Nonlinear modes in the harmonic $\mathcal{PT}$-symmetric potential

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We study the families of nonlinear modes described by the nonlinear Schrödinger equation with the $\mathcal{PT}$-symmetric harmonic potential $x^2 - 2i\alpha x$. The nonlinear modes found display a number of interesting features. In particular, we have observed that modes bifurcating from different eigenstates of the underlying linear problem can actually belong to the same family of nonlinear modes. We also show that by proper adjustment of the coefficient α it is possible to enhance the stability of small-amplitude and strongly nonlinear modes compared to the well-studied case of the real harmonic potential.

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I. INTRODUCTION

Interest in the stationary modes of the nonlinear Schrödinger equation with a potential was raised about two decades ago in connection with applications to the mean-field dynamics of Bose-Einstein condensates [1] and later on in the context of optical applications [2] and in particular of propagation of dispersion-managed solitons in fibers [3]. Various aspects of the nonlinear modes in a parabolic trapping potential with homogeneous [4–7] and inhomogeneous [8] nonlinearities have been intensively studied. A comprehensive analysis of the structure of the nonlinear modes and their stability can be found in [5–7]. Further, taking into account that interaction of a particle with a potential in practice is not absolutely elastic and energy losses are possible, the authors of [9] addressed nonlinear modes in a complex parabolic potential $(1 - i)x^2$ supported by a homogeneous gain. Due to its dissipative nature, the complex parabolic potential has properties very different from those of its real counterpart. In particular, for fixed parameters of the dissipative model, the stable nonlinear modes appear as isolated attractors and do not constitute continuous families. Another interesting feature of the complex parabolic potential is that in the limit of strong defocusing (or repulsive) nonlinearity, the so-called Thomas-Fermi approximation of the model is described by the balance between the losses and the gain. This is not the case for the conservative potential, where the behavior of the nonlinear modes in the Thomas-Fermi limit is determined by the balance between the dispersion (or diffraction) and the nonlinearity.

In the meantime, recently there has been a rapidly increasing interest [10] in linear and nonlinear properties of systems with potentials obeying the so-called $\mathcal{PT}$ symmetry. This interest was initiated by the paper [11], and more recently by the experimental observation of $\mathcal{PT}$ symmetry breaking in optics [12], as well as by several theoretical suggestions of realization of $\mathcal{PT}$-symmetric optical systems [13].

Nonlinear extensions of the $\mathcal{PT}$-symmetric structures were first considered in [14]. Later on the nonlinear modes were studied in periodic [15], Gaussian [16], and sech^2-shaped [17] $\mathcal{PT}$-symmetric potentials, as well as in a harmonic trap with a rapidly decaying $\mathcal{PT}$-symmetric imaginary component [18]. We also mention studies of gap solitons in $\mathcal{PT}$-symmetric optical lattices combined with real superlattices [19] and optical defect modes in $\mathcal{PT}$-symmetric potentials [20]. The modes and their stability in systems with $\mathcal{PT}$-symmetrically modulated nonlinearity landscapes have been recently reported in [21].

It turns out, however, that nonlinear modes in a $\mathcal{PT}$-symmetric parabolic trap have not received any attention, so far, while such a potential, namely, $(x - i\alpha)^2$, has been introduced and well studied in linear theory [22–24]. Meantime, as will be shown below, the nonlinear modes in the $\mathcal{PT}$-symmetric harmonic potential display rather unusual properties, which cannot be observed either in conservative or in dissipative potentials of a general kind. The main aim of the present work is to perform a detailed study of such modes.

The rest of the paper is organized as follows. In the next section we introduce the nonlinear model with a $\mathcal{PT}$-symmetric harmonic potential and briefly discuss its physical relevance. In Sec. III we discuss some properties of the underlying linear model. Next, in Secs. IV and V we report the families of nonlinear modes, as well as a detailed investigation of their stability. Section VI concludes the paper.

II. THE MAIN MODEL

Our main object in this paper is the nonlinear Schrödinger equation with a $\mathcal{PT}$-symmetric parabolic potential:

$$iq_z = -q_{xx} + (x^2 - 2i\alpha x)q - \sigma|q|^2 q, \qquad (1)$$

where $\alpha \geqslant 0$ and $\sigma = 1$ and $\sigma = -1$ correspond to focusing and defocusing nonlinearities (hereafter we use terminology relevant to optical applications). Physically the dimensionless Eq. (1) naturally appears as an equation modeling beam guidance in a medium whose refractive index $n(x) = n_r(x) + in_i(x)$ has parabolic modulation of the real part, $n_r(x) = x^2$, and linear modulation of the imaginary part, $n_i(x) = 2\alpha x$. Even more generally, any smooth enough symmetric profile of the refractive index $n_r(x) = n_r(-x)$ and antisymmetric modulation of its imaginary part $n_i(x) = -n_i(-x)$ leads to the model (1) if a guided beam is narrow enough to allow for the use of the first-order terms of the Taylor expansion of the complex index $n(x)$.

