functions (symbols of the operators) are defined on the "classical" phase space of the system [6–9].

In a different setting it was established [10,11] that the symplectic [12–15] and spin [16,17] tomographies, which furnish alternative formulations of quantum mechanics and quantum field theory [18], can be described as well within a star-product scheme. Moreover, in [11] different known star-product schemes were presented in a unified form. In these schemes the symbols of the operators are defined in terms of a special family of operators using the trace formula (what we sometimes call the "dequantization" map because of its original meaning in the Wigner-Weyl formalism), while the reconstruction of operators in terms of their symbols (the "quantization" map) is determined using another family of operators. These two families determine completely the star-product scheme, including the kernel of the star product.

The development of the tomographic formalism is important from the point of view of available applications in quantum tomography. The measuring optical tomogram of photon quantum states by a homodyne photon detector [19–22] provides the possibility to get direct information on the characteristics of the states such as their purity as well as to check the uncertainty relations [23] containing the purity [24] and the state non-Gaussianity parameters [25]. The optical tomogram found in these experiments contains complete information on quantum states. It can be used to study all the quantum effects without intermediate calculations of the Wigner function or another quasiprobability which is usually considered the object to be found in experiments with homodyne-detecting photon states. But in the framework of the formalism applied in the optical tomography one needs an explicit and simple method to calculate means, dispersions, and the highest moment of physical observables. To do this one has to develop the tools to work with the tomographic symbols of the corresponding operators associated with the physical observables.

One of the advantages of the optical tomograms of quantum states is the fact that the tomograms are fair probability distributions. In view of this the statistical characteristics of the physical observables can be expressed by using standard formulas of probability theory. In these formulas the function corresponding to the physical observable is integrated with the weight function, which is the tomographic probability distribution function. The function corresponding to the quantum observable is known [26] to be the so-called dual tomographic symbol of the operator describing the observable. But the optical dual tomographic symbols are generalized functions, and they can have the form of singular or regular generalized functions. Obtaining the explicit expression for the generalized functions of the physical observables is an important ingredient to get the physical information from the photon state optical tomograms which one measures in experiments with homodyne photon detectors [19].

The aim of our work is to find the explicit expressions of most physically interesting operators and their dual symbols in the optical tomography representation, which is necessary for practical calculations. This paper is organized as follows. In Sec. II we review the optical tomography representation of quantum states. In Sec. III the correspondence rules for physical operators in the optical tomography representation are found. In Sec. IV the general formalism of the symbols of the operators is presented in the optical tomography representation. The expressions for the dual symbols of the operators in terms of singular generalized functions and for the kernel of their star product are presented. In Sec. V the representation of the dual symbols of the operators in terms of regular generalized functions is given.

II. OPTICAL TOMOGRAPHY REPRESENTATION OF QUANTUM STATES

In this section we give a short review of the tomographic representation of quantum mechanics by using a so-called optical tomogram [27,28]. As we mentioned above for the photon states this tomogram is measured experimentally [19,21]. For microwave photons the state tomography is also realized in experiments [29].

If we have the density matrix of the quantum state $\hat{\rho}$, the optical tomogram is defined as

$$w(\vec{X},\vec{\theta},t) = \mathrm{Tr}\{\hat{\rho}(t)\delta(\vec{X} - \hat{\vec{X}}(\vec{\theta}))\}, \quad (1)$$

where $\delta(\vec{X} - \hat{\vec{X}}(\vec{\theta}))$ is a Dirac delta function with arguments corresponding to n degrees of freedom and $\hat{\vec{X}}(\vec{\theta})$ is a vector operator rotated in phase space quadrature components,

$$\hat{\vec{X}}(\vec{\theta}) = \left(\hat{q}_1 \cos\theta_1 + \hat{p}_1 \frac{\sin\theta_1}{m_1\omega_{01}}; \cdots ; \hat{q}_n \cos\theta_n + \hat{p}_n \frac{\sin\theta_n}{m_n\omega_{0n}}\right), \quad (2)$$

where m_i and ω_{0i} are constants that have the dimensions of mass and frequency and are chosen for reasons of convenience for the Hamiltonian of a quantum system under study. For simplicity of the formulas let us choose the system of measurements so that $m_\sigma = \omega_{0\sigma} = \hbar = 1$.

The definition of the optical tomogram can be given also in the form

$$w(\vec{X},\vec{\theta},t) = \langle \vec{X},\vec{\theta}|\hat{\rho}(t)|\vec{X},\vec{\theta}\rangle, \quad (3)$$

where $|\vec{X},\vec{\theta}\rangle$ is an eigenvector of the Hermitian operator (2) for the eigenvalue $\vec{X}$. In the coordinate representation, solving a differential equation for the components X_i,

$$\hat{X}_i(\theta_i)|X_i,\theta_i\rangle = X_i|X_i,\theta_i\rangle,$$

and taking into account that $|\vec{X},\vec{\theta}\rangle = |X_1,\theta_1\rangle \cdots |X_n,\theta_n\rangle$, we obtain

$$\langle \vec{q}\,|\vec{X},\vec{\theta}\rangle = \frac{1}{(2\pi)^{n/2}} \prod_{\sigma=1}^{n} \frac{1}{\sqrt{|\sin\theta_\sigma|}} \times \exp\left\{ i \frac{X_\sigma q_\sigma - \frac{q_\sigma^2 - X_\sigma^2}{2}\cos\theta_\sigma}{\sin\theta_\sigma} + i\frac{\pi}{4}\cos\theta_\sigma \right\}, \quad (4)$$

where the normalization is chosen so that $\langle \vec{X}',\vec{\theta}|\vec{X},\vec{\theta}\rangle = \delta(\vec{X}' - \vec{X})$. In addition, the phase factor is chosen so that $\langle \vec{q}|\vec{X},\vec{\theta}\rangle \to \delta(\vec{q} - \vec{X})$ when $\vec{\theta} \to 0$ and $\langle \vec{q}|\vec{X},\vec{\theta}\rangle \to (2\pi)^{-n/2}\exp(i\vec{q}\cdot\vec{X})$ when $\vec{\theta} \to \pi/2$.

For an arbitrary function on phase space $f(\vec{q},\vec{p})$ we introduce the notation of the Radon transform of this function,

$$\mathcal{R}[f(\vec{q},\vec{p})](\vec{X},\vec{\theta}) = \int \frac{f(\vec{q},\vec{p})}{(2\pi)^n} \prod_{\sigma=1}^{n} \delta(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\sin\theta_\sigma) d^n q\, d^n p. \quad (5)$$

In terms of the Wigner function [30] the tomogram $w(\vec{X},\vec{\theta},t)$ is expressed by the Radon transform [31]:

$$\begin{aligned} w(\vec{X},\vec{\theta},t) &= \mathcal{R}[W(\vec{q},\vec{p},t)](\vec{X},\vec{\theta}) \\ &= \int \frac{W(\vec{q},\vec{p},t)}{(2\pi)^{2n}} \prod_{\sigma=1}^{n} e^{-i\eta_\sigma(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\sin\theta_\sigma)} d^n\eta\, d^n q\, d^n p \\ &= \int \frac{W(\vec{q},\vec{p},t)}{(2\pi)^n} \prod_{\sigma=1}^{n} \delta(X_\sigma - q_\sigma\cos\theta_\sigma \\ &\quad - p_\sigma\sin\theta_\sigma) d^n q\, d^n p, \end{aligned} \quad (6)$$

where the Wigner function $W(\vec{q},\vec{p},t)$ is associated with the density matrix $\rho(\vec{q},\vec{q}\,',t)$ in the coordinate representation by the standard relations

$$W(\vec{q},\vec{p},t) = \int \rho(\vec{q} + \vec{u}/2, \vec{q} - \vec{u}/2, t) e^{-i\vec{p}\cdot\vec{u}} d^n u, \quad (7)$$

$$\rho(\vec{q},\vec{q}\,',t) = \frac{1}{(2\pi)^n} \int W\left(\frac{\vec{q}+\vec{q}\,'}{2}, \vec{p}, t\right) e^{i\vec{p}\cdot(\vec{q}-\vec{q}\,')} d^n p. \quad (8)$$

