

# Linear dynamics of quantum-classical hybrids

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A formulation of quantum-classical hybrid dynamics is presented which concerns the direct coupling of classical and quantum mechanical degrees of freedom. It is of interest for applications in quantum mechanical approximation schemes and may be relevant for the foundations of quantum mechanics, in particular, when it comes to experiments exploring the quantum-classical border. The present linear theory differs from the nonlinear ensemble theory of Hall and Reginatto but shares with it the fulfillment of all consistency requirements discussed in the literature, while earlier attempts have failed in this respect. Our work is based on the representation of quantum mechanics in the framework of classical analytical mechanics by A. Heslot, showing that notions of states in phase space, observables, Poisson brackets, and related canonical transformations can be naturally extended to quantum mechanics. This is suitably generalized for quantum-classical hybrids here.

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## I. INTRODUCTION

The hypothetical direct coupling of quantum (QM) mechanical and classical (CL) degrees of freedom—*hybrid dynamics*—presents a departure from QM mechanics that has been researched for decades for practical as well as theoretical reasons. In particular, the standard Copenhagen interpretation has led to the unresolved measurement problem, which, together with the fact that QM mechanics needs such interpretation in order to be operationally well defined, may indicate that it deserves amendments. In this context, it has been recognized early on that a theory which dynamically bridges the QM-CL divide should have an impact on the measurement problem [1], besides being essential for attempts to describe consistently the interaction between QM matter and CL space time [2].

Numerous works have appeared in the attempt to formulate hybrid dynamics in a satisfactory way. However, they have generally been found to be deficient for one reason or another. This has led to various no-go theorems accompanying a list of desirable properties or consistency requirements (see, e.g., Refs. [3,4]):

- (1) Conservation of energy.
- (2) Conservation and positivity of probability.
- (3) Separability of QM and CL subsystems in the absence of their interaction, recovering the correct QM and CL equations of motion.
- (4) Consistent definitions of states and observables; existence of a Lie bracket structure on the algebra of observables that suitably generalizes Poisson and commutator brackets.
- (5) Existence of canonical transformations generated by the observables; invariance of the CL sector under canonical transformations performed on the QM sector only, and vice versa.
- (6) Existence of generalized Ehrenfest relations (i.e., the correspondence limit) which, for bilinearly coupled CL and QM oscillators, are to assume the form of the CL equations of motion (“Peres-Terno benchmark” test [5]).

These issues have been reviewed in recent works by Hall and Reginatto. Furthermore, there, they have introduced the first viable theory of hybrid dynamics that agrees with all the points listed above [6–8]. Their ensemble theory is based on configuration space, which entails a certain nonlinearity of the action functional from which it is derived. This nonlinearity leads to effects and a proposal to possibly falsify the theory experimentally [9]. We comment on this issue in due course (Sec. VC).

In fact, the aim of the present paper is to set up an alternative theory of hybrid dynamics, which is based on notions of phase space. This is partly motivated by work on related topics of general linear dynamics and CL path integrals [10,11]. Presently, we extend the work of Heslot, who has demonstrated that QM mechanics can entirely be rephrased in the language and formalism of CL analytical mechanics [12]. We thus introduce unified notions of states on phase space, observables, canonical transformations, and a generalized QM-CL Poisson bracket in particular. This leads to an intrinsically linear hybrid theory, which fulfills all consistency requirements as well.

It may be worthwhile to comment on the relevance of hybrid dynamics, even if one is *not* inclined to modify certain ingredients of QM theory. There is clearly practical interest in various forms of hybrid dynamics, in particular, in nuclear, atomic, or molecular physics. The Born-Oppenheimer approximation, for example, is based on a separation of interacting slow and fast degrees of freedom of a compound object. The former are treated as being approximately CL; the latter, as being of a QM mechanical nature. Furthermore, mean-field theory, based on the expansion of QM mechanical variables into a CL part plus QM fluctuations, leads to another approximation scheme and another form of hybrid dynamics. This has been reviewed more generally for macroscopic QM phenomena in Ref. [13].

In all these cases hybrid dynamics is considered as an *approximate description* of an intrinsically QM mechanical object. This can lead to new insights, for example, to an alternative derivation of geometric forces and Berry’s phase [14]. Such considerations are, and will become increasingly, important for the precise manipulation of QM mechanical objects by apparently, and for all practical purposes, CL means, especially in the mesoscopic regime.

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Furthermore, the backreaction effect of QM fluctuations on CL degrees of freedom might be of considerable importance, in particular, if they originate in physically distinct ways. We recall continuing discussions of the “semiclassical” Einstein equation coupling the CL metric of space time to the expectation value of the energy-momentum tensor of quantized matter fields. Can this be made into a consistent hybrid theory leaving gravity unquantized? This has recently been re-examined, for example, in Refs. [6,8,15,16]; various related aspects have been discussed, for example, in Refs. [17–23].

Finally, concerning speculative ideas about the emergence of QM mechanics from a coarse-grained deterministic dynamics (see, e.g., Refs. [24–26], with numerous references to earlier work), the backreaction problem can be more provocatively stated as the problem of the interplay of fluctuations among underlying deterministic and emergent QM mechanical degrees of freedom. Or, to put it differently: *Can QM mechanics be seeded?*

The remainder of the paper is organized as follows. In Sec. II, the results of Heslot’s work are represented, in order to make the paper self-contained, and we frequently refer to it in what follows. In Sec. III, we introduce hybrid phase-space ensembles and, in particular, the QM-CL Poisson bracket which is central to our approach; the important issue of separability is resolved, and time evolution discussed. In Sec. IV, hybrid dynamics is studied, incorporating QM-CL interaction. Energy conservation and Ehrenfest relations, especially for bilinearly coupled oscillators, are derived there. In Sec. V, we discuss various aspects of the present theory, in particular, the possibility of having CL-environment-induced decoherence, the QM-CL backreaction, a deviation from the Hall-Reginatto proposal predicted by the hybrid dynamics developed in this paper, and the closure of the algebra of hybrid observables. Section VI presents concluding remarks.

## II. HAMILTONIAN DYNAMICS REVISITED

In Secs. II A and II B, we briefly present some important results drawn from the remarkably clear exposition of CL Hamiltonian mechanics and its generalization incorporating QM mechanics by Heslot [12]. This forms the starting point of our discussion of the hypothetical direct coupling between QM and CL degrees of freedom in Sec. III.

### A. Classical mechanics

The evolution of a CL object is described with respect to its  $2n$ -dimensional phase space, which is identified as its *state space*. A real-valued regular function on the state space defines an *observable*, i.e., a differentiable function on this smooth manifold.

Darboux’s theorem shows that there always exist (local) systems of so-called *canonical coordinates*, commonly denoted  $(x_k, p_k)$ ,  $k = 1, \dots, n$ , such that the *Poisson bracket* of any pair of observables  $f, g$  assumes the standard form [27]:

$$\{f, g\} = \sum_k \left( \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial x_k} \right). \quad (1)$$

This is consistent with  $\{x_k, p_l\} = \delta_{kl}$ ,  $\{x_k, x_l\} = \{p_k, p_l\} = 0$ ,  $k, l = 1, \dots, n$ , and reflects the bilinearity, antisymmetry,

derivation-like product formula, and Jacobi identity which define a Lie bracket operation,  $f, g \rightarrow \{f, g\}$ , mapping a pair of observables to an observable.

Compatibility with the Poisson bracket structure restricts general transformations  $\mathcal{G}$  of the state space to so-called *canonical transformations* which do not change the physical properties of the object under study, e.g., a translation, a rotation, a change of inertial frame, or evolution in time. Such a  $\mathcal{G}$  induces a change in an observable,  $f \rightarrow \mathcal{G}(f)$ , and is an automorphism of the state space compatible with its Poisson bracket structure if and only if, for any pair of observables  $f, g$ ,

$$\mathcal{G}(\{f, g\}) = \{\mathcal{G}(f), \mathcal{G}(g)\}. \quad (2)$$

Due to the Lie-group structure of the set of canonical transformations, it is sufficient to consider infinitesimal transformations generated by the elements of the corresponding Lie algebra. Then an *infinitesimal transformation*  $\mathcal{G}$  is *canonical* if and only if, for any observable  $f$ , the map  $f \rightarrow \mathcal{G}(f)$  is given by  $f \rightarrow f' = f + \{f, g\}\delta\alpha$ , with some observable  $g$ , the so-called *generator* of  $\mathcal{G}$ , and  $\delta\alpha$  an infinitesimal real parameter.

Thus, for the canonical coordinates, in particular, an infinitesimal canonical transformation amounts to

$$x_k \rightarrow x'_k = x_k + \frac{\partial g}{\partial p_k} \delta\alpha, \quad (3)$$

$$p_k \rightarrow p'_k = p_k - \frac{\partial g}{\partial x_k} \delta\alpha, \quad (4)$$

employing the Poisson bracket given in Eq. (1).

This analysis shows the fundamental relation between observables and generators of infinitesimal canonical transformations in CL Hamiltonian mechanics. For example, the energy  $\mathcal{H}_{\text{CL}}$  given by the CL Hamiltonian function is the generator of time evolution: for  $g = \mathcal{H}_{\text{CL}}$  and  $\delta\alpha = \delta t$ , Eqs. (3) and (4) are equivalent to Hamilton’s equations, considering an infinitesimal time step  $\delta t$ .

### B. Quantum mechanics

An important achievement of Heslot’s work is the realization that the analysis summarized in Sec. II A can be generalized and applied to QM mechanics; in particular, the dynamical aspects of QM mechanics thus find a description in CL terms borrowed from Hamiltonian mechanics. Some related ideas have been presented in Ref. [28].