In this paper we are interested in stationary modes, which are sought in the form $q(z,x) = w(x)e^{i\beta z}$, where β is the propagation constant. We consider localized solutions which obey the zero boundary conditions

$$\lim_{|x|\to\infty} |q(z,x)| = 0. \tag{2}$$

For the next consideration it is convenient to introduce the representation $\beta = b - \alpha^2$, where b is a new parameter, which allows one to arrive at the following stationary equation:

$$w_{xx} - bw - (x - i\alpha)^2 w + \sigma|w|^2 w = 0. \tag{3}$$

Recalling that the existence of the modes implies a balance between the diffraction and the nonlinearity, as well as between gain and losses, we also rewrite Eq. (3) in the hydrodynamic form

$$\rho_{xx} - (b - \alpha^2 - x^2)\rho + \sigma\rho^3 - \frac{j^2}{\rho^3} = 0, \tag{4a}$$

$$j_x = -2\alpha x \rho^2, \tag{4b}$$

where $\rho(x) = |w(x)|$ is the field modulus, while $j(x) = \theta_x(x)\rho^2(x)$, with $\theta(x) = \arg w(x)$, is the real-valued current. From (II) one readily concludes that both $\rho(x)$ and $j(x) = \theta_x(x)\rho^2(x)$ are even functions. The current $j(x)$ has a local maximum at $x = 0$, while $\rho(x)$ has either a local maximum or a local minimum at $x = 0$. Moreover, it follows from Eq. (2) that $j \to 0$ at $x \to \infty$, and hence taking into account that $j_x(x)$ does not change sign for $x \neq 0$, we deduce from Eq. (4a) that $j(x)$ does not become zero at any finite x, and hence the same is valid for $\rho(x)$ [since otherwise the last term in Eq. (4a) would give a singularity]. The absence of zeros of the field contrasts with the known behavior of the nonlinear modes in a real harmonic potential, while is known for the linear $\mathcal{PT}$-symmetric modes [23], which are briefly outlined in the next section.

III. LINEAR MODES

Let us recall some relevant properties of the linear problem [22–24]:

$$\mathcal{L}_n\tilde{w}_n = 0, \qquad \mathcal{L}_n = \frac{d^2}{dx^2} - \tilde{b}_n - (x - i\alpha)^2, \tag{5}$$

which can be formally obtained by setting $\sigma = 0$ in Eq. (3). Hereafter a tilde distinguishes solutions of the linear problem. The set of eigenvalues of the problem (5) does not depend on α and consists of an equidistant sequence $\tilde{b}_n = -(2n+1)$, $n = 0,1,\ldots$. The corresponding eigenfunctions can be written as $\tilde{w}_n(x) = c_n\tilde{\psi}_n(x - i\alpha)$, where $\tilde{\psi}_n(x) = H_n(x)e^{-x^2/2}$ is the nth Gauss-Hermite mode, $\int \tilde{\psi}_n(x)\tilde{\psi}_m^*(x)dx = \delta_{n,m}\sqrt{\pi}2^n n!$, $H_n(x)$ is the nth Hermite polynomial, and c_n are positive coefficients providing the normalization condition $\int \tilde{w}_n(x)\tilde{w}_n^*(x)dx = 1$ (hereafter we omit the integration limits wherever the integration is over the whole real axis, and the asterisk denotes complex conjugation).

