Nonlinear σ model for optical media with linear absorption or gain

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(Received 30 January 2012; published 24 April 2012)

In the framework of the Keldysh technique, we formulate the nonlinear σ model for disordered optical media with linear absorption or gain. The effective action for fluctuations of the matrix field about the saddle point acquires an extra term due to the nonconservative nature of the system. We determine the disorder-averaged Green's function correlator, which has a diffusion pole modified by a finite absorption or gain rate. The diffusion coefficient is found to be close to its value for conservative systems in the relevant range of parameters. In the medium with gain, the random-lasing threshold depends on the sample size.

DOI: 10.1103/PhysRevA.85.043838 PACS number(s): 42.25.Dd, 03.70.+k, 42.55.Zz

I. INTRODUCTION

The transport of waves through disordered matter has been a topic of recurring interest ever since the discovery of the Anderson localization in electronic systems [1]. Analogous phenomena have been subsequently studied for the transport of classical [2–5], matter [6,7], and even seismic waves [8].

The research on classical-wave propagation in disordered media has been motivated by the conjectured possibility of the localization of light. Results such as the enhancement of dwell times due to resonant scatterers (and, hence, lower energy-transport velocities [9]) and the correction term in the Ward identity due to frequency-dependent scattering potentials [10] have shown that, while retaining many similarities, the behavior of light in disordered media differs from that of electrons in several important aspects. One of these aspects concerns the propagation of light in nonconservative disordered media. Such systems can be physically realized, for example, as random lasers [11–13], which have received much attention recently. A promising research direction in this context are theories that combine description of wave propagation through disordered medium with the nonlinear laser equations [14–16].

The properties of light diffusion in absorbing media were studied using the photon transport equation [17–19]. In particular, it was argued that in the parameter range of validity of the diffusion equation, the diffusion coefficient is close to its value in the conservative medium. The treatment of light propagation starting from the wave equation has been mainly conducted via the self-consistent diagrammatic theory [20,21]. Interesting results such as corrections to the bare diffusion coefficient due to the additional terms in the Ward identity [22,23] and dynamics of Anderson localization in quasi-one-dimensional geometry [24] and open three-dimensional media [25] have been obtained by these methods. An alternative description of classical wave propagation is provided by the so-called effective models of disordered systems, commonly known as the nonlinear σ model [26,27] (NLSM). Originally developed for electronic systems, the (supersymmetric) NLSM describing light propagation in a *conservative* disordered medium was derived in Refs. [2,28]. Later, the effects of an open boundary on the diffusion coefficient were studied [29] by using a similar model. Unlike the self-consistent theory of transport, effective models have not yet been applied to optical systems with absorption or gain. The effective models can be useful, for example, in describing special properties of light localization in such systems [30–32].

In the present work we formulate the Keldysh nonlinear σ model [33] for the propagation of electromagnetic waves in *nonconservative* disordered media in the diffusive regime. Systems with absorption or gain are relatively simple to treat in the Keldysh formalism, which makes it possible to define an action needed for the field-theoretical description. By following the general scheme as outlined in Ref. [33] for electronic systems, we derive an effective NLSM action where we obtain a term due to nonconservativeness of the medium. A similar contribution was found in Ref. [29]; in that case the term originated from the openness of the system.

Furthermore, by using the standard methods [33], we show that the light propagation can be described by a diffusion equation for a nonconservative medium. The conditions under which the NLSM yields the diffusion equation are found to be equivalent to the restrictions imposed in the theory of transport equation [17]. Similarly, the diffusion coefficient that we derive is almost independent of the absorption or gain under these conditions. For the amplifying medium, we discuss the applicability of the linear-gain approximation and determine the threshold of random lasing.