#### 1. Preliminaries

To begin with, we recall that the Schrödinger equation and its adjoint can be obtained by requiring the variation with respect to state vector  $|\Psi\rangle$  and adjoint state vector  $\langle\Psi|$ , respectively, of the following action  $S$  to vanish:

$$S := \int dt \langle\Psi(t)|(i\partial_t - \hat{H})|\Psi(t)\rangle \equiv \int dt L, \quad (5)$$

which involves the self-adjoint Hamilton operator  $\hat{H}$  pertaining to the physical object under study. The adjoint equation follows after a partial integration, provided the surface terms do not contribute. This is guaranteed by the *normalization*

condition,

$$\langle \Psi(t) | \Psi(t) \rangle \stackrel{!}{=} \text{constant} \equiv 1, \quad (6)$$

which is an essential ingredient of the probability interpretation associated with state vectors. Adding here that state vectors that differ by an *unphysical constant phase* are to be identified, we recover that the *QM mechanical state space* is formed by the rays of the underlying Hilbert space, i.e., forming a complex projective space.

Making use of the Lagrangian  $L$ , defined as the integrand of the above action  $S$ , we define a momentum conjugate to the state vector,

$$\langle \Pi | := \frac{\partial L}{\partial |\dot{\Psi}\rangle} = i \langle \Psi |, \quad (7)$$

with  $|\dot{\Psi}\rangle := \partial_t |\Psi\rangle$ , and obtain the corresponding Hamiltonian function:

$$\langle \Pi | \dot{\Psi} \rangle - L = -i \langle \Pi | \hat{H} | \Psi \rangle =: \mathcal{H}(\Pi, \Psi). \quad (8)$$

Finally, considering Hamilton's equations, deriving from  $\mathcal{H}$ ,

$$\partial_t |\Psi\rangle = \frac{\partial \mathcal{H}}{\partial \langle \Pi |} = -i \hat{H} |\Psi\rangle, \quad (9)$$

$$\partial_t \langle \Pi | = -\frac{\partial \mathcal{H}}{\partial |\Psi\rangle} = i \langle \Pi | \hat{H}, \quad (10)$$

we see indeed that they represent Schrödinger's equation and its adjoint, using  $\langle \Pi | = i \langle \Psi |$ , keeping the essential normalization condition, (6), in mind.

## 2. The oscillator representation

QM mechanical evolution can be described by a unitary transformation,  $|\Psi(t)\rangle = \hat{U}(t - t_0) |\Psi(t_0)\rangle$ , with  $U(t - t_0) = \exp[-i \hat{H}(t - t_0)]$ , which formally solves the Schrödinger equation. It follows immediately that a stationary state, characterized by  $\hat{H} |\phi_i\rangle = E_i |\phi_i\rangle$ , with a real energy eigenvalue  $E_i$ , performs a simple harmonic motion, i.e.,  $|\psi_i(t)\rangle = \exp[-i E_i(t - t_0)] |\psi_i(t_0)\rangle \equiv \exp[-i E_i(t - t_0)] |\phi_i\rangle$ . Henceforth, we assume a denumerable set of such eigenstates of the Hamilton operator.

Having recognized already the Hamiltonian character of the underlying equation(s) of motion, the harmonic motion suggests introducing what may be called *oscillator representation* for any states. More generally, we consider the expansion of any state vector with respect to a complete orthonormal basis,  $\{|\Phi_i\rangle\}$ :

$$|\Psi\rangle = \sum_i |\Phi_i\rangle (X_i + i P_i) / \sqrt{2}, \quad (11)$$

where the generally time-dependent expansion coefficients are explicitly written in terms of real and imaginary parts,  $X_i, P_i$ . Employing this expansion allows us to evaluate more explicitly the Hamiltonian function introduced in Eq. (8), i.e.,  $\mathcal{H} = \langle \Psi | \hat{H} | \Psi \rangle$ :

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \sum_{i,j} \langle \Phi_i | \hat{H} | \Phi_j \rangle (X_i - i P_i) (X_j + i P_j) \\ &=: \mathcal{H}(X_i, P_i). \end{aligned} \quad (12)$$

Choosing especially the set of energy eigenstates,  $\{|\phi_i\rangle\}$ , as the basis for the expansion, we obtain

$$\mathcal{H}(X_i, P_i) = \sum_i \frac{E_i}{2} (P_i^2 + X_i^2); \quad (13)$$

hence the name *oscillator representation*. The simple reasoning leading to this result clearly indicates that  $(X_i, P_i)$  may play the role of *canonical coordinates* in the description of a QM mechanical object and its evolution with respect to the state space. However, several points need to be clarified, in order to validate this interpretation.

First, with  $(X_i, P_i)$  as canonical coordinates and  $\mathcal{H}$  as the Hamiltonian function, we verify that the Schrödinger equation is recovered by evaluating  $|\dot{\Psi}\rangle = \sum_i |\Phi_i\rangle (\dot{X}_i + i \dot{P}_i) / \sqrt{2}$  according to Hamilton's equations of motion:

$$\begin{aligned} \dot{X}_i &= \frac{\partial \mathcal{H}(X_j, P_j)}{\partial P_i} \\ &= -\frac{i}{2} \sum_j [H_{ij}(X_j + i P_j) - (X_j - i P_j) H_{ji}], \end{aligned} \quad (14)$$

$$\begin{aligned} \dot{P}_i &= -\frac{\partial \mathcal{H}(X_j, P_j)}{\partial X_i} \\ &= -\frac{1}{2} \sum_j [H_{ij}(X_j + i P_j) + (X_j - i P_j) H_{ji}], \end{aligned} \quad (15)$$

where  $H_{ij} := \langle \Phi_i | \hat{H} | \Phi_j \rangle = H_{ji}^*$ . Inserting these terms and using Eq. (11) leads to  $|\dot{\Psi}\rangle = -i \hat{H} |\Psi\rangle$ , as expected. Using  $\mathcal{H}$  in the special form given by Eq. (13), we see that a zero mode with  $E_i = 0$  automatically leads to  $(X_i, P_i) = \text{constant}$ .

Second, the *constraint*  $\mathcal{C} := \langle \Psi | \Psi \rangle \stackrel{!}{=} 1$  [cf. Eq. (6)] becomes

$$\mathcal{C}(X_i, P_i) = \frac{1}{2} \sum_i (X_i^2 + P_i^2) \stackrel{!}{=} 1. \quad (16)$$

Thus, the vector with components given by the canonical coordinates  $(X_i, P_i)$ ,  $i = 1, \dots, N$ , is constrained to the surface of a  $2N$ -dimensional sphere with radius  $\sqrt{2}$ . This constraint obviously presents a major difference to CL Hamiltonian mechanics.

Following the discussion in Sec. II A, it is natural to introduce also here a *Poisson bracket* for any two *observables* on the *spherically compactified state space*, i.e., real-valued regular functions  $F, G$  of the coordinates  $(X_i, P_i)$ :

$$\{F, G\} = \sum_i \left( \frac{\partial F}{\partial X_i} \frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial X_i} \right) \quad (17)$$

[cf. Eq. (1)]. Then, as before, the Hamiltonian acts as the generator of time evolution of any observable  $O$ , i.e.,

$$\frac{dO}{dt} = \partial_t O + \{O, \mathcal{H}\}. \quad (18)$$

In particular, it is straightforward to verify with the help of Eqs. (14) and (15) that the constraint, Eq. (16), is conserved under the Hamiltonian flow:

$$\frac{d\mathcal{C}}{dt} = \{\mathcal{C}, \mathcal{H}\} = 0. \quad (19)$$

Therefore, it is sufficient to impose this constraint, which implements the normalization of the QM mechanical state, on the initial condition of time evolution.

It remains to demonstrate the *compatibility* of the notion of *observable* introduced here—as in CL mechanics [cf. the discussion leading to Eq. (2) and thereafter]—with the one adopted in QM mechanics. This concerns, in particular, the implementation of *canonical transformations* and the role of observables as their generators.

### 3. Canonical transformations and quantum observables

The Hamiltonian function has been introduced as an observable in Eq. (12), which provides a direct relation to the corresponding QM observable, namely, the expectation value of the self-adjoint Hamilton operator. This is an indication of the general structure to be discussed now.

Referring to Sec. III of Heslot's work [12] for details of the derivations, here we summarize the main points, which are useful in the following.

*a. Compatibility of unitary transformations and Poisson structure.* The canonical transformations discussed in Sec. II A represent automorphisms of the CL state space which are compatible with the Poisson brackets. In QM mechanics, automorphisms of the Hilbert space are implemented by unitary transformations,  $|\Psi'\rangle = \hat{U}|\Psi\rangle$ , with  $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = 1$ . This implies a transformation of the canonical coordinates here, i.e., of the expansion coefficients  $(X_i, P_i)$  introduced in Eq. (11):

$$\begin{aligned} |\Psi'\rangle &= \sum_{i,j} |\Phi_i\rangle \langle \Phi_i | \hat{U} | \Phi_j \rangle \frac{X_j + iP_j}{\sqrt{2}} \\ &= \sum_i |\Phi_i\rangle \frac{X'_i + iP'_i}{\sqrt{2}}. \end{aligned} \quad (20)$$

Splitting the matrix elements  $U_{ij} := \langle \Phi_i | \hat{U} | \Phi_j \rangle$  into real and imaginary parts and separating Eq. (20) accordingly, using orthonormality of the basis, yields the transformed coordinates in terms of the original ones. Then a simple calculation, employing the Poisson bracket defined in Eq. (17), shows that  $\{X'_i, P'_j\} = \delta_{ij}$  and  $\{X'_i, X'_j\} = \{P'_i, P'_j\} = 0$ , as before. The fundamental Poisson brackets remain invariant under unitary transformations. More generally, this implies [27] that  $U(\{F, G\}) = \{U(F), U(G)\}$  [cf. Eq. (2)]. Thus, *unitary transformations on Hilbert space are canonical transformations on the  $(X, P)$  state space.*