Unlike in the conservative case $\alpha = 0$, for $\alpha > 0$ the eigenfunctions $\tilde{w}_n(x)$ are not orthogonal. Using the relation (see, e.g., [25]) $H_n(x + x_0) = \sum_{k=0}^n C_n^k(2x_0)^{n-k}H_k(x)$ where $C_n^k = n!/[k!(n-k)!]$ are the binomial coefficients, for any n and m one finds

$$\int \tilde{w}_n(x)\tilde{w}_m^*(x)dx$$
$$= c_n c_m e^{\alpha^2}\sqrt{\pi}\sum_{k=0}^{p} C_n^k C_m^k 2^k k!(-1)^{n-k}(2i\alpha)^{n+m-2k}$$
$$= c_n c_m e^{\alpha^2}\sqrt{\pi}2^{(n+m+g)/2}i^{3n+m}\alpha^g p! L_p^{(g)}(-2\alpha^2),$$

where $p = \min(n,m)$, $g = |n - m|$, and $L_p^{(g)}(x)$ is the generalized Laguerre polynomial. Setting $n = m$ we obtain the expression for the normalization coefficients c_n:

$$c_n = \frac{e^{-\alpha^2/2}}{\sqrt{\sqrt{\pi}2^n n! L_n(-2\alpha^2)}}. \tag{6}$$

For $\alpha = 0$ the eigenfunctions $\tilde{w}_n(x)$ are real valued (up to an irrelevant phase shift). Moreover, $\tilde{w}_n(x)$ is an even (odd) function if n is even (odd). For $\alpha \neq 0$ the eigenfunctions are complex valued and are neither even nor odd. Instead, they can be chosen to have even real parts and odd imaginary parts.

IV. BIFURCATIONS OF NONLINEAR MODES

Turning now to the nonlinear problem, we observe that the eigenvalues $\tilde{b}_n$, $n = 0,1,\ldots$, are the bifurcation points where families of nonlinear modes branch off from the zero solution $w(x) \equiv 0$. The nonlinear modes $w_n(x)$ belonging to the nth family have the same symmetry as the corresponding linear eigenfunction $\tilde{w}_n(x)$. In the vicinity of the nth bifurcation point, the nonlinear modes $w_n(x)$ can be described by means of asymptotic expansions

$$w_n(x) = \varepsilon\tilde{w}_n + O(\varepsilon^3), \quad b_n = \tilde{b}_n + \sigma\varepsilon^2 b_n^{(2)} + o(\varepsilon^2), \tag{7}$$

where $\varepsilon \ll 1$ is a formal small parameter. Since $\tilde{w}_n(x)$ were chosen normalized, in the leading order the total energy flow $U = \int |w_n(x)|^2 dx$ (hereafter all the integrals are taken over the whole real axis) is equal to ε: $U \sim \varepsilon^2$. The solvability condition for the ε^3-order equation yields

$$b_n^{(2)} = \frac{\int \tilde{w}_n^3(x)\tilde{w}_n^*(x)dx}{\int \tilde{w}_n^2(x)dx}. \tag{8}$$

Since for $\alpha = 0$ the eigenfunctions $\tilde{w}_n(x)$ are real valued, one has that $b_n^{(2)} > 0$ for any n. For $\alpha > 0$ the eigenfunctions $\tilde{w}_n(x)$ are complex valued. However, parity of their real and imaginary parts ensures that $b_n^{(2)}$ is nevertheless real for any n and α. It is straightforward to obtain explicit expressions for $b_n^{(2)}$. For the two lowest families ($n = 0$ and $n = 1$) one has

$$b_0^{(2)} = \frac{e^{(1/2)\alpha^2}}{\sqrt{2\pi}}, \quad b_1^{(2)} = \frac{3e^{(1/2)\alpha^2}}{4\sqrt{2\pi}}\frac{1 + 2\alpha^2 - \alpha^4}{1 + 2\alpha^2}. \tag{9}$$

It follows from Eqs. (9) that $b_0^{(2)}$ is positive for all α while $b_1^{(2)}$ is positive for small α, but becomes negative for $\alpha > \sqrt{1+\sqrt{2}}$. Regarding the next families, we have found that for $n = 2$ the coefficient $b_2^{(2)}$ changes sign twice. For $n = 3$, however, the coefficient $b_3^{(2)}$ changes sign only once, becoming negative for all sufficiently large α.