II. KELDYSH APPROACH TO LIGHT PROPAGATION

A. Partition function for nonconservative medium

We consider an optical medium defined by a complex dielectric constant $\epsilon(\mathbf{r},\omega)$. Restricting our theory to the transverse magnetic (TM) modes in two dimensions, we describe the electric field by its normal component $E_\omega(\mathbf{r}) = i\omega A_\omega(\mathbf{r})$ (in the Coulomb gauge), where $A_\omega(\mathbf{r})$ is the normal component of the vector potential. We use the Gaussian units with the velocity of light $c \equiv 1$. The transversality condition $\nabla \cdot (\epsilon \mathbf{A}) = 0$, with ϵ varying in two dimensions, leads to the two-dimensional wave equation

$$[\nabla^2 + \epsilon(\mathbf{r},\omega)\,\omega^2] A_\omega(\mathbf{r}) = 0. \quad (1)$$

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For real $\epsilon(\mathbf{r},\omega)$, this equation, and its complex conjugate, can be obtained by setting to zero the functional derivatives

$$\frac{\delta S}{\delta A} = 0, \quad \frac{\delta S}{\delta A^*} = 0 \tag{2}$$

of the action (Hamilton principal function)

$$S[A,A^*] = \frac{1}{16\pi}\int d\mathbf{r}\,\frac{d\omega}{2\pi}[\epsilon(\mathbf{r},\omega)\,\omega^2|A_\omega(\mathbf{r})|^2 - |\nabla A_\omega(\mathbf{r})|^2] = \frac{1}{16\pi}\int d\mathbf{r}\,\frac{d\omega}{2\pi}A^*_\omega(\mathbf{r})[\epsilon(\mathbf{r},\omega)\,\omega^2 + \nabla^2]A_\omega(\mathbf{r}), \tag{3}$$

treating A and A^* as independent functions. The action can be rewritten in the representation-free operator notation as

$$S[A,A^\dagger] = \frac{1}{16\pi}A^\dagger G^{-1}A, \tag{4}$$

where the inverse Green's function operator $G^{-1} = \epsilon(\mathbf{r},\omega)\,\omega^2 + \nabla^2$ in the $(\mathbf{r},\omega)$ representation and A $[A^\dagger]$ is the Hilbert-space vector $A_\omega(\mathbf{r})$ $[A^*_\omega(\mathbf{r})]$.

In order to construct the quantum Hamiltonian, one expresses the energy of the system in terms of the vector potential. $A_\omega(\mathbf{r})$ and $A^*_\omega(\mathbf{r})$ are then expanded in the normal modes of the system, and the expansion coefficients become the photon annihilation and creation operators.

In the Keldysh field-theoretical approach [33,34] we calculate the partition function

$$Z = \mathrm{Tr}\,(U\rho), \tag{5}$$

where ρ is the density matrix at time $t = -\infty$, with $\mathrm{Tr}\,\rho = 1$, and

$$U = \mathrm{T}_\mathcal{C}\exp\left[-i\int_\mathcal{C} H(t)\,dt\right], \quad \hbar \equiv 1, \tag{6}$$

is the time-evolution operator along the Keldysh contour $\mathcal{C}$. The contour begins at $t = -\infty$, where the state of the system is known, and then goes forward in time up to $t = \infty$, where it turns back and goes to $t = -\infty$. $\mathrm{T}_\mathcal{C}$ denotes the time ordering along the contour. The Hamiltonian $H(t)$ is switched on adiabatically, starting from a trivial Hamiltonian $H(-\infty)$. $H(t)$ is the same on the forward and backward branches of the contour. This condition leads to $U = 1$ and, hence, $Z = 1$. If the source terms that are different on the two branches are added to the Hamiltonian, then $Z \neq 1$. The (functional) derivatives of the type $\delta Z[J]/\delta J|_{J=0}$ with respect to the sources J generate averages with the density matrix propagated from $t = -\infty$ to relevant times.