*b. Self-adjoint operators as observables.* Any infinitesimal unitary transformation  $\hat{U}$  can be generated by a self-adjoint operator  $\hat{G}$ , such that

$$\hat{U} = 1 - i\hat{G}\delta\alpha, \quad (21)$$

which will lead to the QM mechanical relation between an observable and a self-adjoint operator, replacing the CL construction in Sec. II A. In fact, straightforward calculation along the lines of item *a.* above, splitting matrix elements of  $\hat{G}$ , with  $G_{ji}^* = G_{ij}$ , into real and imaginary parts, shows that

in the present case we have

$$X_i \rightarrow X'_i = X_i + \frac{\partial \langle \Psi | \hat{G} | \Psi \rangle}{\partial P_i} \delta\alpha, \quad (22)$$

$$P_i \rightarrow P'_i = P_i - \frac{\partial \langle \Psi | \hat{G} | \Psi \rangle}{\partial X_i} \delta\alpha. \quad (23)$$

Due to the phase arbitrariness— $\hat{U}$  and  $\hat{U} \cdot \exp(i\theta)$ , with constant phase  $\theta$ , are physically equivalent—the operator  $\hat{G}$  is defined up to an additive constant. This constant is naturally chosen such that the relation between an observable  $G$ , defined in analogy to Sec. II A, and a self-adjoint operator  $\hat{G}$  can be inferred from Eqs. (22) and (23),

$$G(X_i, P_i) = \langle \Psi | \hat{G} | \Psi \rangle, \quad (24)$$

by comparison with the CL result, Eqs. (3) and (4). In conclusion, a *real-valued regular function  $G$  of the state is an observable if and only if there exists a self-adjoint operator  $\hat{G}$  such that Eq. (24) holds.* Note that all QM observables are *quadratic forms* in the  $X_i$ 's and  $P_i$ 's. This explains that there are much fewer of them than in the corresponding CL case.

*c. Commutators as Poisson brackets.* Relation (24) between observables and self-adjoint operators is linear and admits  $\hat{1}$  as unit operator, since  $\langle \Psi | \Psi \rangle \stackrel{!}{=} 1$ . Therefore, the addition of observables and multiplication by a scalar of observables are well defined and translate into the corresponding expressions for the operators. One may then consider the Poisson bracket, (17), of two observables and demonstrate the important result [12],

$$\{F, G\} = \langle \Psi | \frac{1}{i} [\hat{F}, \hat{G}] | \Psi \rangle, \quad (25)$$

with both sides of the equality considered as functions of the variables  $X_i, P_i$ , of course, and with the commutator defined as usual,  $[\hat{F}, \hat{G}] := \hat{F}\hat{G} - \hat{G}\hat{F}$ . This shows that the *commutator is a Poisson bracket with respect to the  $(X, P)$  state space* and relates the algebra of observables, in the sense of the CL construction of Sec. II A, to the algebra of self-adjoint operators in QM mechanics.

*d. Normalization, phase arbitrariness, and admissible observables.* Coming back to the normalization condition  $\langle \Psi | \Psi \rangle \stackrel{!}{=} 1$ , which compactifies the state space [cf. the constraint, Eq. (16)], it must be preserved under infinitesimal canonical transformations, since it belongs to the structural characteristics of the state space. By Eqs. (22)–(24), an infinitesimal canonical transformation generated by an observable  $G$  leads to  $\mathcal{C}(X_i, P_i) \rightarrow \mathcal{C}(X'_i, P'_i)$ , with

$$\begin{aligned} \mathcal{C}(X'_i, P'_i) &= \mathcal{C}(X_i, P_i) + \sum_j \left( \frac{\partial G}{\partial P_j} X_j - \frac{\partial G}{\partial X_j} P_j \right) \delta\alpha \\ &\quad + \mathcal{O}(\delta\alpha^2). \end{aligned} \quad (26)$$

Therefore, a *necessary* condition which observables must fulfill is the vanishing of the term  $\propto \delta\alpha$  here, i.e., the *invariance of the constraint* under such transformations,  $\mathcal{C}(X'_i, P'_i) \stackrel{!}{=} \mathcal{C}(X_i, P_i)$ . It is *not sufficient*, since the product  $G_1 G_2$  of two observables—which fulfill this condition individually and, therefore, their product as well—does not necessarily represent an observable: the corresponding self-adjoint operators do not necessarily commute, i.e., generally we have  $(\hat{G}_1 \hat{G}_2)^\dagger =$



$G_2 G_1 \neq G_1 G_2$ . Incidentally, the condition of the vanishing second term on the right-hand side of Eq. (26) follows also more generally, via Eq. (24), from the requirement that any observable  $G$  is invariant under an infinitesimal phase transformation  $|\Psi\rangle \rightarrow |\Psi\rangle \cdot \exp(i\delta\theta)$ , with constant  $\delta\theta$ ,  $G(X'_i, P'_i) \stackrel{!}{=} G(X_i, P_i)$ . Conversely, assuming this *phase invariance* of the observables, we recover that Hilbert-space vectors differing by an arbitrary constant phase are indistinguishable and represent the same physical state.

We note that any observable  $G$  with an expansion as in Eq. (12) automatically satisfies the invariance requirements of item d., above, the vanishing of the second term on the right-hand side of Eq. (26), in particular. Explicit calculation shows

$$\{\mathcal{C}, G\} = \sum_j \left( \frac{\partial G}{\partial P_j} X_j - \frac{\partial G}{\partial X_j} P_j \right) = 0, \quad (27)$$

assuming that

$$G(P_i, X_i) := \langle \Psi | \hat{G} | \Psi \rangle = \frac{1}{2} \sum_{i,j} G_{ij}(X_i - iP_i)(X_j + iP_j), \quad (28)$$

and where  $G_{ij} := \langle \Phi_i | \hat{G} | \Phi_j \rangle = G_{ji}^*$ , for a self-adjoint operator  $\hat{G}$ .

In conclusion, QM mechanics shares with CL mechanics an even-dimensional state space, a Poisson structure, and a related algebra of observables. Yet it differs essentially by a restricted set of observables and the requirements of phase invariance and normalization, which compactify the underlying Hilbert space to the complex projective space formed by its rays.

### III. HYBRID PHASE-SPACE ENSEMBLES

So far, we have described the Hamiltonian formalism of CL mechanics and its generalization, which covers QM mechanics, by adding more structure to the relevant state space. With the Hamiltonian equations of motion at hand, we could proceed to study the evolution and direct coupling of CL and QM objects. However, it is convenient to study the evolution of ensembles over the state (or phase) space instead [10,11]. Last but not least, this will allow us to include QM mechanical mixed states, thus generalizing beyond the above tacitly assumed pure states.

In this section, we still *neglect interactions between CL and QM sectors* of a combined system, the study of which will lead us to truly QM-CL hybrids only in Sec. IV. We describe a *QM-CL hybrid ensemble* by a real-valued, positive semidefinite, normalized, and possibly time-dependent regular function, the *probability distribution*  $\rho$ , on the *Cartesian product state space* canonically coordinated by  $2(n+N)$ -tuples  $(x_k, p_k; X_i, P_i)$ ; we reserve variables  $x_k, p_k$ ,  $k = 1, \dots, n$  for the CL sector (cf. Sec. II A) and variables  $X_i, P_i$ ,  $i = 1, \dots, N$  for the QM sector (cf. Sec. II B). One or the other sector of the state space may eventually be infinite-dimensional. A physical realization of such an ensemble can be imagined as a collection of representatives of the combined system with different initial conditions.

In order to qualify as *observable*, the distribution  $\rho$  additionally has to obey the constraint induced by the extra structure of the QM sector of state space (see Sec. II B 3 d.). Evaluating the expectation of the corresponding self-adjoint, positive semidefinite, trace-normalized *density operator*  $\hat{\rho}$  in a generic state  $|\Psi\rangle$ , Eq. (11), we have  $\rho(x_k, p_k; X_i, P_i) := \langle \Psi | \hat{\rho}(x_k, p_k) | \Psi \rangle$ , and

$$\rho(x_k, p_k; X_i, P_i) = \frac{1}{2} \sum_{i,j} \rho_{ij}(x_k, p_k)(X_i - iP_i)(X_j + iP_j), \quad (29)$$

with  $\rho_{ij}(x_k, p_k) := \langle \Phi_i | \hat{\rho}(x_k, p_k) | \Phi_j \rangle = \rho_{ji}^*(x_k, p_k)$ . This assures that  $\rho$  (or the marginal QM distribution obtained by integrating over the CL variables), as the generator of a canonical transformation, does not violate the normalization constraint and phase invariance; in particular, it follows that  $\{\mathcal{C}, \rho\} = 0$  [cf. Eq. (27)].

Furthermore, *positive semidefiniteness* of  $\rho$  imposes constraints on any other observable  $G$  (g) of the QM (CL) sector, which can generate a canonical transformation. Considering infinitesimal transformations in both sectors [cf. Eqs. (3) and (4) and Eqs. (22) and (24)], we obtain  $\rho(x_k, p_k; X_i, P_i) \rightarrow \rho(x'_k, p'_k; P'_i, X'_i)$ , with

$$\begin{aligned} \rho(x_k, p_k; X_i, P_i) &= \rho(x_k, p_k; P_i, X_i) \\ &+ (\partial_{x_k} \rho \partial_{p_k} g - \partial_{p_k} \rho \partial_{x_k} g) \delta \alpha_{\text{CL}} \\ &+ (\partial_{X_i} \rho \partial_{P_i} G - \partial_{P_i} \rho \partial_{X_i} G) \delta \alpha_{\text{QM}} \\ &+ O(\delta \alpha_{\text{CL}}^2, \delta \alpha_{\text{QM}}^2, \delta \alpha_{\text{CL}} \delta \alpha_{\text{QM}}). \end{aligned} \quad (30)$$

Now, if and where the distribution  $\rho$  vanishes, also the first-order terms  $\propto \delta \alpha_{\text{CL}}$  and  $\propto \delta \alpha_{\text{QM}}$  have to vanish, since otherwise  $\rho$  can be made to decrease below 0 by suitably choosing the signs of these independent infinitesimal parameters. This is particularly relevant for the time evolution generated by a QM-CL hybrid Hamiltonian, discussed in Sec. IV.