From Eqs. (9) we also arrive at another interesting observation: the coefficients $b_{0,1}^{(2)}$ grow exponentially fast with α. This, in particular, means that $\lim_{\alpha\to\infty} \frac{\partial U}{\partial b}\big|_{b=\tilde{b}_{0,1}} = 0$. Taking into

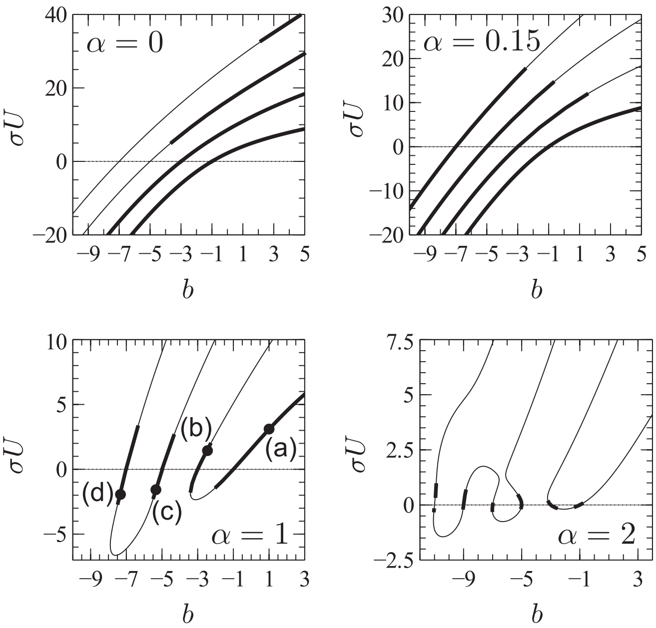

FIG. 1. The lowest families of nonlinear modes for different α. The fragments of curves corresponding to stable nonlinear modes are shown in bold. The nonlinear modes indicated with the points (a)–(d) in the panel $\alpha = 1$ are explicitly shown in Fig. 2.

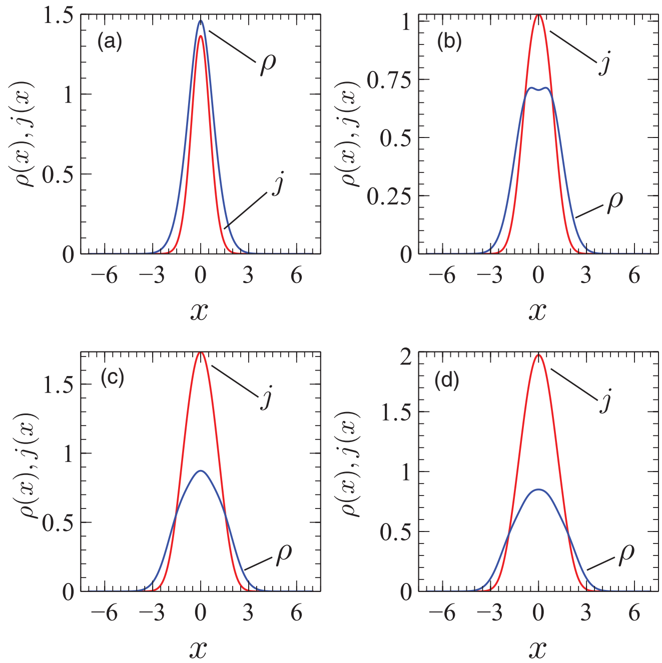

FIG. 2. (Color online) The modulus $\rho(x)$ and the current $j(x)$ for stable nonlinear modes corresponding to $\alpha = 1$. Panels (a)–(d) correspond to nonlinear modes indicated by the points (a)–(d) in the panel $\alpha = 1$ of Fig. 1.

account that the coefficients $b_0^{(2)}$ and $b_1^{(2)}$ have opposite signs for $\alpha \gg 1$, one can expect that for large α the nonlinear modes bifurcating from $\tilde{b}_0$ and $\tilde{b}_1$ merge (or intersect) at some value of the energy flow U.

The latter situation seems to be counterintuitive and strongly contrasts with what is known for the conservative harmonic potential, where the modes bifurcating from different eigenstates of the linear problem do not merge. In order to check this issue we performed a direct numerical study of the families of nonlinear modes. The characteristic results are summarized in Fig. 1, where the families of nonlinear modes are shown on the plane $(b,\sigma U)$ for several different values of α. Respectively, the modes corresponding to the focusing (defocusing) nonlinearity are situated above (below) the axis $\sigma U = 0$, which is indicated with the dashed line.