Relation (6) can be reversed using the symmetry property of the optical tomogram,

$$w(\vec{X},\vec{\theta},t) = w((-1)^{k_\sigma} X_\sigma, \theta_\sigma + \pi k_\sigma, t), \quad k_\sigma = 0, \pm 1, \pm 2, \ldots . \quad (9)$$

After calculations we can write

$$W(\vec{q},\vec{p},t) = \frac{1}{(2\pi)^n} \int_0^\pi d^n\theta \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} w(\vec{X},\vec{\theta},t) \times \prod_{\sigma=1}^{n} |\eta_\sigma| e^{i\eta_\sigma(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\sin\theta_\sigma)} d^n\eta\, d^n X. \quad (10)$$

Thus, the tomogram $w(\vec{X},\vec{\theta},t)$ contains all the information about the quantum state.

One can give the physical interpretation of the optical tomogram (see, e.g., [32]) for an example classical particle. If one considers the probability density of the particle state on the phase space (q,p), one can obtain the marginal probability distribution of only position q integrating the probability distribution over momentum p. We can rotate the reference frame axes in the phase space to get a new rotated reference frame. Then the tomogram $w(X,\theta)$ is the probability distribution of the position X measured in the reference frame in phase space with axes q', p' rotated by angle θ with respect to initial axes (q,p). For many degrees of freedom the optical tomogram $w(X_1,\ldots,X_n,\theta_1,\ldots,\theta_n)$ is a joint probability distribution of the random position measured each in its own reference frame, which is rotated by the angle θ_i in the phase space of the system.

From this interpretation (we give an example for one degree of freedom) simple formulas for moments of position and momentum are as follows:

$$\langle \hat{q}^k \rangle = \int w(X,\theta = 0) X^k dX,$$

$$\langle \hat{p}^k \rangle = \int w(X,\theta = \pi/2) X^k dX.$$

The optical tomography of photon-added coherent states, even and odd coherent states, and thermal states was discussed in Ref. [33].

The evolution equation and the equation for stationary states for the optical tomogram of the state of a quantum system with a potential field were obtained in [34], along with the Liouville equation with an arbitrary potential field in the optical tomography representation. In [35] the evolution equation for the optical tomogram of a quantum system with an arbitrary spinless Hamiltonian was obtained. In [36] the evolution equation of the optical tomogram of quantum systems with an arbitrary quadratic Hamiltonian was solved, and the optical propagator for such systems was found.

III. THE CORRESPONDENCE RULES FOR PHYSICAL OPERATORS IN THE OPTICAL TOMOGRAPHY REPRESENTATION

Using relations (7) and (8) between the density matrix $\rho(\vec{q},\vec{q}\,')$ and the Wigner function $W(\vec{q},\vec{p})$ for any operator $\hat{A}$ acting on the density matrix, one can find the corresponding operator in the Wigner-Weyl representation $(\hat{A})_W$ acting on the Wigner function. It is well known that (see, e.g., [32])

$$\begin{aligned}
q_\sigma \rho(\vec{q},\vec{q}\,') &\leftrightarrow (\hat{q}_\sigma)_W W(\vec{q},\vec{p}) = \left(q_\sigma + \frac{i}{2}\frac{\partial}{\partial p_\sigma}\right) W(\vec{q},\vec{p}),\\
q'_\sigma \rho(\vec{q},\vec{q}\,') &\leftrightarrow (\hat{q}'_\sigma)_W W(\vec{q},\vec{p}) = \left(q_\sigma - \frac{i}{2}\frac{\partial}{\partial p_\sigma}\right) W(\vec{q},\vec{p}),\\
\frac{\partial \rho(\vec{q},\vec{q}\,')}{\partial q_\sigma} &\leftrightarrow \left(\frac{\partial}{\partial q_\sigma}\right)_W W(\vec{q},\vec{p}) = \left(\frac{1}{2}\frac{\partial}{\partial q_\sigma} + i p_\sigma\right) W(\vec{q},\vec{p}),\\
\frac{\partial \rho(\vec{q},\vec{q}\,')}{\partial q'_\sigma} &\leftrightarrow \left(\frac{\partial}{\partial q'_\sigma}\right)_W W(\vec{q},\vec{p}) = \left(\frac{1}{2}\frac{\partial}{\partial q_\sigma} - i p_\sigma\right) W(\vec{q},\vec{p}).
\end{aligned} \tag{11}$$

Using relation (6) between the Wigner function and the optical tomogram one can find the correspondence rules for the operators acting on the Wigner function and the optical tomogram.

Let $\hat{\mathcal{A}}$ be an operator acting on the Wigner function. We define the Radon transform of this operator $\hat{\mathcal{R}}[\hat{\mathcal{A}}](\vec{X},\vec{\theta})$ as follows:

$$\hat{\mathcal{R}}[\hat{\mathcal{A}}](\vec{X},\vec{\theta})\, w(\vec{X},\vec{\theta}) = \mathcal{R}[\hat{\mathcal{A}} W(\vec{q},\vec{p})](\vec{X},\vec{\theta}). \tag{12}$$

For any operator $\hat{A}$ acting on the density matrix one can find the corresponding operator in the optical tomography representation $(\hat{A})_w$ acting on the optical tomogram. With definition (12) it can be written as

$$(\hat{A})_w = \hat{\mathcal{R}}[(\hat{A})_W](\vec{X},\vec{\theta}), \tag{13}$$

where $(\hat{A})_W$ is operator $\hat{A}$ in the Wigner-Weyl representation.

Let us find the operator $\hat{\mathcal{R}}[q_i](\vec{X},\vec{\theta})$, which is the Radon transform of operator (12) of the product of ith coordinate q_i by the Wigner function $W(\vec{q},\vec{p})$. For this we note that for angle derivatives of the tomogram (6) we have (we restore the dimensional constants)

$$\begin{aligned}
\frac{\partial}{\partial \theta_i} w(\vec{X},\vec{\theta}) = \frac{\partial}{\partial \theta_i}\mathcal{R}\left[W(\vec{q},\vec{p})\right](\vec{X},\vec{\theta}) &= \frac{1}{(2\pi\hbar)^n}\int W(\vec{q},\vec{p})\left(q_i \sin\theta_i - p_i\frac{\cos\theta_i}{m_i\omega_{0i}}\right)\\
&\times \delta'\left(X_i - q_i\cos\theta_i - p_i\frac{\sin\theta_i}{m_i\omega_{0i}}\right)\prod_{\sigma\neq i}^{n}\delta\left(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_{0\sigma}}\right) d^nq\, d^np,
\end{aligned} \tag{14}$$

where δ' is the derivative of the Dirac δ function. For position derivatives of the tomogram we obtain the integral expression

$$\begin{aligned}
\frac{\partial}{\partial X_i}\mathcal{R}[W(\vec{q},\vec{p})](\vec{X},\vec{\theta}) &= \frac{1}{(2\pi\hbar)^n}\int W(\vec{q},\vec{p})\delta'\left(X_i - q_i\cos\theta_i - p_i\frac{\sin\theta_i}{m_i\omega_{0i}}\right)\\
&\times \prod_{\sigma\neq i}^{n}\delta\left(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_{0\sigma}}\right) d^nq\, d^np.
\end{aligned} \tag{15}$$