#### A. The probability density and marginal distributions

Finally, we remark that the relation between an observable in  $(X, P)$  space and a self-adjoint operator, Eq. (24), can be written as  $G(X_i, P_i) = \text{Tr}(|\Psi\rangle\langle\Psi| \hat{G})$ , which shows explicitly the role of a QM pure state as one-dimensional projector, in this context. In order to illuminate the meaning of the probability density  $\rho$ , we may then use the representation of  $\hat{\rho}$  in terms of its eigenstates,  $\hat{\rho} = \sum_j w_j |j\rangle\langle j|$ , and obtain

$$\begin{aligned} \rho(x_k, p_k; X_i, P_i) &= \sum_j w_j(x_k, p_k) \text{Tr}(|\Psi\rangle\langle\Psi| |j\rangle\langle j|) \\ &= \sum_j w_j(x_k, p_k) |\langle j | \Psi \rangle|^2, \end{aligned} \quad (31)$$

with  $0 \leq w_j \leq 1$  and  $\sum_j \int \Pi_i(dx_i dp_i) w_j(x_k, p_k) = 1$ .

We see that  $\rho(x_k, p_k; X_i, P_i)$ , when properly normalized, is the probability density to find in the hybrid ensemble the QM state  $|\Psi\rangle$ , parametrized by  $X_i, P_i$  through Eq. (11), *together with* the CL state described by the coordinates  $(x_k, p_k)$  of a point in CL phase space.

The probability density allows us to evaluate expectations of QM, CL, or hybrid observables in the usual way. Particularly

useful also are the *marginal* (or *reduced*) distributions defined by

$$\rho_{\text{CL}}(x_k, p_k) := \Gamma_N^{-1} \int_{\delta S_{2N}(\sqrt{2})} \Pi_j(dX_j dP_j) \rho(x_k, p_k; X_i, P_i), \quad (32)$$

$$\rho_{\text{QM}}(X_i, P_i) := \int \Pi_l(dx_l dp_l) \rho(x_k, p_k; X_i, P_i), \quad (33)$$

where  $\Gamma_N$  denotes a normalization factor, to be determined shortly, and the integration in Eq. (32) extends over the surface of a  $2N$ -dimensional sphere of radius  $\sqrt{2}$ , in accordance with Eq. (16); the integration in Eq. (33) extends over all the state space of the CL subsystem. The convergence of the integrals is assured by the positive semidefiniteness and normalizability of  $\rho$ ; the underlying assumption is that the CL subsystem occupies essentially only a finite region of its phase space, while the QM subsystem is constrained by the normalization of its state.

More explicitly, using the representation given in Eq. (31), we calculate for a state vector  $|\Psi\rangle$ , expanded according to Eq. (11),

$$\rho_{\text{CL}}(x_k, p_k) = \sum_j w_j(x_k, p_k) \sum_{i_1, i_2} \langle j | \Phi_{i_1} \rangle \langle \Phi_{i_2} | j \rangle \cdot I_{i_1 i_2}, \quad (34)$$

with a remaining surface integral defined by

$$I_{ab} := \Gamma_N^{-1} \int_{\delta S_{2N}(\sqrt{2})} \Pi_c(dX_c dP_c) (X_a + i P_a)(X_b - i P_b) \quad (35)$$

and evaluated as follows:

$$\begin{aligned} I_{ab} &= \delta_{ab} \Gamma_N^{-1} \int \Pi_c(dX_c dP_c) \delta \left( 2 - \sum_i (X_i^2 + P_i^2) \right) \\ &\quad \times (X_a^2 + P_a^2) \\ &= \delta_{ab} \frac{2}{N \Gamma_N} \int d\Omega_{2N} \int_0^\infty dR R^{2N-1} \delta((R + \sqrt{2})(R - \sqrt{2})) \\ &= \delta_{ab}, \end{aligned} \quad (36)$$

making use of isotropy and, in particular, replacing  $X_a^2 + P_a^2$  with  $\sum_a (X_a^2 + P_a^2)/N = 2/N$  under the integral; in the end, we employ  $2N$ -dimensional spherical coordinates, where  $\Omega_{2N}$  denotes the spherical angle, and choose the normalization factor appropriately:

$$\Gamma_N := \frac{N}{2^{N-1} \Omega_{2N}} = \frac{N!}{(2\pi)^N}. \quad (37)$$

Thus, we obtain from Eq. (34) the expected simple result,

$$\rho_{\text{CL}}(x_k, p_k) = \sum_j w_j(x_k, p_k), \quad (38)$$

using completeness and orthonormality of the bases.

### B. Quantum-classical Poisson bracket and separability

The result of the calculation in Eq. (30) suggests introducing a *generalized Poisson bracket*, when considering observables defined on the Cartesian product state space of CL and QM

sectors as follows:

$$\{A, B\}_\times := \{A, B\}_{\text{CL}} + \{A, B\}_{\text{QM}} \quad (39)$$

$$\begin{aligned} &:= \sum_k \left( \frac{\partial A}{\partial x_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial x_k} \right) \\ &\quad + \sum_i \left( \frac{\partial A}{\partial X_i} \frac{\partial B}{\partial P_i} - \frac{\partial A}{\partial P_i} \frac{\partial B}{\partial X_i} \right), \end{aligned} \quad (40)$$

for any two observables  $A, B$ . It is bilinear and antisymmetric, leads to a derivation-like product formula, and obeys the Jacobi identity, since the right-hand side of Eq. (40) can be written in standard form as a single sum, after relabeling the canonical coordinates.

Let us say that an observable “belongs” to the CL (QM) sector if it is constant with respect to the canonical coordinates of the QM (CL) sector. Then the generalized Poisson bracket has the following additional important properties.

(1) It reduces to the Poisson brackets introduced in Eqs. (1) and (17), respectively, for pairs of observables that belong either to the CL or to the QM sector.

(2) It reduces to the appropriate one of the former brackets if one of the observables belongs only to either one of the two sectors.

(3) It reflects the *separability* of the CL and QM sectors, since  $\{A, B\}_\times = 0$  if  $A$  and  $B$  belong to different sectors.

The physical relevance of separability can be expressed as the following requirement: *If a canonical transformation is performed on the QM (CL) sector only, then all observables that belong to the CL (QM) sector should remain unaffected.* This is indeed the case, as we demonstrate directly by examining the behavior of the reduced CL (QM) probability distribution under such transformations.

Performing in the QM sector, for example, an infinitesimal canonical transformation on the integral of Eq. (32), we obtain

$$\begin{aligned} \rho_{\text{CL}}(x_k, p_k) &\rightarrow \Gamma_N^{-1} \int_{\delta S'_{2N}(\sqrt{2})} \Pi_j(dX'_j dP'_j) \rho(x_k, p_k; X'_i, P'_i) \\ &= \int \rho \left( x_k, p_k; X_i + \frac{\partial G(X_i, P_i)}{\partial P_i} \delta\alpha, P_i - \frac{\partial G(X_i, P_i)}{\partial X_i} \delta\alpha \right) \\ &= \int \rho(x_k, p_k; X_i, P_i) \\ &\quad + \int (\partial_{X_i} \rho \partial_{P_i} G - \partial_{P_i} \rho \partial_{X_i} G) \delta\alpha + \mathcal{O}(\delta\alpha^2) \\ &= \rho_{\text{CL}}(x_k, p_k) + \mathcal{O}(\delta\alpha^2), \end{aligned} \quad (41)$$

where we abbreviated  $\Gamma_N^{-1} \int_{\delta S_{2N}(\sqrt{2})} \Pi_j(dX_j dP_j) \equiv \int$  and used the well-known invariance of the phase-space volume element and of the constraint surface, by Eq. (27), together with Eqs. (22)–(24); furthermore, the last equality follows from the fact that the *integral of a Poisson bracket* of observables over QM state space vanishes:

$$\begin{aligned} \int \{A, B\}_{\text{QM}} &= \int \langle \Psi | \frac{1}{i} [\hat{A}, \hat{B}] | \Psi \rangle = \int \text{Tr} \left( |\Psi\rangle \langle \Psi| \frac{1}{i} [\hat{A}, \hat{B}] \right) \\ &= \text{Tr} \left( \frac{1}{i} [\hat{A}, \hat{B}] \right) = 0, \end{aligned} \quad (42)$$

using Eq. (25), followed by a calculation similar to the one leading from Eq. (31) to Eq. (38), via Eqs. (32) and Eqs. (34)–(37). In the present case, incidentally, we have that  $\{A, B\}_{\text{QM}} \equiv \{\rho, G\}_{\text{QM}} = \{\rho, G\}_\times$ , since  $G$  belongs to the QM sector.

Thus, we find invariance of  $\rho_{\text{CL}}$  under infinitesimal and, hence, finite canonical transformations in the QM sector. Consequently, the expectation of any CL observable  $g_{\text{CL}}$ , defined by

$$\langle g_{\text{CL}} \rangle := \int \Pi_l(dx_l dp_l) g_{\text{CL}}(x_k, p_k) \rho_{\text{CL}}(x_k, p_k), \quad (43)$$

is invariant. Similarly, one shows that  $\rho_{\text{QM}}$  is invariant under canonical transformations in the CL sector and, thus, expectations of QM observables as well.

Separability, as demonstrated here, has been a crucial issue in discussions of earlier attempts to formulate a consistent QM-CL hybrid dynamics; see, for example, Refs. [3, 7, 15], and references therein.