For the sake of comparison, in the left upper panel of Fig. 1 we show the families of nonlinear modes for the well-studied real harmonic oscillator [4,6,7], which in our case corresponds to $\alpha = 0$. On increasing α (see the other panels of Fig. 1), we observe that already at $\alpha = 1$ in the defocusing medium the nonlinear modes bifurcating from $\tilde{b}_0 = -1$ and $\tilde{b}_1 = -3$ (as well as the ones bifurcating from $\tilde{b}_2 = -5$ and $\tilde{b}_3 = -7$) indeed appear to be connected in a single family. For larger α (e.g., for $\alpha = 2$) the structure of the nonlinear modes becomes more complicated and the higher families (the ones bifurcating from $\tilde{b}_4 = -9$ and $\tilde{b}_5 = -11$) also become involved in creation of a single family snaking through the linear eigenstates with $n = 2$, 3, 4, and 5. For $\alpha = 2$ one can see the connection of the modes not only in the defocusing but also in the focusing medium. Since $\alpha = 2 > \sqrt{1+\sqrt{2}}$, Eqs. (9) imply that the coefficient $b_1^{(2)}$ is negative for $\alpha = 2$, and thus, in contrast to the cases $\alpha = 0$, 0.15, and 1, the slope $\partial(\sigma U)/\partial b$ is negative in the vicinity of the bifurcation from the point $\tilde{b}_1$. In Fig. 2 we show the field modulus $\rho(x)$ and the superfluid current $j(x)$ for several stable nonlinear modes corresponding to $\alpha = 1$. In accordance with the discussion in Sec. II, both $\rho(x)$ and $j(x)$ are even functions, and for all the shown modes the current $j(x)$ has a maximum at $x = 0$. The field modulus $\rho(x)$ has a maximum at $x = 0$ for the nonlinear modes (a), (c), and (d). For the nonlinear mode (b) the field modulus has a local minimum at $x = 0$.

It is interesting to observe that the described behavior of the modes allows one to suggest that it is possible to use continuous deformation to transform one of the modes of the conventional linear harmonic oscillator to another one having different parity. Indeed, to this end it is enough to properly change the strength of the nonconservative potential α and the intensity of the beam U. The stability of the modes, important for any practical realization of such a deformation, is discussed in the next section.

V. STABILITY OF THE NONLINEAR MODES

A. Analytical results

Now we turn to analysis of the linear stability of the modes. Following the standard procedure, we use the substitution $q(z,x) = e^{i\beta z}[w(x) + u(x)e^{i\omega z} + v^*(x)e^{-i\omega^* z}]$ and arrive at the eigenvalue problem

$$\mathbf{L}\mathbf{p} = \omega\mathbf{p}, \tag{10}$$

where

$$\mathbf{L} = \begin{pmatrix} L + 2\sigma|w_n|^2 & \sigma w_n^2 \\ -\sigma(w_n^2)^* & -L^\dagger - 2\sigma|w_n|^2 \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} u \\ v \end{pmatrix},$$

$L = d^2/dx^2 - b - (x - i\alpha)^2$, and $L^\dagger$ is the Hermitian adjoint operator. The nonlinear mode $w_n(x)$ is unstable if there exists an eigenvalue ω such that Im $\omega < 0$.

It is straightforward to check the properties of the operator $\mathbf{L}$ as follows. If ω is an eigenvalue of $\mathbf{L}$ with an eigenvector $(u(x),v(x))^T$, then $-\omega^*$ is also an eigenvalue with the eigenvector $(v^*(x),u^*(x))^T$. Employing the symmetry of the nonlinear modes $[w_n(x) = w_n^*(-x)]$ one finds that ω^* is also an eigenvalue with the eigenvector $(u^*(-x),v^*(-x))$. Also, $\omega = 0$ is always an eigenvalue of the operator $\mathbf{L}$. The eigenvector corresponding to $\omega = 0$ reads $(w_n(x), -w_n^*(x))^T$.

Let us now analyze the spectrum of the operator $\mathbf{L}$ in the vicinity of the nth bifurcation point. In the linear limit (i.e., for $\varepsilon = 0$) the operator $\mathbf{L}$ acquires the form

$$\mathbf{L} = \tilde{\mathbf{L}}_n \equiv \begin{pmatrix} \mathcal{L}_n & 0 \\ 0 & -\mathcal{L}_n^\dagger \end{pmatrix}, \tag{11}$$