We define the inverse derivative (antiderivative) of the tomogram by using the Heaviside step function $\Theta(X_i - X'_i)$:

$$\left[\frac{\partial}{\partial X_i}\right]^{-1} w(\vec{X},\vec{\theta}) = \int \Theta(X_i - X'_i)[w(\vec{X},\vec{\theta})]_{X_i = X'_i}\, dX'_i. \tag{16}$$

With the help of (6) this antiderivative can be expressed as follows:

$$\begin{aligned}
\left[\frac{\partial}{\partial X_i}\right]^{-1} w(\vec{X},\vec{\theta}) = \left[\frac{\partial}{\partial X_i}\right]^{-1}\mathcal{R}[W(\vec{q},\vec{p})](\vec{X},\vec{\theta}) &= \frac{1}{(2\pi\hbar)^n}\int \Theta(X_i - X'_i) W(\vec{q},\vec{p})\\
&\times \delta\left(X'_i - q_i\cos\theta_i - p_i\frac{\sin\theta_i}{m_i\omega_{0i}}\right)\prod_{\sigma\neq i}^{n}\delta\left(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_{0\sigma}}\right) d^nq\, d^np\, dX'_i.
\end{aligned} \tag{17}$$

With the help of Eqs. (6), (14), and (17) we can write

$$\begin{aligned}
\left[\frac{\partial}{\partial X_i}\right]^{-1}\frac{\partial}{\partial \theta_i} w(\vec{X},\vec{\theta}) = \left[\frac{\partial}{\partial X_i}\right]^{-1}\frac{\partial}{\partial \theta_i}\mathcal{R}\left[W(\vec{q},\vec{p})\right](\vec{X},\vec{\theta}) &= \frac{1}{(2\pi\hbar)^n}\int \Theta(X_i - X'_i) W(\vec{q},\vec{p})\left(q_i\sin\theta_i - p_i\frac{\cos\theta_i}{m_i\omega_{0i}}\right)\\
&\times \delta'\left(X'_i - q_i\cos\theta_i - p_i\frac{\sin\theta_i}{m_i\omega_{0i}}\right)\prod_{\sigma\neq i}^{n}\delta\left(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_{0\sigma}}\right) d^nq\, d^np\, dX'_i\\
&= \frac{1}{(2\pi\hbar)^n}\int W(\vec{q},\vec{p})\left(q_i\sin\theta_i - p_i\frac{\cos\theta_i}{m_i\omega_{0i}}\right)\prod_{\sigma=1}^{n}\delta\left(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_{0\sigma}}\right) d^nq\, d^np,
\end{aligned} \tag{18}$$

where we evaluated the integral over dX_i'. From the theory of generalized functions of slow growth we know

$$Y_i\delta(\vec{Y}) = Y_i\prod_{\sigma=1}^{n}\delta(Y_\sigma) = 0.$$

Substituting in this formula the argument

$$Y_\sigma = X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_{0\sigma}},$$

multiplying it by the Wigner function $W(\vec{q},\vec{p})$, and integrating the result over $d^nqd^np/(2\pi\hbar)^n$, we will find

$$\frac{1}{(2\pi\hbar)^n}\int W(\vec{q},\vec{p})\left(X_i - q_i\cos\theta_i - p_i\frac{\sin\theta_i}{m_i\omega_{0i}}\right)\times\prod_{\sigma=1}^{n}\delta\left(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_{0\sigma}}\right)d^nqd^np = 0. \quad (19)$$

Using formulas (6), (12), (18), and (19), after simple calculations we obtain

$$\begin{aligned}\mathcal{R}[q_iW(\vec{q},\vec{p})](\vec{X},\vec{\theta}) &= \hat{\mathcal{R}}[q_i](\vec{X},\vec{\theta})\,w(\vec{X},\vec{\theta})\\ &= \left\{\sin\theta_i\left[\frac{\partial}{\partial X_i}\right]^{-1}\frac{\partial}{\partial\theta_i} + X_i\cos\theta_i\right\}w(\vec{X},\vec{\theta}).\end{aligned} \quad (20)$$

Similarly, using formulas (6), (12), (14), (18), (19), and well-known properties of the functions $\delta(Y)$ and $\delta'(Y)$, we can find

$$\begin{aligned}\mathcal{R}[p_iW(\vec{q},\vec{p})](\vec{X},\vec{\theta}) &= \hat{\mathcal{R}}[p_i](\vec{X},\vec{\theta})\,w(\vec{X},\vec{\theta})\\ &= m_i\omega_{0i}\left\{-\cos\theta_i\left[\frac{\partial}{\partial X_i}\right]^{-1}\frac{\partial}{\partial\theta_i} + X_i\sin\theta_i\right\}w(\vec{X},\vec{\theta}),\end{aligned}$$

$$\begin{aligned}\mathcal{R}\left[\frac{\partial}{\partial q_i}W(\vec{q},\vec{p})\right](\vec{X},\vec{\theta}) &= \hat{\mathcal{R}}\left[\frac{\partial}{\partial q_i}\right](\vec{X},\vec{\theta})\,w(\vec{X},\vec{\theta})\\ &= \cos\theta_i\frac{\partial}{\partial X_i}w(\vec{X},\vec{\theta}),\\ \mathcal{R}\left[\frac{\partial}{\partial p_i}W(\vec{q},\vec{p})\right](\vec{X},\vec{\theta}) &= \hat{\mathcal{R}}\left[\frac{\partial}{\partial p_i}\right](\vec{X},\vec{\theta})\,w(\vec{X},\vec{\theta})\\ &= \frac{\sin\theta_i}{m_i\omega_{0i}}\frac{\partial}{\partial X_i}w(\vec{X},\vec{\theta}).\end{aligned} \quad (21)$$

Here we used the notations $\hat{\mathcal{R}}[\partial/\partial q_i](\vec{X},\vec{\theta}),\hat{\mathcal{R}}[\partial/\partial p_i](\vec{X},\vec{\theta})$ in the sense of definition (12).

If the product of the direct and the inverse Radon transform is unity, then the explicit form in the optical tomography representation of the product of the operators is equal to the product of these operators in the optical tomography representation. Moreover, suppose we have a set of operators $\{\hat{\mathcal{A}}_{ik}\}$ acting on the set of functions $W(\vec{q},\vec{p})\in\mathcal{S}^{2n}$, where $\mathcal{S}^{2n}$ is a space of well-behaved test functions (for instance, Schwartz space [37]), and where, for any $W(\vec{q},\vec{p})\in\mathcal{S}^{2n}$, we have $\hat{\mathcal{A}}_{ik}W(\vec{q},\vec{p})\in\mathcal{S}^{2n}$ for any $\hat{\mathcal{A}}_{ik}\in\{\hat{\mathcal{A}}_{ik}\}$, then we can write

$$\begin{aligned}&\mathcal{R}\left[\sum_i C_i\prod_k(\hat{\mathcal{A}}_{ik})^{l_k}W(\vec{q},\vec{p})\right](\vec{X},\vec{\theta})\\ &\quad= \sum_i C_i\prod_k\left(\hat{\mathcal{R}}[\hat{\mathcal{A}}_{ik}^{l_k}](\vec{X},\vec{\theta})\right)^{l_k}\mathcal{R}[W(\vec{q},\vec{p})](\vec{X},\vec{\theta}).\end{aligned} \quad (22)$$

Using the formulas given in this paragraph, it is possible to find the explicit form of any interesting practice operators in the optical tomography representation.