### C. Time evolution of noninteracting quantum-classical ensembles

In order to illustrate another aspect of the separability of CL and QM sectors, as long as there is *no* interaction between them, we consider the time evolution of the probability distribution generated by the total Hamiltonian function  $\mathcal{H}_\Sigma$ ,

$$\mathcal{H}_\Sigma(x_k, p_k; X_i, P_i) := \mathcal{H}_{\text{CL}}(x_k, p_k) + \mathcal{H}_{\text{QM}}(X_i, P_i), \quad (44)$$

where  $\mathcal{H}_{\text{CL}}$  denotes an assumed Hamiltonian function for the CL sector, while the Hamiltonian function  $\mathcal{H}_{\text{QM}}$  for the QM sector has been detailed above [cf. Eqs. (12) and (13)].

Based on Hamilton's equations for both sectors and equipped with the generalized Poisson bracket of Eq. (39), we can invoke Liouville's theorem to obtain the evolution equation:

$$-\partial_t \rho = \{\rho, \mathcal{H}_\Sigma\}_\times. \quad (45)$$

Clearly, this equation admits a factorizable solution  $\rho(x_k, p_k; X_i, P_i; t) \equiv \rho(x_k, p_k; t) \rho(X_i, P_i; t)$ , provided that the initial condition has this property. *No spurious correlations* are produced by the evolution, which corresponds to  $\{\mathcal{H}_{\text{CL}}, \mathcal{H}_{\text{QM}}\}_\times = 0$ , by construction.

In other words, the CL and QM sectors evolve independently, as if the respective other sector was absent, and maintain their CL or QM nature, as long as they do not interact.

## IV. QUANTUM-CLASSICAL HYBRID DYNAMICS

Following the preparations in Sec. III, which concerns QM-CL composite systems, however, without interaction of the CL and QM sectors, we propose here the generalization to include such a hypothetical coupling and study the consistency and consequences of such truly hybrid systems.

This discussion concerns hybrid ensembles, or specific hybrid states, and their dynamics. Given the generalized Poisson bracket, introduced in Eq. (39), we have to incorporate a hybrid interaction term  $\mathcal{I}$  in the total Hamiltonian function

$\mathcal{H}_\Sigma$ , which serves as the generator of time evolution, as before. Therefore, we replace the definition of Eq. (44), with  $\mathcal{H}_\Sigma \equiv \mathcal{H}_\Sigma(x_k, p_k; X_i, P_i)$ , with

$$\mathcal{H}_\Sigma := \mathcal{H}_{\text{CL}}(x_k, p_k) + \mathcal{H}_{\text{QM}}(X_i, P_i) + \mathcal{I}(x_k, p_k; X_i, P_i). \quad (46)$$

For  $\mathcal{H}_\Sigma$  to be an *observable*, it is necessary that the hybrid interaction qualifies as observable, in particular. Further properties of  $\mathcal{H}_\Sigma$  are detailed in due course.

Then the evolution equation is of the same form as Eq. (45):  $-\partial_t \rho = \{\rho, \mathcal{H}_\Sigma\}_\times$ , where, henceforth, the Hamiltonian function includes the interaction term  $\mathcal{I}$ , unless stated otherwise. Here, the *positive semidefiniteness* of  $\rho$  holds for the same reason as for the case of the CL Liouville equation, namely, that the underlying dynamics is described by a Hamiltonian flow.

### A. Energy conservation

Having proposed  $\mathcal{H}_\Sigma$  as the generator of time evolution, it also provides the natural candidate for the *conserved energy* of the hybrid system. Since  $\mathcal{H}_\Sigma$  is assumed not to be explicitly time dependent, we find [cf. with the general structure of Eq. (18)]

$$\frac{d\mathcal{H}_\Sigma}{dt} = \{\mathcal{H}_\Sigma, \mathcal{H}_\Sigma\}_\times = 0, \quad (47)$$

an immediate consequence of the antisymmetry of the generalized Poisson bracket. Note that in the absence of a CL subsystem, this result reduces to the conservation of the expectation of the QM Hamilton operator, as it should. More generally, in the absence of QM-CL interactions, the CL and QM mechanical energies simply add.

### B. Generalized Ehrenfest relations for hybrids

Here we show that the Poisson structure built into the present theory of hybrid systems, in particular, in the form of underlying Hamiltonian equations of motion, translates into generalizations of Ehrenfest relations for coordinate and momentum observables. We consider hybrid systems described by a generic CL *Hamiltonian function* and a QM mechanical *Hamiltonian operator*, respectively:

$$\mathcal{H}_{\text{CL}} := \sum_k \frac{p_k^2}{2} + v(x_l), \quad (48)$$

$$\hat{H}_{\text{QM}} := \frac{\hat{P}^2}{2} + V(\hat{X}), \quad (49)$$

where  $v(x_l) \equiv v(x_1, \dots, x_n)$  and  $V$  denote relevant potentials, together with a *self-adjoint hybrid interaction operator*  $\hat{I}(x_k, p_k; \hat{X}, \hat{P})$ ; note that symmetrical (Weyl) ordering is necessary, concerning the noncommuting operators  $\hat{X}$  and  $\hat{P}$ . We set all masses equal to 1 here, for simplicity, but introduce them explicitly in the particular case of coupled oscillators below. By Eq. (24), this gives rise to the *Hamiltonian function*  $\mathcal{H}_\Sigma$ ,

$$\begin{aligned} \mathcal{H}_\Sigma &= \sum_k \frac{p_k^2}{2} + v(x_l) + \langle \Psi | \left( \frac{\hat{P}^2}{2} + V(\hat{X}) \right) | \Psi \rangle \\ &\quad + \langle \Psi | \hat{I}(x_k, p_k; \hat{X}, \hat{P}) | \Psi \rangle \\ &=: \mathcal{H}_{\text{CL}}(x_k, p_k) + \mathcal{H}_{\text{QM}}(X_i, P_i) + \mathcal{I}(x_k, p_k; X_i, P_i), \end{aligned} \quad (50)$$

when evaluated in a pure state  $|\Psi\rangle$ , invoking the oscillator representation of Eq. (11). Correspondingly, we define *coordinate and momentum observables*, in the sense of our earlier construction in Sec. II B, pertaining to the QM subsystem:

$$X(X_i, P_i) := \langle \Psi | \hat{X} | \Psi \rangle, \quad P(X_i, P_i) := \langle \Psi | \hat{P} | \Psi \rangle. \quad (51)$$

With these definitions in place, we proceed to determine the equations of motion by following the rules of Hamiltonian dynamics.

The equations of motion for the CL *observables*  $x_k, p_k$  are

$$\dot{x}_k = \{x_k, \mathcal{H}_\Sigma\}_\times = p_k + \partial_{p_k} \mathcal{I}(x_k, p_k; X_i, P_i), \quad (52)$$

$$\begin{aligned} \dot{p}_k &= \{p_k, \mathcal{H}_\Sigma\}_\times \\ &= -\partial_{x_k} v(x_k) - \partial_{x_k} \mathcal{I}(x_k, p_k; X_i, P_i). \end{aligned} \quad (53)$$

Similarly, we obtain for the QM *variables*  $X_i, P_i$ , which are *not observables*,

$$\begin{aligned} \dot{X}_i &= \{X_i, \mathcal{H}_\Sigma\}_\times \\ &= \partial_{P_i} \mathcal{H}_{\text{QM}}(X_j, P_j) + \partial_{P_i} \mathcal{I}(x_k, p_k; X_j, P_j) \end{aligned} \quad (54)$$

$$= E_i P_i + \partial_{P_i} \mathcal{I}(x_k, p_k; X_j, P_j), \quad (55)$$

$$\begin{aligned} \dot{P}_i &= \{P_i, \mathcal{H}_\Sigma\}_\times \\ &= -\partial_{X_i} \mathcal{H}_{\text{QM}}(X_j, P_j) - \partial_{X_i} \mathcal{I}(x_k, p_k; X_j, P_j) \end{aligned} \quad (56)$$

$$= -E_i X_i - \partial_{X_i} \mathcal{I}(x_k, p_k; X_j, P_j), \quad (57)$$

where Eqs. (55) and (57) follow, if the oscillator expansion is performed with respect to the stationary states of  $\hat{H}_{\text{QM}}$  [cf. Eqs. (12) and (13) in Sec. II B 2]. Notably, Eqs. (52) and (53) together with Eqs. (54) and (56), or together with Eqs. (55) and (57), form a *closed set* of  $2(n + N)$  equations, where  $n$  denotes the number of CL degrees of freedom, and  $N$  the dimension of the QM Hilbert space (assumed to be denumerable, if not finite).

However, in distinction, the *generalized Ehrenfest relations* for the QM *observables*  $X, P$ , defined in Eqs. (51), are obtained as follows:

$$\begin{aligned} \dot{X} &= \{X, \mathcal{H}_\Sigma\}_\times = \{X, \mathcal{H}_\Sigma\}_{\text{QM}} = -i \langle \Psi | [\hat{X}, \hat{H}_{\text{QM}} + \hat{I}] | \Psi \rangle \\ &= P - i \langle \Psi | [\hat{X}, \hat{I}(x_k, p_k; \hat{X}, \hat{P})] | \Psi \rangle, \end{aligned} \quad (58)$$

$$\begin{aligned} \dot{P} &= \{P, \mathcal{H}_\Sigma\}_\times = \{P, \mathcal{H}_\Sigma\}_{\text{QM}} \\ &= -i \langle \Psi | [\hat{P}, \hat{H}_{\text{QM}} + \hat{I}] | \Psi \rangle \\ &= -\langle \Psi | V'(\hat{X}) | \Psi \rangle - i \langle \Psi | [\hat{P}, \hat{I}(x_k, p_k; \hat{X}, \hat{P})] | \Psi \rangle, \end{aligned} \quad (59)$$

where we used Eq. (25), in order to replace Poisson brackets with commutators and the explicit form of  $\hat{H}_{\text{QM}}$ , Eq. (49);  $V'$  denotes the appropriate first derivative of the potential function  $V$ . Equations (58) and (59) together with Eqs. (52) and (53) do *not form a closed set of equations*, since the expectation of a function of observables generally does not equal the function of the expectations of the observables, as in Ehrenfest's theorem in QM mechanics [29].