where $\mathcal{L}_n$ is defined in (5). The spectrum of the operator $\tilde{\mathbf{L}}_n$ consists of two sequences. The eigenvalues and eigenvectors of the first sequence read $\omega_{n,k}^{(I)} = 2(n-k)$, $\mathbf{p}_{n,k}^{(I)} = (\tilde{w}_k(x),0)^T$, $k = 0,1,\ldots$. The second sequence reads $\omega_{n,k}^{(II)} = -2(n-k)$, $\mathbf{p}_{n,k}^{(II)} = (0,\tilde{w}_k^*(x))^T$, $k = 0,1,\ldots$. First, we notice that the operator $\tilde{\mathbf{L}}_n$ has a double zero eigenvalue $\omega_{n,n}^{(I)} = \omega_{n,n}^{(II)} = 0$. Generically, passing from the linear limit $\varepsilon = 0$ to $\varepsilon > 0$, a double eigenvalue splits into two simple eigenvalues. However, in the case at hand, the splitting of the double zero eigenvalue cannot occur. Indeed, if the zero eigenvalue splits into two simple ones, they will be either both real and of opposite signs or complex conjugated. Either of these possibilities means that for $\varepsilon \neq 0$ the eigenvalue $\omega = 0$ is no longer in the spectrum of the operator $\mathbf{L}$. This, however, contradicts the properties of the operator $\mathbf{L}$ established above. Thus, for $\varepsilon \neq 0$ the operator $\mathbf{L}$ also has the double zero eigenvalue.

Besides of the double zero eigenvalue, the operator $\tilde{\mathbf{L}}_n$ has $2n$ double eigenvalues: $\Omega_{n,k} = \omega_{n,k}^{(I)} = \omega_{n,2n-k}^{(II)}$, where k runs from 0 to $2n$ except for $k = n$. Again, the double eigenvalue $\Omega_{n,k}$ generically splits into two simple eigenvalues, which will be either both real or complex conjugated. At the same time, the opposite double eigenvalue $\Omega_{n,2n-k} = -\Omega_{n,k}$ will split in the same manner. Since the double eigenvalues $\Omega_{n,k}$ and $\Omega_{n,2n-k}$ behave in the same way, it is sufficient to analyze only n positive double eigenvalues $\Omega_{n,k}$ which correspond to $k = 0,\ldots,n-1$. The double eigenvalue $\Omega_{n,k}$ is semisimple; the corresponding eigenvectors read $\mathbf{p}_{n,k}^{(I)} = (\tilde{w}_k,0)^T$ and $\mathbf{p}_{n,2n-k}^{(II)} = (0,\tilde{w}_{2n-k}^*)^T$.

In order to examine splitting of the double eigenvalues, we employ Eqs. (7), which yield the following asymptotic expansion for the linear stability operator: $\mathbf{L} = \tilde{\mathbf{L}}_n + \sigma\varepsilon^2\mathbf{L}_n^{(2)} + o(\varepsilon^2)$, where

$$\mathbf{L}_n^{(2)} = \begin{pmatrix} -\tilde{b}_n^{(2)} + 2|\tilde{w}_n|^2 & \tilde{w}_n^2 \\ -(\tilde{w}_n^2)^* & \tilde{b}_n^{(2)} - 2|\tilde{w}_n|^2 \end{pmatrix}. \tag{12}$$

Following the standard arguments of perturbation theory for linear operators [22], in order to explore the behavior of a double eigenvalue $\Omega_{n,k}$ we introduce a 2×2 matrix

$$\mathbf{M}_{n,k} = \begin{pmatrix} \frac{\langle \mathbf{L}_n^{(2)}\mathbf{p}_{n,k}^{(I)},\mathbf{p}_{n,k}^{(I)*}\rangle}{\langle \mathbf{p}_{n,k}^{(I)},\mathbf{p}_{n,k}^{(I)*}\rangle} & \frac{\langle \mathbf{L}_n^{(2)}\mathbf{p}_{n,2n-k}^{(II)},\mathbf{p}_{n,k}^{(I)*}\rangle}{\langle \mathbf{p}_{n,k}^{(I)},\mathbf{p}_{n,k}^{(I)*}\rangle} \\ \frac{\langle \mathbf{L}_n^{(2)}\mathbf{p}_{n,k}^{(I)},\mathbf{p}_{n,2n-k}^{(II)*}\rangle}{\langle \mathbf{p}_{n,2n-k}^{(II)},\mathbf{p}_{n,2n-k}^{(II)*}\rangle} & \frac{\langle \mathbf{L}_n^{(2)}\mathbf{p}_{n,2n-k}^{(II)},\mathbf{p}_{n,2n-k}^{(II)*}\rangle}{\langle \mathbf{p}_{n,2n-k}^{(II)},\mathbf{p}_{n,2n-k}^{(II)*}\rangle} \end{pmatrix},$$