Thus for position $\hat{\vec{q}}$ and momentum $\hat{\vec{p}}$ operators, with the help of formulas (11), (13), (20), and (21), one can find their components in the optical tomography representation:

$$(\hat{q}_i)_w = \hat{\mathcal{R}}[(\hat{q}_i)_W](\vec{X},\vec{\theta}) = \hat{\mathcal{R}}\left[q_i + \frac{i}{2}\frac{\partial}{\partial p_i}\right](\vec{X},\vec{\theta}) = \sin\theta_i\left[\frac{\partial}{\partial X_i}\right]^{-1}\frac{\partial}{\partial\theta_i} + X_i\cos\theta_i + \frac{i}{2}\frac{\hbar}{m_i\omega_{0i}}\sin\theta_i\frac{\partial}{\partial X_i}, \quad (23)$$

$$(\hat{p}_i)_w = -i\hbar\hat{\mathcal{R}}\left[\left(\frac{\partial}{\partial q_i}\right)_W\right](\vec{X},\vec{\theta}) = m_i\omega_{0i}\left(-\cos\theta_i\left[\frac{\partial}{\partial X_i}\right]^{-1}\frac{\partial}{\partial\theta_i} + X_i\sin\theta_i\right) - \frac{i\hbar}{2}\cos\theta_i\frac{\partial}{\partial X_i}. \quad (24)$$

Multiplying these operators and taking into account property (22), we can find

$$\begin{aligned}(\hat{q}_i^2)_w &= (\hat{q}_i)_w(\hat{q}_i)_w = \sin^2\theta_i\left[\frac{\partial}{\partial X_i}\right]^{-2}\left(\frac{\partial^2}{\partial\theta_i^2}+1\right) + X_i\left[\frac{\partial}{\partial X_i}\right]^{-1}\left(\sin 2\theta_i\frac{\partial}{\partial\theta_i} - \sin^2\theta_i\right) + X_i^2\cos^2\theta_i\\ &\quad + i\frac{\hbar}{m_i\omega_{0i}}\left\{\sin^2\theta_i\frac{\partial}{\partial\theta_i} + \frac{\sin 2\theta_i}{2}\left(1 + X_i\frac{\partial}{\partial X_i}\right)\right\} - \frac{1}{4}\frac{\hbar^2}{m_i^2\omega_{0i}^2}\sin^2\theta_i\frac{\partial^2}{\partial X^2},\\ (\hat{p}_i^2)_w &= (\hat{p}_i)_w(\hat{p}_i)_w = m_i^2\omega_{0i}^2\left\{\cos^2\theta_i\left[\frac{\partial}{\partial X_i}\right]^{-2}\left(\frac{\partial^2}{\partial\theta_i^2}+1\right) - X_i\left[\frac{\partial}{\partial X_i}\right]^{-1}\left(\sin 2\theta_i\frac{\partial}{\partial\theta_i} + \cos^2\theta_i\right) + X_i^2\sin^2\theta_i\right\}\\ &\quad + i\hbar m_i\omega_{0i}\left\{\cos^2\theta_i\frac{\partial}{\partial\theta_i} - \frac{\sin 2\theta_i}{2}\left(1 + X_i\frac{\partial}{\partial X_i}\right)\right\} - \frac{\hbar^2}{4}\cos^2\theta_i\frac{\partial^2}{\partial X^2},\\ (\hat{q}_i\hat{p}_i)_w &= (\hat{q}_i)_w(\hat{p}_i)_w = m_i\omega_{0i}\left\{-\frac{\sin 2\theta_i}{2}\left[\frac{\partial}{\partial X_i}\right]^{-2}\left(\frac{\partial^2}{\partial\theta_i^2}+1\right) + X_i\left[\frac{\partial}{\partial X_i}\right]^{-1}\left(\frac{\sin 2\theta_i}{2} - \cos 2\theta_i\frac{\partial}{\partial\theta_i}\right) + X_i^2\frac{\sin^2\theta_i}{2}\right\}\\ &\quad - i\hbar\left\{\frac{X_i}{2}\frac{\partial}{\partial X_i}\cos 2\theta_i + \frac{\sin 2\theta_i}{2}\frac{\partial}{\partial\theta_i} - \sin^2\theta_i\right\} + \frac{\hbar^2}{8m_i\omega_{0i}}\sin 2\theta_i\frac{\partial^2}{\partial X^2},\end{aligned} \quad (25)$$

where we denote the designation

$$\begin{aligned}&\left[\frac{\partial}{\partial X_i}\right]^{-2} w(\vec{X},\vec{\theta})\\&= \left[\frac{\partial}{\partial X_i}\right]^{-1}\left[\frac{\partial}{\partial X_i}\right]^{-1} w(\vec{X},\vec{\theta})\\&= \int \Theta(X_i - X_i'')\Theta(X_i'' - X_i')[w(\vec{X},\vec{\theta})]_{X_i=X_i'}dX_i''dX_i'\\&= \int (X_i - X_i')\Theta(X_i - X_i')[w(\vec{X},\vec{\theta})]_{X_i=X_i'}dX_i'. \qquad (26)\end{aligned}$$

Similarly, we define higher orders of the operator $[\partial/\partial X_i]^{-1}$.

Let us find the momentum operator in the optical tomography representation. As known in the density matrix representation $\hat{l} = -i\hbar[\vec{q},\nabla_{\vec{q}}]$, i.e., $\hat{l}_1 = \hat{q}_2\hat{p}_3 - \hat{p}_2\hat{q}_3$, $\hat{l}_2$ and $\hat{l}_3$ are given by the relation for $\hat{l}_1$ by cyclic replacement of indices. In the Wigner representation (hereafter the dimensional constants are taken to be unity)

$$\begin{aligned}(\hat{l}_1)_W = -i\Bigg\{&\frac{q_2}{2}\frac{\partial}{\partial q_3} + iq_2p_3 + \frac{i}{4}\frac{\partial^2}{\partial q_3\partial p_2} - \frac{p_3}{2}\frac{\partial}{\partial p_2}\\&-\frac{q_3}{2}\frac{\partial}{\partial q_2} - iq_3p_2 - \frac{i}{4}\frac{\partial^2}{\partial p_3\partial q_2} + \frac{p_2}{2}\frac{\partial}{\partial p_3}\Bigg\},\end{aligned}$$

and the corresponding Wigner-Weyl symbol of this operator is

$$W_{\hat{l}_1}(\vec{q},\vec{p}) = q_2p_3 - q_3p_2.$$

In the optical tomography representation, after calculations with the help of (23) and (24), we get

$$\begin{aligned}(\hat{l}_1)_w = -i\Bigg\{&\frac{1}{2}\left(\sin\theta_2\left[\frac{\partial}{\partial X_2}\right]^{-1}\frac{\partial}{\partial\theta_2} + X_2\cos\theta_2\right)\cos\theta_3\frac{\partial}{\partial X_3}\\&+i\left(\sin\theta_2\left[\frac{\partial}{\partial X_2}\right]^{-1}\frac{\partial}{\partial\theta_2} + X_2\cos\theta_2\right)\left(-\cos\theta_3\left[\frac{\partial}{\partial X_3}\right]^{-1}\frac{\partial}{\partial\theta_3} + X_3\sin\theta_3\right)\\&+\frac{i}{4}\sin\theta_2\frac{\partial}{\partial X_2}\cos\theta_3\frac{\partial}{\partial X_3} + \frac{\sin\theta_2}{2}\frac{\partial}{\partial X_2}\left(\cos\theta_3\left[\frac{\partial}{\partial X_3}\right]^{-1}\frac{\partial}{\partial\theta_3} - X_3\sin\theta_3\right)\Bigg\} + i\{2\leftrightarrow 3\}. \qquad (27)\end{aligned}$$

Components $(\hat{l}_2)_w$ and $(\hat{l}_3)_w$ are given by (27) by cyclic replacement of indices.