### C. Bilinearly coupled oscillators

We consider here a *set* of CL oscillators coupled bilinearly to *one* QM oscillator, choosing, for example,

$$\mathcal{H}_{\text{CL}} := \sum_k \left( \frac{1}{2m_k} p_k^2 + \frac{m_k \omega_k^2}{2} x_k^2 \right), \quad (60)$$

$$\hat{H}_{\text{QM}} := \frac{1}{2M} \hat{P}^2 + \frac{M\Omega^2}{2} \hat{X}^2, \quad (61)$$

$$\hat{I} := \hat{X} \sum_k \lambda_k x_k, \quad (62)$$

where we have introduced masses  $m_k, M$ , frequencies  $\omega_k, \Omega$ , and coupling constants  $\lambda_k$ .

In this case, the equations of motion for the CL observables together with the *generalized Ehrenfest relations* of Sec. IV B reduce to a simple *closed set* of equations:

$$\dot{x}_k = \frac{1}{m_k} p_k, \quad (63)$$

$$\dot{p}_k = -m_k \omega_k^2 x_k - \lambda_k X, \quad (64)$$

$$\dot{X} = \frac{1}{M} P, \quad (65)$$

$$\dot{P} = -M\Omega^2 X - \sum_k \lambda_k x_k, \quad (66)$$

with the QM observables  $X := \langle \Psi | \hat{X} | \Psi \rangle$  and  $P := \langle \Psi | \hat{P} | \Psi \rangle$  [cf. Eqs. (51)]. Here, the *backreaction* of QM on CL subsystem appears, *as if* the CL subsystem were coupled to another CL oscillator.

In view of Eqs. (63)–(66), we find that our theory passes the “Peres-Terno benchmark” test for interacting QM-CL hybrid systems [5], which, so far, has been achieved only by the configuration ensemble theory of Hall and Reginatto [6–8].

## V. DISCUSSION

The proposed theory describing QM-CL hybrid systems certainly raises a number of questions, some of which we address here.

### A. Classical-environment-induced decoherence

Well-known studies of environment-induced decoherence describe the effects that an environment of QM degrees of freedom has on the coherence properties of a QM subsystem coupled to it [31,32]. In particular, the Feynman-Vernon or Caldeira-Leggett models and, more generally, models of QM Brownian motion have been studied in this respect [33]. Here we suggest considering the situation where the QM environment is replaced with a CL one. We find that a CL environment similarly can produce decoherence in a generic model.

For simplicity, we consider a QM object characterized by a two-dimensional Hilbert space, a “q-bit,” which is coupled bilinearly to a set of CL oscillators. The oscillators are described by Eq. (60), as before. The Hamiltonian function of the q-bit presents the simplest example of the oscillator expansion of a QM Hamiltonian,

$$\mathcal{H}(X_i, P_i) := \sum_{i=1,2} \frac{E_i}{2} (P_i^2 + X_i^2), \quad (67)$$



when expanding with respect to the energy eigenstates,  $\{|\phi_1\rangle, |\phi_2\rangle\}$  [cf. Eqs. (11)–(13)]. The bilinear QM-CL coupling is defined by

$$\hat{I} := \hat{\Sigma} \sum_k \lambda_k x_k, \quad (68)$$

similarly as in Eq. (62); here  $\hat{\Sigma}$  denotes an observable of the q-bit.

In this case, the closed set of dynamical equations of motion becomes

$$\dot{x}_k = p_k/m_k, \quad (69)$$

$$\dot{p}_k = -m_k \omega_k^2 x_k - \lambda_k \langle \Psi | \hat{\Sigma} | \Psi \rangle, \quad (70)$$

$$\dot{X}_i = E_i P_i, \quad (71)$$

$$\dot{P}_i = -E_i X_i - \frac{\partial \langle \Psi | \hat{\Sigma} | \Psi \rangle}{\partial X_i} \sum_k \lambda_k x_k, \quad (72)$$

analogous to Eqs. (52), (53), (55), and (57), respectively, and where we have

$$\langle \Psi | \hat{\Sigma} | \Psi \rangle = \frac{1}{2} \sum_{i,j=1,2} \langle \phi_i | \hat{\Sigma} | \phi_j \rangle (X_i - iP_i)(X_j + iP_j). \quad (73)$$

Equations (69) and (70) are solved by employing the retarded Green's function for the equation of motion of a driven harmonic oscillator. This yields

$$x_k(t) = x_k^{(0)}(t) - \lambda_k \int_{-\infty}^t ds \frac{\sin \omega_k(t-s)}{m_k \omega_k} \langle \Psi(s) | \hat{\Sigma} | \Psi(s) \rangle, \quad (74)$$

with the harmonic term  $x_k^{(0)}(t) := a_k \cos(\omega_k t) + b_k \sin(\omega_k t)$  and where the real coefficients  $a_k$  and  $b_k$  are determined by initial conditions.

Furthermore, Eqs. (71) and (72) can be combined into second-order form:

$$\ddot{X}_i + E_i^2 X_i = -E_i \xi(t) \sum_{j=1,2} [\text{Re}(\Sigma_{ij}) X_j - \text{Im}(\Sigma_{ij}) \dot{X}_j / E_j], \quad (75)$$

introducing the matrix elements  $\Sigma_{ij} := \langle \phi_i | \hat{\Sigma} | \phi_j \rangle = \Sigma_{ji}^*$ , the real and imaginary parts of which enter. Thus, we obtain a system of  $N = 2$  coupled oscillator equations, where the coupling terms are *nonlinear and non-Markovian* through the function

$$\begin{aligned} \xi(t) &:= \sum_k \lambda_k x_k(t) \\ &= \sum_k \left[ \lambda_k x_k^{(0)}(t) - \lambda_k^2 \sum_{i,j=1,2} \Sigma_{ij} \int_{-\infty}^t ds \frac{\sin \omega_k(t-s)}{m_k \omega_k} \right. \\ &\quad \left. \times (X_i(s) X_j(s) + \dot{X}_i(s) \dot{X}_j(s) / (E_i E_j)) \right] \end{aligned} \quad (76)$$

by Eqs. (71), (73), and (74).

Let us reduce the above model to a crudely simplified version, neglecting presumably much of the rich dynamics

described by Eqs. (75) and (76). For sufficiently *weak coupling*, we drop the non-Markovian terms, i.e., terms  $\propto \lambda_k^2$ . Furthermore, we choose  $\Sigma_{11} = \Sigma_{22} \equiv 0$  and  $\Sigma_{12} = \Sigma_{21} \equiv 1$ . This simplifies the equations to describe two oscillators which are symmetrically coupled to each other through a periodic or, in the case of CL oscillators with incommensurate frequencies, quasiperiodic function  $\xi$ . Under the additional assumption of *slow CL oscillators*, i.e., with frequencies that are low compared to those of the QM oscillators, the resulting equations are solved by ( $i = 1, 2$ )

$$X_i = A_i \cos(\Omega_i t) + B_i \sin(\Omega_i t), \quad (77)$$

and  $P_i = \dot{X}_i / E_i$ ; the real coefficients  $A_i$ ,  $B_i$  are determined by initial conditions, and, to leading nonvanishing order in  $\xi$ , the characteristic frequencies are given by

$$\Omega_1 := E_1 + \frac{\xi^2 E_2}{2(E_1^2 - E_2^2)}, \quad \Omega_2 := E_2 - \frac{\xi^2 E_1}{2(E_1^2 - E_2^2)}. \quad (78)$$

Choosing, for illustration, initial conditions such that the expansion coefficients in  $|\Psi\rangle = \sum_{i=1,2} |\phi_i\rangle (X_i + iP_i) / \sqrt{2}$  [cf. Eq. (11)] are real at  $t = 0$ , we obtain the off-diagonal elements of the corresponding density matrix  $\hat{\rho} := |\Psi\rangle \langle \Psi|$  in the form

$$\begin{aligned} \langle \phi_1 | \hat{\rho} | \phi_2 \rangle &= \langle \phi_2 | \hat{\rho} | \phi_1 \rangle^* = (X_1 + iP_1)(X_2 - iP_2) / 2 \\ &= e^{i(\Omega_2 - \Omega_1)t} \left( 1 - \frac{\xi^2}{4E_1 E_2} \right) \\ &\quad - \frac{\xi^2}{4(E_1^2 - E_2^2)} \left( \frac{E_2}{E_1} e^{i(\Omega_1 + \Omega_2)t} - \frac{E_1}{E_2} e^{-i(\Omega_1 + \Omega_2)t} \right) \\ &\quad + O(\xi^4). \end{aligned} \quad (79)$$

We note that there is a term  $\propto i\xi^2 t$  contributing to the argument of each exponential [cf. Eqs. (78)]. This can lead to *decoherence by dephasing* in the following way.

If the non-negative function  $\xi^2(t)$  is sufficiently irregular (depending on the frequency distribution of environmental oscillators), we may treat it as a random variable and average the result of Eq. (79) correspondingly. We consider the leading term, while the others can be dealt with similarly. Thus, writing  $\Omega_2 - \Omega_1 = \delta E + \xi^2 / 2\delta E$ , with  $\delta E := E_2 - E_1$ , we have to evaluate the dimensionless function  $f$ ,

$$f(t) := \int_0^\infty d\Omega P(\Omega) e^{i\Omega t}, \quad (80)$$

with  $\Omega \equiv \xi^2 / 2\delta E$ , and where  $P$  represents the appropriately normalized distribution of the values of  $\Omega$ . Now, there are *continuous distributions*, such that  $f(t) \rightarrow 0$ , for  $t \rightarrow \infty$ , for example, a constant distribution over a finite range of  $\Omega$ , an exponential distribution, or a Gaussian distribution. In these circumstances, the leading term (similarly to the others) gives a decaying contribution, i.e.,  $\propto f(t)$ , to the off-diagonal density matrix element  $\langle \phi_1 | \hat{\rho} | \phi_2 \rangle$ , after averaging.