where $\langle \mathbf{a},\mathbf{b}\rangle = \int \mathbf{b}^\dagger(x)\mathbf{a}(x)dx$ for any two column vectors $\mathbf{a}$ and $\mathbf{b}$. If both the eigenvalues of the matrix $\mathbf{M}_{n,k}$ are real, then the simple eigenvalues emerging from $\Omega_{n,k}$ are real, at least for $\varepsilon \geqslant 0$ sufficiently small. If such a situation takes place for all $k = 0,1,\ldots,n-1$, then one can state that the nonlinear modes $w_n(x)$ belonging to the nth family are stable in the linear limit. On the other hand, if for some k the matrix $\mathbf{M}_{n,k}$ has a complex eigenvalue, then the double eigenvalue $\Omega_{n,k}$ gives rise to a pair of complex-conjugated eigenvalues. This is sufficient to conclude that the nonlinear modes of the nth family are unstable in the linear limit. For $n = 0$ no double eigenvalues $\Omega_{n,k}$ exists. Therefore the lowest family $n = 0$ is always stable in the linear limit.

Taking into account the symmetry of the eigenfunctions $\tilde{w}_n(x)$, one finds that the entries of the matrix $\mathbf{M}_{n,k}$ have the form

$$(\mathbf{M}_{n,k})_{1,1} = -b_n^{(2)} + 2\frac{\int |\tilde{w}_n|^2\tilde{w}_k^2 dx}{\int \tilde{w}_k^2 dx},$$

$$(\mathbf{M}_{n,k})_{2,2} = b_n^{(2)} - 2\frac{\int |\tilde{w}_n|^2\tilde{w}_{2n-k}^2 dx}{\int \tilde{w}_{2n-k}^2 dx},$$

$$(\mathbf{M}_{n,k})_{2,1} = -\frac{\int \tilde{w}_n^2\tilde{w}_k^*\tilde{w}_{2n-k} dx}{\int \tilde{w}_{2n-k}^2 dx},$$

$$(\mathbf{M}_{n,k})_{1,2} = \frac{\int \tilde{w}_n^2\tilde{w}_k\tilde{w}_{2n-k}^* dx}{\int \tilde{w}_k^2 dx}.$$

One also observes that all these entries are real.

Using the above expressions, the matrices $\mathbf{M}_{n,k}$ as well as their eigenvalues can be found explicitly. One observes that for any n and k an expression for the eigenvalues of the matrix $\mathbf{M}_{n,k}$ contains a term $\sqrt{P_{n,k}(\alpha)}$, where $P_{n,k}(\alpha)$ is a polynomial with real coefficients. Such polynomials are different for different n and k and are computable explicitly. Their properties (for $n = 1,2,\ldots,5$) are summarized in Table I.

The terms $\sqrt{P_{n,k}(\alpha)}$ represent the only possibility for the eigenvalues eventually to have a nonzero imaginary part. Splitting of the double eigenvalue $\Omega_{n,k}$ for $\alpha = 0$ is determined by the sign of $P_{n,k}(0)$, while the behavior of $\Omega_{n,k}$ in the limit $\alpha \gg 1$ is determined by the sign of the leading coefficient of the polynomial $P_{n,k}(\alpha)$.

Turning now to Table I, the following comments can be given: (i) the degree of the polynomial $P_{n,k}(\alpha)$, denoted by $D_{n,k}$, obeys the relation $D_{n,k} = 12n - 4k$; (ii) more importantly, the leading coefficients of all the considered polynomials are positive. This means that for any n there exists a critical value $\alpha_n^{\rm cr}$ such that for all $\alpha > \alpha_n^{\rm cr}$ the nth family is stable in the linear limit, even if this family is unstable in the case of the real harmonic potential (i.e., for $\alpha = 0$). For $n = 0$ and $n = 1$ the critical values are zero: $\alpha_0^{\rm cr} = \alpha_1^{\rm cr} = 0$. These families are stable in the linear limit both for the case of the real harmonic oscillator ($\alpha = 0$) and in the $\mathcal{PT}$-symmetric case for any α. For the next families, $n = 2$ and $n = 3$, Table I yields $\alpha_2^{\rm cr} \approx 3.60$, and $\alpha_3^{\rm cr} \approx 4.17$. This means that, although these latter families are unstable in the linear limit for $\alpha = 0$, they become stable in the linear limit for α sufficiently large. The same situation holds for the families $n = 4$ and $n = 5$. Moreover we conjecture that it also persists for all higher families.