The creation and annihilation operators acting on the density matrix in the coordinate representation have the form

$$\hat{a} = \frac{1}{\sqrt{2}}\left(q + \frac{\partial}{\partial q}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(q - \frac{\partial}{\partial q}\right). \qquad (28)$$

So, in the optical tomography representation we can write

$$\begin{aligned}(\hat{a}_i)_w &= \frac{\exp(i\theta_i)}{\sqrt{2}}\left\{\frac{1}{2}\frac{\partial}{\partial X_i} + X_i - i\left[\frac{\partial}{\partial X_i}\right]^{-1}\frac{\partial}{\partial\theta_i}\right\}, \qquad (29)\\(\hat{a}_i^\dagger)_w &= \frac{\exp(-i\theta_i)}{\sqrt{2}}\left\{-\frac{1}{2}\frac{\partial}{\partial X_i} + X_i + i\left[\frac{\partial}{\partial X_i}\right]^{-1}\frac{\partial}{\partial\theta_i}\right\}.\end{aligned}$$

For the number of quanta operator $\hat{N}_i = \hat{a}_i^\dagger\hat{a}_i$ in ith mode of an n-dimensional oscillator we have

$$\hat{N}_i\rho(\vec{q},\vec{q}\,') = \hat{a}_i^\dagger\hat{a}_i\rho(\vec{q},\vec{q}\,') = \frac{1}{2}\left\{q_i^2 - \frac{\partial^2}{\partial q_i^2} - 1\right\}\rho(\vec{q},\vec{q}\,'),$$

and in the Wigner representation,

$$\begin{aligned}(\hat{N}_i)_W W(\vec{q},\vec{p}) &= (\hat{a}_i^\dagger)_W(\hat{a}_i)_W W(\vec{q},\vec{p})\\&= \frac{1}{2}\Bigg\{q_i^2 - \frac{1}{4}\left(\frac{\partial^2}{\partial p_i^2} + \frac{\partial^2}{\partial q_i^2}\right)\\&\quad + iq_i\frac{\partial}{\partial p_i} - ip_i\frac{\partial}{\partial q_i} + p_i^2 - 1\Bigg\} W(\vec{q},\vec{p}).\end{aligned}$$

Using formulas (11) or taking the product of two operators (29), we arrive at

$$\begin{aligned}&(\hat{N}_i)_w w(\vec{X},\vec{\theta})\\&= (\hat{a}_i^\dagger\hat{a}_i)_w w(\vec{X},\vec{\theta})\\&= \frac{1}{2}\Bigg\{\left[\frac{\partial}{\partial X_i}\right]^{-2}\left(\frac{\partial^2}{\partial\theta_i^2} + 1\right) + X_i^2\\&\quad - X_i\left[\frac{\partial}{\partial X_i}\right]^{-1} - \frac{1}{4}\frac{\partial^2}{\partial X_i^2} + i\frac{\partial}{\partial\theta_i} - 1\Bigg\} w(\vec{X},\vec{\theta}). \qquad (30)\end{aligned}$$

The operator $(\hat{N}_i)_w$ acts on the functions $w_n(\vec{X},\vec{\theta})$ of the harmonic oscillator according to the following formula:

$$(\hat{N}_i)_w w_{n_i}(\vec{X},\vec{\theta}) = n_i w_{n_i}(\vec{X},\vec{\theta}), \qquad (31)$$

where n_i is the number of quanta in the ith mode. Note that, at a derivation from the correspondence rules we actually used, the functions $W(\vec{q},\vec{p})$ belong to a space of well-behaved test functions $\mathcal{S}^{2n}$ (for instance, Schwartz space [37]), on which the space of the generalized functions of slow growth $\mathcal{S}'^{2n}$ can be constructed.

IV. GENERAL FORMALISM OF THE SYMBOLS OF OPERATORS

The relation between the density matrix and the tomogram can be represented in the form

$$\begin{aligned}w(\vec{X},\vec{\theta}) &= \mathrm{Tr}\{\hat{\rho}\hat{U}(\vec{X},\vec{\theta})\},\\\hat{\rho} &= \int w(\vec{X},\vec{\theta})\hat{D}(\vec{X},\vec{\theta})d^nX\,d^n\theta,\end{aligned}$$

where

$$\hat{U}(\vec{X},\vec{\theta}) = \prod_{\sigma=1}^{n} \delta(X_\sigma \hat{1} - \hat{q}_\sigma \cos\theta_\sigma - \hat{p}_\sigma \sin\theta_\sigma),$$

$$\hat{D}(\vec{X},\vec{\theta}) = \int \prod_{\sigma=1}^{n} \frac{|\eta_\sigma|}{2\pi} e^{i\eta_\sigma(X_\sigma - \hat{q}_\sigma \cos\theta_\sigma - \hat{p}_\sigma \sin\theta_\sigma)} d^n\eta$$

are the dequantizer and quantizer operators, respectively. These operators satisfy the orthogonality and completeness conditions.

$$\mathrm{Tr}\{\hat{U}(\vec{X},\vec{\theta})\hat{D}(\vec{X}',\vec{\theta}')\} = \prod_{\sigma=1}^{n} \delta(X_\sigma \cos(\theta_\sigma - \theta'_\sigma) - X'_\sigma)\delta(\sin(\theta_\sigma - \theta'_\sigma)), \quad (32)$$

$$\int \hat{D}_{\hat{\vec{q}}'\hat{\vec{p}}'}(\vec{X},\vec{\theta})\hat{U}_{\hat{\vec{q}}\hat{\vec{p}}}(\vec{X},\vec{\theta})\, d^nX\, d^n\theta = \delta(\hat{\vec{q}} - \hat{\vec{q}}\,')\delta(\hat{\vec{p}} - \hat{\vec{p}}\,'). \quad (33)$$

Let us associate the symbol $w_{\hat{A}}(\vec{X},\vec{\theta})$ with the arbitrary operator $\hat{A}$ by the definition

$$w_{\hat{A}}(\vec{X},\vec{\theta}) = \mathrm{Tr}\{\hat{A}\hat{U}(\vec{X},\vec{\theta})\}.$$

Taking into account the completeness condition (33), we can write the inverse relation,

$$\hat{A} = \int w_{\hat{A}}(\vec{X},\vec{\theta})\hat{D}(\vec{X},\vec{\theta}) d^nX\, d^n\theta.$$

The action of operator $\hat{A}$ on the density matrix can be written in the tomographic representation as the integral operator,

$$\mathrm{Tr}\{\hat{A}\hat{\rho}\hat{U}(\vec{X},\vec{\theta})\} = \int w_{\hat{A}}(\vec{X}',\vec{\theta}\,')w(\vec{X}'',\vec{\theta}\,'')\mathrm{Tr}\{\hat{D}(\vec{X}',\vec{\theta}\,') \times \hat{D}(\vec{X}'',\vec{\theta}\,'')\hat{U}(\vec{X},\vec{\theta})\} d^nX'\, d^n\theta'\, d^nX''\, d^n\theta''.$$

The average value of operator $\hat{A}$ is

$$\mathrm{Tr}\{\hat{A}\hat{\rho}\} = \int w(\vec{X},\vec{\theta})\mathrm{Tr}\{\hat{A}\hat{D}(\vec{X},\vec{\theta})\} d^nX\, d^n\theta = \int w(\vec{X},\vec{\theta}) w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta}) d^nX\, d^n\theta,$$

where we denote the designation for the dual symbol of operator $\hat{A}$,

$$w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta}) = \mathrm{Tr}\{\hat{A}\hat{D}(\vec{X},\vec{\theta})\}. \quad (34)$$