This indicates a decoherence mechanism which is effectively quite similar to “fundamental energy decoherence,” which was reviewed recently in Ref. [19].

### B. Quantum-classical backreaction

QM-CL backreaction, in particular, the effect of QM fluctuations on the CL subsystem, has always been an important topic for various proposals of QM-CL hybrid dynamics and its applications. This concerns improvements of approximation methods and applications, for example, in “semiclassical gravity,” studying the effects of QM fluctuations of matter on the CL metric of space time (see Refs. [2,6,8,16], with numerous references to related work).

Our formalism consistently *incorporates all QM fluctuations*, even if they are not explicitly visible, unlike in many approaches where fluctuations are added by hand, in some approximation. Presently, the QM dynamics is treated exactly in terms of a complete set of canonical variables, for example,  $(X_i, P_i)_{i=1,\dots,N}$  in the closed set of dynamical equations (52)–(57). As long as no approximations are applied to these equations, their solutions allow us to evaluate exactly all quantities which reflect the fluctuations associated with a pure QM state  $|\Psi\rangle$ , such as the typical variance  $\Delta X^2 := \langle \Psi | \hat{X}^2 | \Psi \rangle - \langle \Psi | \hat{X} | \Psi \rangle^2$ . This follows from the fact that *all QM observables* can be expanded in the oscillator representation [recall Eqs. (11), (24), and (28)], with the expansion coefficients given by the solutions of the deterministic equations. Thus, for example,  $\Delta X^2$  becomes a function of the canonical variables.

The QM variables do not fluctuate in a given pure state. By the QM-CL Poisson brackets and ensuing equations of motion (“Hamilton’s equations”), they are coupled to the CL variables (observables), which, therefore, do not show fluctuations either.

However, this admits the possibility that the initial conditions of the hybrid dynamical equations, in particular, for the QM subsystem, are determined by the fluctuating outcome of a certain preparation or measurement. In this case, if only statistical or conditional information about the initial state of the system is available, the CL observables, generally, will reflect corresponding fluctuations. For example, we can evaluate a correlation function of CL observables to find

$$\langle x_a x_b \rangle := \int \Pi_I(dx_I dp_I) x_a x_b \rho_{\text{CL}}(x_k, p_k) \neq \langle x_a \rangle \langle x_b \rangle,$$

with the help of the reduced distribution  $\rho_{\text{CL}}$  introduced in Eq. (32). This distribution function is determined by the solution of the Liouville equation for the full density  $\rho$  of the interacting hybrid system [cf. Eq. (45)]; it could be, furthermore, conditioned by a selected outcome of some QM measurement(s) specifying the initial state.

### C. Hybrid observables, separable interactions, and quantum-classical Poisson brackets

It is a common feature of either QM or CL systems that particular forms of interaction among subsystems allow the separation of degrees of freedom into noninteracting subsets. Generally, this is associated with the existence of *symmetries* of the compound system, such as translation or rotation invariance.

A recent study of a translation-invariant harmonic interaction between a QM and a CL particle revealed that—according to the hybrid theory proposed by Hall and Reginatto—there

arises an irreducible coupling between center-of-mass and relative motion [6]. This is contrary to what happens if both particles are treated as either CL or QM mechanical and has been traced to the inherent nonlinearity of their proposal. The action functional, from which the equations of motion are derived, “knows” which variables belong to the QM and CL sectors, respectively, and mixing them by coordinate transformations produces the coupling. Consequently, such a system has been proposed as a prospective testing ground, where their theory could be falsified experimentally [9].

This issue can also be examined in the light of the present linear hybrid theory. Specializing the system of bilinearly coupled oscillators from Sec. IV C as

$$\mathcal{H}_{\text{CL}} := \frac{1}{2m} p^2, \quad \hat{\mathcal{H}}_{\text{QM}} := \frac{1}{2M} \hat{P}^2, \quad \hat{I} := \lambda(x \cdot \hat{\mathbf{I}} - \hat{X})^2, \quad (81)$$

we reconsider the example from Ref. [9]; here  $\hat{\mathbf{I}}$  denotes the unit operator on the Hilbert space of the QM subsystem.

As before, the Hamiltonian function which generates time evolution of the composite system,  $\mathcal{H}_{\Sigma} := \mathcal{H}_{\text{CL}} + \langle \Psi | (\hat{\mathcal{H}}_{\text{QM}} + \hat{I}) | \Psi \rangle$ , is conserved by construction,  $d\mathcal{H}_{\Sigma}/dt = 0$  (see Sec. IV A). Furthermore, we know from Sec. IV C that the generalized Ehrenfest equations for bilinearly coupled oscillators in terms of the CL observables, here  $x$  and  $p$ , and of the QM observables,  $X := \langle \Psi | \hat{X} | \Psi \rangle$  and  $P := \langle \Psi | \hat{P} | \Psi \rangle$ , form a closed set [cf. Eqs. (63)–(66)]. These equations of motion are nothing but Hamilton’s equations for the “CL” Hamiltonian function:

$$\mathcal{H}_{\Sigma}^{\text{cl}}(x, p; X, P) := \frac{1}{2m} p^2 + \frac{1}{2M} P^2 + \lambda(x - X)^2, \quad (82)$$

which implies that also  $\mathcal{H}_{\Sigma}^{\text{cl}}$  is conserved,  $d\mathcal{H}_{\Sigma}^{\text{cl}}/dt = 0$ . Then it follows that the energy carried by QM fluctuations is separately conserved:

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2M} (\langle \hat{P}^2 \rangle - \langle \hat{P} \rangle^2) + \lambda(\langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2) \right) \\ = \frac{d}{dt} (\mathcal{H}_{\Sigma} - \mathcal{H}_{\Sigma}^{\text{cl}}) = 0, \end{aligned} \quad (83)$$

where  $\langle \dots \rangle \equiv \langle \Psi | \dots | \Psi \rangle$ . Note that all mixed QM-CL terms cancel and no energy is transferred between the (fluctuations of the) QM and the CL subsystems.

We observe that the “CL” Hamiltonian  $\mathcal{H}_{\Sigma}^{\text{cl}}$  is separable by transforming the variables  $x, p$  and  $X, P$  to center-of-mass and relative variables:

$$s := (MX + mx)/\sigma, \quad p_s := P + p, \quad (84)$$

$$r := X - x, \quad p_r := \mu \left( \frac{1}{M} P - \frac{1}{m} p \right), \quad (85)$$

with the total and reduced masses defined by  $\sigma := M + m$  and  $\mu := Mm/(M + m)$ , respectively. Thus, we find  $\mathcal{H}_{\Sigma}^{\text{cl}} = (p_s^2/2\sigma) + (p_r^2/2\mu) + \lambda r^2$ , not surprisingly.

At this point, it seems worthwhile to assess the character of these transformations with respect to the fundamental QM-CL Poisson brackets, defined in Eqs. (39) and (40) (Sec. III B), on which our theory is based. We have  $\{x, p\}_{\times} = 1$  and  $\{X, P\}_{\times} = \langle [\hat{X}, \hat{P}] / i \rangle = 1$ , using Eq. (25); similarly, we find

that the brackets of all other pairs of these variables vanish. Thus, we may consider  $x, p$  and  $X, P$  as two pairs of canonical phase-space coordinates. Furthermore, one may check that also the two pairs of center-of-mass and relative variables,  $s, p_s$  and  $r, p_r$ , respectively, form pairs of canonical coordinates under the QM-CL Poisson brackets. Therefore, transformations (84) and (85) are consistent *canonical transformations*. An immediate consequence is that the “CL Hamiltonians” describing center-of-mass and relative motion are separately conserved as well.

It must be emphasized that the separation of the “CL” degrees of freedom,  $x, p$  and  $X, P$ , or of the corresponding center-of-mass and relative variables, from the full set of canonical variables  $x, p$  and  $X_i, P_i$  is an *accident of the harmonic interaction*. Independently of the hybrid coupling between a CL and a QM mechanical particle [cf. Eqs. (81)], any coupling other than constant, linear (in one dimension), or harmonic translation-invariant coupling would not allow such separation, even if both particles were treated QM mechanically.

In general, the “mean-field” variables  $X \equiv \langle \hat{X} \rangle$  and  $P \equiv \langle \hat{P} \rangle$  will always couple to “correlation functions,” such as  $\langle \hat{X}^2 \rangle$ ,  $\langle \hat{X} \hat{P} + \hat{P} \hat{X} \rangle$ ,  $\langle \hat{P}^2 \rangle$ , or more complicated ones, depending on the kind of interaction. This phenomenon of QM mechanics is not particular to hybrid dynamics.

While the separation of degrees of freedom in the case of translation- or rotation-invariant potentials in QM mechanics can always be completed at the operator level, the hybrid dynamics presented here necessitates the consideration of “CL” canonical variables on which to perform any canonical transformations consistently with the underlying Poisson bracket structure. This seems to limit separability to certain potentials, as we have just seen.

We conclude that a composite system of a QM and a CL particle with harmonic translation-invariant interaction, or some analog of this, does not allow experimental falsification of our formulation of hybrid dynamics. We find no coupling between relative and center-of-mass motion, contrary to the proposal in Refs. [6,9]. However, anharmonic interactions need to be studied in this context and may lead to experimentally accessible signatures of the linear QM-CL hybrid dynamics.