TABLE I. Properties of the polynomials $P_{n,k}(\alpha)$. Here $D_{n,k}$ is the degree of the polynomial, $S_{n,k}$ is the sign of the leading coefficient, and $s_{n,k}$ is the sign of the constant term $P_{n,k}(0)$. Approximate values of all the positive roots are also reported.

n	k	D	S	s	Positive roots					
1	0	12	+	+	No positive roots					
2	0	24	+	−	0.05	2.47	2.54	3.21	3.60	
	1	20	+	+	No positive roots					
3	0	36	+	−	0.12					
	1	32	+	−	0.05	1.68	1.94	3.18	4.17	
	2	28	+	+	No positive roots					
4	0	48	+	+	0.08	0.14	3.35	3.40	4.77	4.82
	1	44	+	−	0.11					
	2	40	+	−	0.05	3.64	4.66			
	3	36	+	+	No positive roots					
5	0	60	+	+	0.12	0.14				
	1	56	+	−	0.12	1.74	2.20	5.14	5.24	
	2	52	+	−	0.10	2.41	2.58			
	3	48	+	−	0.05	2.84	2.92	4.10	5.10	
	4	44	+	+	No positive roots					

B. Numerical results

Passing to the numerical study of the stability (see Fig. 1), we first recall some results known for the real harmonic potential, which in our model corresponds to $\alpha = 0$. The nonlinear modes that belong to the two lowest families ($n = 0$ and $n = 1$) are always stable. The families $n = 2$ and $n = 3$ are unstable in the linear limit and for small and moderate values of U. However, both the latter families become stable if the nonlinearity is sufficiently strong (for a stability analysis of the modes in a strongly nonlinear defocusing medium see [7]). In the defocusing medium, the value of U that has to be exceeded for the families $n = 2$ and $n = 3$ to become stable is large and does not fit within the scope of the panel $\alpha = 0$ of Fig. 1.

In the next panel of Fig. 1 we consider the case $\alpha = 0.15$. For this value of α it follows from Table I that for any $n = 1,2,\ldots,5$ the nth family is stable in the linear limit. Turning to the stability of the nonlinear modes of arbitrary amplitude, we observe that the lowest family $n = 0$ is stable in the whole considered region of parameters. The same situation holds for the family $n = 1$ but in the defocusing medium only. For $\sigma = 1$ this family loses stability at sufficiently strong nonlinearity. The most interesting results, however, are obtained for the families $n = 2$ and $n = 3$. In contrast to their counterparts for the real harmonic oscillator, these families are stable in the linear limit. Moreover, they remain stable at least for small and moderate values of U. In the defocusing medium, the families $n = 2$ and $n = 3$ appeared to be stable in the whole explored region. In the focusing medium, we found the critical values of nonlinearity after which the onset of instability occurs. It is interesting that in a certain sense the situations for the real oscillator and for the $\mathcal{PT}$-symmetric one are opposite: for $\alpha = 0$ the families $n = 2$ and $n = 3$ are unstable in the linear limit but become stable in the focusing medium for U sufficiently large. Vice versa, for $\alpha = 0.15$, those families are stable in the linear limit but lose stability in the focusing medium for large U. At this stage we emphasize that only a finite range of b and U has been considered in our numerics, and, in principle, the families of nonlinear modes may change stability for stronger values of nonlinearity which have not been considered here.

Next, we considered $\mathcal{PT}$-symmetric harmonic potentials with stronger imaginary components, $\alpha = 1$ and $\alpha = 2$. One can deduce from Table I that for $\alpha = 1$ the families $n = 1,2,\ldots,5$ are stable in the linear limit, while for $\alpha = 2$ the families $n = 1,\ldots,4$ are stable in the linear limit and the family $n = 5$ is unstable. However, for both $\alpha = 1$ and $\alpha = 2$ all the considered families lose stability for relatively small values of U. One observes that the larger α, the smaller is the nonlinearity strength that is sufficient for the destabilization to occur.

VI. CONCLUSION

To conclude, we have performed an analysis of the structure and stability of the lowest families of nonlinear modes in the nonlinear Schrödinger equation with a parabolic $\mathcal{PT}$-symmetric potential. We have found a number of striking features, not observable for the cases of conservative and dissipative parabolic potentials. Among these features we emphasize transformation of the families bifurcating from the different eigenstates of the underlying linear problem to a single family; enhancement of the stability in the linear limit comparing to the standard case of the real harmonic oscillator; and the possibility that proper choice of the strength of the nonconservative part α makes nonlinear modes that are unstable for $\alpha = 0$ become stable in the $\mathcal{PT}$-symmetric case.

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