With the help of (33) operator $\hat{A}$ can be found from its dual symbol:

$$\hat{A} = \int w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta})\hat{U}(\vec{X},\vec{\theta}) d^nX\, d^n\theta.$$

The symbol $w_{\hat{A}}(\vec{X},\vec{\theta})$ and the corresponding dual symbol $w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta})$ are associated by the relations

$$w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta}) = \int w_{\hat{A}}(\vec{X}',\vec{\theta}\,')\mathrm{Tr}\{\hat{D}(\vec{X}',\vec{\theta}\,')\hat{D}(\vec{X},\vec{\theta})\} d^nX'\, d^n\theta',$$

$$w_{\hat{A}}(\vec{X},\vec{\theta}) = \int w^{(d)}_{\hat{A}}(\vec{X}',\vec{\theta}\,')\mathrm{Tr}\{\hat{U}(\vec{X}',\vec{\theta}\,')\hat{U}(\vec{X},\vec{\theta})\} d^nX'\, d^n\theta'.$$

The dual symbol of the product of two operators $\hat{A}$ and $\hat{B}$ is equal to the star product with the corresponding kernel,

$$w^{(d)}_{\hat{A}\hat{B}}(\vec{X},\vec{\theta}) = w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta}) * w^{(d)}_{\hat{B}}(\vec{X},\vec{\theta}) = \int K^{(d)}(\vec{X},\vec{\theta};\vec{X}',\vec{\theta}\,';\vec{X}'',\vec{\theta}\,'') w^{(d)}_{\hat{A}}(\vec{X}',\vec{\theta}\,') \times w^{(d)}_{\hat{B}}(\vec{X}'',\vec{\theta}\,'') d^nX'\, d^n\theta'\, d^nX''\, d^n\theta'', \quad (35)$$

where

$$K^{(d)}(\vec{X},\vec{\theta};\vec{X}',\vec{\theta}\,';\vec{X}'',\vec{\theta}\,'') = \mathrm{Tr}\{\hat{U}(\vec{X}',\vec{\theta}\,')\hat{U}(\vec{X}'',\vec{\theta}\,'')\hat{D}(\vec{X},\vec{\theta})\}. \quad (36)$$

This formula can be transformed to a form suitable for practical use as follows:

$$K^{(d)}(\vec{X},\vec{\theta};\vec{X}',\vec{\theta}\,';\vec{X}'',\vec{\theta}\,'') = \frac{1}{(2\pi)^{2n}} \int \prod_{\sigma=1}^{n} \delta(X'_\sigma - q_\sigma \cos\theta'_\sigma - p_\sigma \sin\theta'_\sigma) \times \delta(X''_\sigma - q_\sigma \cos\theta''_\sigma - p_\sigma \sin\theta''_\sigma)|\eta_\sigma| \times \exp\{i\eta_\sigma(X_\sigma - q_\sigma \cos\theta_\sigma - p_\sigma \sin\theta_\sigma)\} \times \exp\left\{ i\eta_\sigma^2 \frac{\sin(\theta_\sigma - \theta'_\sigma)\sin(\theta_\sigma - \theta''_\sigma)}{\sin(\theta'_\sigma - \theta''_\sigma)} \right\} d^n\eta\, d^nq\, d^np. \quad (37)$$

From the definition of dual symbol (34) for operators $\hat{1}$, $\hat{q}$, and $\hat{p}$ after calculations we arrive at

$$w^{(d)}_{\hat{1}}(\vec{X},\vec{\theta}) = \delta(\sin(\vec{\theta} - \vec{\theta}_0)), \quad \theta_{0i} \in [0,\pi],$$

$$w^{(d)}_{\hat{q}_i}(\vec{X},\vec{\theta}) = X_i \cos\theta_i \delta(\sin\theta_i)\delta(\sin(\vec{\theta}_{\sigma\neq i} - \vec{\theta}_{0\sigma\neq i})),$$

$$w^{(d)}_{\hat{p}_i}(\vec{X},\vec{\theta}) = X_i \delta(\theta_i - \pi/2)\delta(\sin(\vec{\theta}_{\sigma\neq i} - \vec{\theta}_{0\sigma\neq i})),$$

$$w^{(d)}_{\hat{q}_i\hat{p}_i}(\vec{X},\vec{\theta}) = \left[X_i^2\delta(\theta_i - \pi/4) - \frac{1}{2}X_i^2\delta(\sin\theta_i) - \frac{1}{2}X_i^2\delta(\theta_i - \pi/2) + \frac{i}{2\pi} \right]\delta(\sin(\vec{\theta}_{\sigma\neq i} - \vec{\theta}_{0\sigma\neq i})).$$

One can also find the symbols of other operators.

V. REPRESENTATION OF SYMBOLS OF OPERATORS IN TERMS OF REGULAR GENERALIZED FUNCTIONS

The dual symbols of the operators for the optical tomogram allow the representation in terms of regular generalized functions. The dual symbol $w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta})$ of some operator $\hat{A}$ defines the linear continuous functional on the set of optical tomographic distribution functions $w(\vec{X},\vec{\theta})$, which are from the Schwartz space in $\vec{X}$ and are infinitely differentiable in $\vec{\theta}$, belonging to the space of well-behaved test functions. Thus, the set of $w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta})$ actually defines the set of generalized functions on this space. Obviously, the equality of two symbols of one operator has to be defined as the functional equality or the equality of two generalized functions, i.e., two symbols are equal to each other when, for any tomogram $w(\vec{X},\vec{\theta})$ from the space of well-behaved test functions, we have the equality of the values of the corresponding functionals, denoted by these symbols. Thus there is a set of symbols for any operator $\hat{A}$ which are equal to each other in the meaning of generalized functions.

In the previous paragraph we have given the general expression for the dual symbol of an arbitrary operator and presented the singular forms of some operators. Singular forms of operators are convenient for analytical calculations, but for the numerical calculations and for processing experimental data the representation of the symbols in the form of regular generalized functions can be preferable.

If $(\hat{A})_W$ is an arbitrary operator in the Wigner representation with the existing average value, then the integral

$$\int \mathcal{R}[(\hat{A})_W W(\vec{q},\vec{p})](\vec{X},\vec{\theta})d^nX = \langle\hat{A}\rangle$$

does not depend on $\vec{\theta}$. Taking into account the definition of the dual symbol of the operator, we can write

$$\begin{aligned}\langle\hat{A}\rangle &= \int w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta})w(\vec{X},\vec{\theta})d^nXd^n\theta\\ &= \frac{1}{\pi^n}\int \mathcal{R}[(\hat{A})_W W(\vec{q},\vec{p})](\vec{X},\vec{\theta})d^nXd^n\theta\\ &= \frac{1}{\pi^n}\int \hat{\mathcal{R}}[(\hat{A})_W](\vec{X},\vec{\theta})w(\vec{X},\vec{\theta})d^nXd^n\theta,\end{aligned} \tag{38}$$

where $\hat{\mathcal{R}}[(\hat{A})_W](\vec{X},\vec{\theta})$ is an explicit form of operator $\hat{A}$ in the optical tomography representation which we found in Sec. III. In turn, the continuous linear functionals (generalized functions) of the form

$$\frac{1}{\pi^n}\int d^nXd^n\theta\hat{\mathcal{R}}[(\hat{A})_W](\vec{X},\vec{\theta})\,w(\vec{X},\vec{\theta})$$

acting on the set of optical tomograms of $w(\vec{X},\vec{\theta})$, found with the above rules, can be easily represented in the form of regular generalized functions.