#### D. The classical $\times$ almost-classical algebra of hybrid observables

In Secs. IIA and IIB, we introduced the notions of CL and QM observables, respectively, relevant for the considerations of this paper. Furthermore, in Sec. III B, we introduced the fundamental QM-CL hybrid Poisson bracket,  $\{A, B\}_\times := \{A, B\}_{\text{CL}} + \{A, B\}_{\text{QM}}$ . Following Eqs. (39) and (40), we pointed out three important properties of this bracket, last but not not least related to separability. However, we tacitly assumed that a fourth case would not need further mention, which can arise for two genuine hybrid observables:

Let  $A \equiv A(x_k, p_k; X_i, P_i)$ ,  $B \equiv B(x_k, p_k; X_i, P_i)$  be *both hybrid observables*, i.e., both are quadratic forms in the  $X_i$ ’s and  $P_i$ ’s and both are not completely independent of the  $x_k$ ’s and  $p_k$ ’s. If, furthermore, one observable, say  $A$ , depends on any pair of canonical variables, say  $x \equiv x_{k'}$  and  $p \equiv p_{k'}$ ,

and  $B$  also depends on  $x$  or  $p$ , then the “CL part” of the bracket,  $\{A, B\}_{\text{CL}}$ , generates terms which do *not* qualify as observable with respect to the QM sector.

Such terms are of the general form

$$\begin{aligned} & \sum_{i,i',j,j'} M_{i,j,i',j'}(x_k, p_k) (X_i - iP_i)(X_j + iP_j) \\ & \quad \times (X_{i'} - iP_{i'})(X_{j'} + iP_{j'}) \\ & = 4 \sum_{i,i',j,j'} \langle \Psi | \Phi_i \rangle \langle \Psi | \Phi_{i'} \rangle M_{i,j,i',j'}(x_k, p_k) \langle \Phi_j | \Psi \rangle \langle \Phi_{j'} | \Psi \rangle, \end{aligned} \quad (86)$$

where we used the oscillator expansion, Eq. (11), and

$$M_{i,j,i',j'}(x_k, p_k) := \frac{1}{4} \sum_k \left( \frac{\partial A_{ij}}{\partial x_k} \frac{\partial B_{i'j'}}{\partial p_k} - \frac{\partial A_{ij}}{\partial p_k} \frac{\partial B_{i'j'}}{\partial x_k} \right)$$

using the related expansion for observables  $A$  and  $B$  [cf. Eq. (28)].

Generally, iterations of such brackets will implicitly contribute to the solution,  $\rho \equiv \rho(x_k, p_k; X_i, P_i)$ , of the evolution equation,  $-\partial_t \rho = \{\rho, \mathcal{H}_\Sigma\}_\times$ , in the presence of a true hybrid coupling, cf. Sec. IV. Thus, multiple factors involving the state vector  $|\Psi\rangle$  and its adjoint, or multiple pairs like  $(X_i - iP_i)(X_j + iP_j)$ , will enter. In this way, evolution of hybrid observables, of the density  $\rho$  in particular, can induce a structural change: while continuing to be CL observables, they do not remain QM observables (quadratic forms in  $X_i$ ’s and  $P_i$ ’s). They fall outside of the product algebra generated by the observables to which we confined ourselves, so far.

We note that the assumption of a product algebra covering the observables of a hybrid system was essential for the no-go theorem put forth in Ref. [34], which ruled out a class of hybridization models. However, this assumption can be criticized as being too restrictive from the QM point of view [7]. Here we assume that

The algebra of hybrid observables is closed under the QM-CL Poisson bracket  $\{, \}_\times$  operation—a physical hypothesis.

Referring to the phase-space coordinates  $(X_i, P_i)$ , we define an *almost-CL observable* as a real-valued regular function of pairs of factors like  $(X_i - iP_i)(X_j + iP_j)$ , such as on the left-hand side of Eq. (86), subject to the constraint  $\mathcal{C}(X_i, P_i) = \frac{1}{2} \sum_i (X_i^2 + P_i^2) \stackrel{!}{=} 1$ . This normalization constraint [cf. Eq. (16) in Sec. II B 2] is preserved under the evolution, since  $\{\mathcal{C}, \mathcal{H}_\Sigma\}_\times = 0$ , in the presence of QM-CL hybrid interaction as defined in Sec. IV.

According to this definition, QM observables (quadratic forms in phase-space coordinates; cf. Sec. II B) form a subset of almost-CL observables which, in turn, form a subset of CL observables (real-valued regular functions of phase-space coordinates; cf. Sec. II A). Furthermore, we may now say that members of the complete algebra of hybrid observables, generally, are CL with respect to coordinates  $(x_k, p_k)$  and *almost CL* with respect to coordinates  $(X_i, P_i)$ .

This leads us to speculate about a physical consequence of the enlarged CL  $\times$  almost-CL algebra for interacting QM-CL hybrids, as illustrated by the following *Gedankenexperiment*.

Consider a QM together with a CL object subject to a transient hybrid interaction. As long as the hybrid interaction is ineffective, both objects evolve independently according to Schrödinger's and Hamilton's equations, respectively. However, once they form an interacting hybrid, the corresponding phase-space density changes from a factorized form, in the absence of any initial correlation, to become an almost-CL-CL hybrid observable. Even if the hybrid interaction eventually disappears, the density possibly maintains a mixed almost-CL-CL character. This agrees with the general structure of the evolution equation yet needs to be understood in detailed examples.

This outcome contradicts the naive expectation that QM and CL objects evolve separately in QM and CL ways, *after* any hybrid interaction has ceased. Two possibilities come to mind. Either persistence of the almost-CL-CL character is a *physical effect* accompanying QM-CL hybrids, if they exist, or our description needs to be augmented with a *reduction mechanism* by which evolving observables return to standard QM or CL form (cf. Sec. II), following a hybrid interaction. Both possibilities seem quite interesting in their own right. We reserve this topic for future study [35].

#### E. Hybrid dynamics and the Wigner function approach

Suppose that a physicist unfamiliar with QM mechanics were presented with the general equations of motion, Eqs. (52)–(57) [plus the normalization constraint, Eq. (16)]. We know that these equations present independent CL and QM sectors, in the absence of a hybrid interaction. However, he or she would naturally interpret them to describe the dynamics of a composite CL object, with part of its phase space compactified, due to the constraint. Thus, he or she finds *perfectly local dynamics*. In fact, *our* knowledge of nonlocal features can be traced to the definition of the canonical coordinates and momenta  $X_i, P_i$ , introduced by the oscillator representation, Eq. (11), since  $X_i/\sqrt{2} = \text{Re} \int dq \Phi_i^*(q)\Psi(q)$  and  $P_i/\sqrt{2} = \text{Im} \int dq \Phi_i^*(q)\Psi(q)$ . Therefore, spatially non-local (and probabilistic) features enter by reference to the QM wave function [36].

In view of this, it might be surprising that our proposed hybrid dynamics passes the set of consistency requirements (cf. Sec. I), in particular, the requirement for conservation and positivity of probability, as we have seen. This must be contrasted with the problems that arise if one maps the QM sector “locally” on a would-be CL phase space by using the Wigner function approach and the corresponding version of the von Neumann equation. The latter differs from the CL Liouville equation by a series of corrections in powers of  $\hbar$  which, in turn, incorporate nonlocal features. It is well known that they spoil the interpretation of the Wigner function as a genuine probability distribution on phase space, since it generally does not remain positive semidefinite (see Ref. [38] for a comprehensive review of probability issues).

Conversely, the dynamics of CL phase-space distributions, typically described by the Liouville equation, can be presented in QM mechanical and, in particular, in path integral form [10,11]. Again, the resulting would-be QM mechanical density matrix corresponding to a CL probability distribution is, generally, not positive semidefinite.

In both cases, the problem is caused by the intermediate Fourier transformation, which apparently is not suited to represent the nonlocality properties appropriately, when formally relating phase space to Hilbert space, and vice versa. Therefore, the Wigner function approach, which one could be tempted to employ, in order to systematically reduce part of a composite QM system to a CL subsystem, thus defining a QM-CL hybrid, unfortunately violates the positivity of probability requirement.

Similar problems were encountered in Ref. [4], where the QM  $\rightarrow$  CL reduction is attempted via coherent “minimum uncertainty” states. In distinction, the oscillator representation allows us to circumvent this difficulty, at the expense of introducing the phase-space structure in an abstract way [39].

## VI. CONCLUDING REMARKS

We have proposed a theory of *QM-CL hybrid dynamics* in this paper. In particular, our considerations are based on the representation of QM mechanics in the framework of CL analytical mechanics by Heslot, who showed that notions of states in phase space, observables, and Poisson brackets can be naturally extended to QM mechanics [12].

Our formulation provides a generalization for the case where QM mechanical and CL degrees of freedom are directly coupled to each other. An important guideline has been to satisfy the complete set of *consistency conditions* mentioned in Sec. I and fulfilled, so far, only by the configuration space ensemble theory of Hall and Reginatto [6–8]; all earlier attempts failed on one point or the other. However, our linear theory deviates from their nonlinear theory in that no “spurious” coupling between center-of-mass and relative motion is found for a two-body system with a harmonic translation-invariant potential.

The latter issue, QM-CL backreaction, CL-environment-induced decoherence, and completion of the algebra of hybrid observables have been discussed in Sec. V, while further interesting topics are left for future work. These include the hypothetical role of hybrid dynamics in measurement processes, seen as the interaction between a CL apparatus and a QM object according to the Copenhagen interpretation, and the effect of CL and QM degrees of freedom on entangled and classically correlated states, respectively, through hybrid interactions.

On the technical side, since hybrid dynamics in the present formulation leads to a Liouville equation, as discussed, the superspace path integral we have recently devised can be readily adapted to it [10,11]. This may be particularly interesting for applications in which hybrid dynamics is considered as an approximation scheme for complex QM systems.

In a more speculative vein, one could wonder about the essential difference between QM and CL state spaces seen here, respectively, in the presence and absence of curvature (cf. Sec. II B 2). Does a properly understood CL limit of QM mechanics, which possibly helps with the measurement problem [40], require a dynamical treatment of this curvature? Conversely, is the hypothetical emergence of QM mechanics from deterministic dynamics related to a dynamical structure of phase space?



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