Let us give one more useful formula for the dual symbol of the product of two operators. Let operators $(\hat{A})_W$ and $(\hat{B})_W$ act on the set $W(\vec{q},\vec{p})\in\mathcal{S}^{2n}$, such that for every $W(\vec{q},\vec{p})\in\mathcal{S}^{2n}$ the functions $(\hat{A})_W W(\vec{q},\vec{p})$ and $(\hat{B})_W W(\vec{q},\vec{p})$ also belong to $\mathcal{S}^{2n}$; then from (22) for the product of two operators and from formula (38) we have equality:

$$\begin{aligned}\langle\hat{A}\hat{B}\rangle &= \int w^{(d)}_{\hat{A}\hat{B}}(\vec{X},\vec{\theta})w(\vec{X},\vec{\theta})d^nXd^n\theta\\ &= \int w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta})\hat{\mathcal{R}}[(\hat{B})_W](\vec{X},\vec{\theta})\mathcal{R}[W(\vec{q},\vec{p})](\vec{X},\vec{\theta})d^nXd^n\theta\end{aligned}$$

or

$$w^{(d)}_{\hat{A}}(\vec{X},\vec{\theta})\hat{\mathcal{R}}[(\hat{B})_W](\vec{X},\vec{\theta}) \longrightarrow w^{(d)}_{\hat{A}\hat{B}}(\vec{X},\vec{\theta}).$$

For example, let us regularize the functional $\pi^{-n}\int\hat{\mathcal{R}}[(\hat{q}_i)_W]w(\vec{X},\vec{\theta})d^nXd^n\theta$; i.e., let us find a regular symbol for operator $\hat{q}_i$ in the optical tomography representation. From (18), (23), and (38) we have the following chain of equalities:

$$\begin{aligned}\langle\hat{q}_i\rangle &= \int w^{(d)}_{\hat{q}_i}(\vec{X},\vec{\theta})w(\vec{X},\vec{\theta})d^nXd^n\theta = \frac{1}{\pi^n}\int(\hat{q}_i)_w w(\vec{X},\vec{\theta})d^nXd^n\theta\\ &= \frac{1}{\pi^n}\int\left\{\sin\theta_i\left[\frac{\partial}{\partial X_i}\right]^{-1}\frac{\partial}{\partial\theta_i} + X_i\cos\theta_i + \frac{i}{2}\frac{\hbar}{m_i\omega_{0i}}\sin\theta_i\frac{\partial}{\partial X_i}\right\}w(\vec{X},\vec{\theta})d^nXd^n\theta\\ &= \frac{1}{\pi^n(2\pi\hbar)^n}\int\sin\theta_i W(\vec{q},\vec{p})\left(q_i\sin\theta_i - p_i\frac{\cos\theta_i}{m_i\omega_{0i}}\right)\prod_{\sigma=1}^n\delta\left(X_\sigma - q_\sigma\cos\theta_\sigma - p_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_{0\sigma}}\right)d^nq\,d^np\,d^nXd^n\theta\\ &\quad+\frac{1}{\pi^n}\int X_i\cos\theta_i w(\vec{X},\vec{\theta})d^nXd^n\theta + 0.\end{aligned} \tag{39}$$

Integrating in this expression the integral with the Wigner function over d^nX and over $d^n\theta$, we can find

$$\begin{aligned}\langle\hat{q}_i\rangle &= \frac{1}{2(2\pi\hbar)^n}\int q_iW(\vec{q},\vec{p})d^nq\,d^np\\ &\quad+\frac{1}{\pi^n}\int X_i\cos\theta_i w(\vec{X},\vec{\theta})d^nXd^n\theta\\ &= \frac{1}{2}\langle\hat{q}_i\rangle + \frac{1}{\pi^n}\int X_i\cos\theta_i w(\vec{X},\vec{\theta})d^nXd^n\theta.\end{aligned}$$

Thus we can write

$$w^{(d)}_{\hat{q}_i}(\vec{X},\vec{\theta}) = \frac{2}{\pi^n}X_i\cos\theta_i\,.$$

Similarly, we find

$$\begin{aligned}w^{(d)}_{\hat{p}_i}(\vec{X},\vec{\theta}) &= \frac{2m_i\omega_{0i}}{\pi^n}X_i\sin\theta_i,\\ w^{(d)}_{\hat{q}_i^2}(\vec{X},\vec{\theta}) &= \frac{X_i^2}{\pi^n}(1+2\cos 2\theta_i),\\ w^{(d)}_{\hat{p}_i^2}(\vec{X},\vec{\theta}) &= \frac{X_i^2m_i^2\omega_{0i}^2}{\pi^n}(1-2\cos 2\theta_i),\\ w^{(d)}_{\hat{q}_i\hat{p}_i}(\vec{X},\vec{\theta}) &= \frac{2m_i\omega_{0i}}{\pi^n}X_i^2\sin 2\theta_i + \frac{i\hbar}{2\pi^n}.\end{aligned}$$

From (27) we can find the symbols of the components of the angular momentum of the particle (we choose $\omega_{01}=\omega_{02}=\omega_{03}=\omega_0$):

$$\begin{aligned}w^{(d)}_{\hat{l}_1}(X_1,X_2,X_3;\theta_1,\theta_2,\theta_3) &= \frac{4m\omega_0}{\pi^3}X_2X_3\sin(\theta_3-\theta_2),\\ w^{(d)}_{\hat{l}_2}(X_1,X_2,X_3;\theta_1,\theta_2,\theta_3) &= \frac{4m\omega_0}{\pi^3}X_3X_1\sin(\theta_1-\theta_3),\\ w^{(d)}_{\hat{l}_3}(X_1,X_2,X_3;\theta_1,\theta_2,\theta_3) &= \frac{4m\omega_0}{\pi^3}X_1X_2\sin(\theta_2-\theta_1).\end{aligned}$$

From formulas (29) and (30) we can find

$$w^{(d)}_{\hat{a}_i}(\vec{X},\vec{\theta}) = \frac{\sqrt{2}}{\pi^n}\sqrt{\frac{m_i\omega_{0i}}{\hbar}}X_i(\cos\theta_i + i\sin\theta_i),$$

$$w^{(d)}_{\hat{a}^\dagger_i}(\vec{X},\vec{\theta}) = \frac{\sqrt{2}}{\pi^n}\sqrt{\frac{m_i\omega_{0i}}{\hbar}}X_i(\cos\theta_i - i\sin\theta_i),$$

$$w^{(d)}_{\hat{a}^\dagger_i\hat{a}_i}(\vec{X},\vec{\theta}) = w^{(d)}_{\hat{N}_i}(\vec{X},\vec{\theta}) = \frac{1}{\pi^n}\left(\frac{m_i\omega_{0i}}{\hbar}X_i^2 - 1/2\right).$$

It is also clear that

$$w^{(d)}_{\hat{1}}(\vec{X},\vec{\theta}) = \frac{1}{\pi^n}.$$

So, if we have the experimental data for the optical tomogram (for instance, in one-dimensional case) in the form of histogram $\{w_{\rm ex}(X_k,\theta_l)\}$ normalized by the conditions

$$\sum_k w_{\rm ex}(X_k,\theta_l)\triangle X = 1, \quad \sum_{k,l} w_{\rm ex}(X_k,\theta_l)\triangle X\triangle\theta = \pi,$$

where $\triangle X = X_{k+1} - X_k$ is a step of the histogram and $\triangle\theta = \theta_{l+1} - \theta_l$ is the distance between the phases at which measurements are taken, then we can approximately find the average value of observable $\hat{A}$ from the formula

$$\langle\hat{A}\rangle \approx \sum_{k,l} w^{(d)}_{\hat{A}}(X_k,\theta_l)w_{\rm ex}(X_k,\theta_l)\triangle X\triangle\theta,$$

and the precision of the average value will be dependent on the accuracy of the experimental tomogram $w_{\rm ex}(X_k,\theta_l)$.

Having different forms of dual symbols of the same operators, we can write test expressions for the experimentally measured tomograms. Thus, for the $\hat{q}$ quadrature we have

$$\langle\hat{q}\rangle = \int X\cos\theta\,\delta(\sin\theta)w_{\rm ex}(X,\theta)dX\,d\theta = \frac{2}{\pi}\int X\cos\theta\, w_{\rm ex}(X,\theta)dX\,d\theta. \quad (40)$$

Similar test expressions can be written for the other operators.

VI. CONCLUSION

To summarize, we point out the main results of this work. We obtained the correspondence rules and explicit expressions for operators of physical quantities in the optical tomography representation. We presented the general formalism for symbols of operators in this representation. We found an explicit expressions for the dual symbols of physical quantities in terms of regular generalized functions, and we gave suggestions for their use for experimental data processing in quantum tomography.

The expressions for operators found in this work provide the possibility of direct calculations of physical quantities from the optical tomogram without transformation of the tomogram into the Wigner function or density matrix. The developed formalism can be used in the homodyne tomography of photon states to study quantum phenomena with the help of the probability distribution as the primary concept of state without using the quasiprobability functions.

In Refs. [23,38–41] suggestions to check the quantum inequalities such as quadrature uncertainty relations and different types of entropic uncertainty relations in experiments with homodyne detection of the photon states [19–22] in microwave photon tomography [29] were proposed. For data processing in this experiment one needs to evaluate the moments of photon quadratures. To evaluate these moments one needs to integrate the optical tomograms (probability distributions) with dual tomographic symbols of the different degrees of the photon quadrature components.

The evaluation can be done by using either regular form of the generalized function corresponding to the photon quadratures or the equivalent (in principle) singular form of the generalized function. On the other hand due to unavoidable inaccuracy of the measured data the regular form of the dual tomographic symbols of the quadratures may provide essential numerical advantages compared to using the singular form of the dual tomographic symbols of physical observables. We will study the application of the results obtained in this work on the explicit form of the tomographic symbols to the analysis of the experiments directed to check the quantum inequalities in a future presentation.

ACKNOWLEDGMENTS

Ya.A.K. and V.I.M. thank the Russian Foundation for Basic Research for partial support under Projects No. 10-02-00312 and No. 11-02-00456.

[1] G. Esposito, G. Marmo, and E. C. G. Sudarshan, *From Classical to Quantum Mechanics* (Cambridge University Press, Cambridge, 2004).
[2] F. Bayen, M. Flato, C. Fronsdal, A. Lechnerovicz, and D. Sternheimer, Lett. Math. Phys. **1**, 521 (1975).
[3] H. Grönewold, Physica (Amsterdam) **12**, 405 (1946).
[4] J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).
[5] C. Zachos, J. Math. Phys. **41**, 5129 (2000).
[6] T. Curtright, T. Uematsu, and C. K. Zachos, J. Math. Phys. **42**, 2396 (2001).
[7] F. A. Berezin, Math. USSR Izv. **8**, 1109 (1974).
[8] A. Voros, Phys. Rev. A **40**, 6814 (1989).
[9] V. V. Dodonov and V. I. Man'ko, in *Invariants and Evolution of Nonstationary Quantum Systems*, Proceedings of the Lebedev Physics Institute, Vol. 183, edited by M. A. Markov (Nova Science, Commack, NY, 1989).
[10] O. V. Man'ko, V. I. Man'ko, and G. Marmo, Phys. Scr. **62**, 446 (2000).
[11] O. V. Man'ko, V. I. Man'ko, and G. Marmo, J. Phys. A: Math. Gen. **35**, 699 (2002).
[12] S. Mancini, V. I. Man'ko, and P. Tombesi, J. Opt. B: Quantum Semiclassical Optics **7**, 615 (1995).
[13] G. M. D'Ariano, S. Mancini, V. I. Man'ko, and P. Tombesi, J. Opt. B: Quantum Semiclassical Optics **8**, 1017 (1996).

[14] S. Mancini, V. I. Man'ko, and P. Tombesi, Phys. Lett. A **213**, 1 (1996).
[15] S. Mancini, V. I. Man'ko, and P. Tombesi, Found. Phys. **27**, 801 (1997).
[16] V. V. Dodonov and V. I. Man'koPhys. Lett. A **229**, 335 (1997).
[17] V. I. Man'ko and O. V. Man'ko, J. Exp. Theor. Phys. **85**, 430 (1997).
[18] V. I. Man'ko, L. Rosa, and P. Vitale, Phys. Lett. B **439**, 328 (1998),
[19] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993).
[20] S. Schiller, G. Breitenbach, S. F. Pereira, T. Müller, and J. Mlynek, Phys. Rev. Lett. **77**, 2933 (1996).
[21] A. I. Lvovsky and M. G. Raymer, Rev. Mod. Phys. **81**, 299 (2009).
[22] T. Kiesel, W. Vogel, V. Parigi, A. Zavatta, and M. Bellini, Phys. Rev. A **78**, 021804 (2008).
[23] V. I. Manko, G. Marmo, A. Porzio, S. Solimeno, and F. Ventriglia, Phys. Scr. **83**, 045001 (2011).
[24] V. V. Dodonov, J. Opt. B: Quantum Semiclassical Optics **4**, R1 (2002).
[25] A. Mandilara, E. Karpov, and N. J. Cerf, arXiv:0910.3473v2 [quant-ph].
[26] O. V. Man'ko, V. I. Man'ko, G. Marno, and P. Vitale, Phys. Lett. A **360**, 522 (2007).
[27] J. Bertrand and P. Bertrand, Found. Phys. **17**, 397 (1987).
[28] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).
[29] C. Eichler, D. Bozyigit, C. Lang, L. Steffen, J. Fink, and A. Wallraff, Phys. Rev. Lett. **106**, 220503 (2011).
[30] E. Wigner, Phys. Rev. **40**, 749 (1932).
[31] J. Radon, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. Nat. kl. **69**, 262 (1917).
[32] A. Ibort, V. I. Manko, G. Marmo, A. Simoni, and F. Ventriglia, Phys. Scr. **79**, 065013 (2009).
[33] Ya. A. Korennoy and V. I. Man'ko, Phys. Rev. A **83**, 053817 (2011).
[34] Ya. A. Korennoy and V. I. Man'ko, J. Russ. Laser Res. **32**, 74 (2011).
[35] Ya. A. Korennoy and V. I. Man'ko, J. Russ. Laser Res. **32**, 338 (2011).
[36] Ya. A. Korennoy and V. I. Man'ko, J. Russ. Laser Res. **32**, 153 (2011).
[37] I. M. Gelfand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. 1.
[38] V. I. Manko, G. Marmo, A. Simoni, A. Stern, E. C. G. Sudarshan, and F. Ventriglia, Phys. Lett. A **351**, 1 (2006).
[39] V. I. Manko, G. Marmo, A. Simoni, and F. Ventriglia, Phys. Scr. **T147**, 014021 (2012).
[40] S. N. Filippov and V. I. Manko, Phys. Rev. A **84**, 033827 (2011).
[41] M. A. Man'ko and V. I. Man'ko, Found. Phys. **41**, 330 (2011).