The partition function can be written in the form of a functional integral over the fields (i.e., the classical functions) A and $A^\dagger$. To this end, we represent the classical action along the Keldysh contour as

$$S_\mathcal{C} = \frac{1}{16\pi}[A^\dagger_+ G^{-1}A_+ - A^\dagger_- G^{-1}A_-] = \frac{1}{16\pi}[(A^{\mathrm{cl}})^\dagger G^{-1}A^{\mathrm{q}} + (A^{\mathrm{q}})^\dagger G^{-1}A^{\mathrm{cl}}], \tag{7}$$

where the subscripts $+$ and $-$ denote the fields on the forward and backward branches respectively of the contour and the so-called classical and quantum fields are defined by

$$A^{\mathrm{cl}} = \frac{1}{\sqrt{2}}(A_+ + A_-), \quad A^{\mathrm{q}} = \frac{1}{\sqrt{2}}(A_+ - A_-). \tag{8}$$

The minus sign in front of the $A^\dagger_- G^{-1}A_-$ term in Eq. (7) takes care of the time reversal on the backward branch, whereas A_- is the representation-free (vector) notation for the function $A_-(\mathbf{r},t)$ with the forward time ordering. It is convenient to consider A^{cl} and A^{q} as components of a single field

$$\hat{A} = \begin{pmatrix} A^{\mathrm{cl}} \\ A^{\mathrm{q}} \end{pmatrix} \tag{9}$$

in the Keldysh space, which is twice the size of the original Hilbert space. (We furnish the vectors and operators in this space with a hat.) Then the contour action can be written in the form (dropping the subscript $\mathcal{C}$)

$$S[\hat{A},\hat{A}^\dagger] = \frac{1}{16\pi}\hat{A}^\dagger\,\hat{G}^{-1}\hat{A}, \tag{10}$$

where $\hat{G}^{-1}$ has a 2×2 matrix structure with zeros on the diagonal and equal off-diagonal elements.

In order to use $S[\hat{A},\hat{A}^\dagger]$ in the functional integral for Z, the operator $\hat{G}^{-1}$ has to be regularized [33] by imposing the causality structure on its matrix:

$$\hat{G}^{-1} = \begin{pmatrix} 0 & (G^{-1})^{\mathrm{A}} \\ (G^{-1})^{\mathrm{R}} & (G^{-1})^{\mathrm{K}} \end{pmatrix}, \tag{11}$$

$$(G^{-1})^{\mathrm{R,A}}_\omega(\mathbf{r}) = \epsilon(\mathbf{r},\omega)\,\omega^2 + \nabla^2 \pm i0^+, \tag{12}$$

$$(G^{-1})^{\mathrm{K}} = (G^{-1})^{\mathrm{R}}F - F(G^{-1})^{\mathrm{A}}. \tag{13}$$

Here, $(G^{-1})^{\mathrm{R,A,K}}$ are the retarded, advanced, and Keldysh components respectively of the inverse Green's function operator. The operator F that parameterizes $(G^{-1})^{\mathrm{K}}$ depends on the thermal distribution. Equation (12) is written under assumption of real $\epsilon(\mathbf{r},\omega)$. In the medium with absorption, it is generalized to

$$(G^{-1})^{\mathrm{R,A}}_\omega(\mathbf{r}) = \epsilon'(\mathbf{r},\omega)\,\omega^2 + \nabla^2 \pm i\epsilon''(\mathbf{r},\omega)\,\omega^2, \tag{14}$$

$$\epsilon'(\mathbf{r},\omega) = \mathrm{Re}\,[\epsilon(\mathbf{r},\omega)], \quad \epsilon''(\mathbf{r},\omega) = \mathrm{Im}\,[\epsilon(\mathbf{r},\omega)], \tag{15}$$

where $\epsilon'' > 0$. In the case of the gain medium, $\epsilon'' < 0$, the time-dependent Green's function is exponentially diverging, and its Fourier transform to the frequency domain does not exist. The frequency representation can be defined with the help of the Laplace transform, which is equivalent to introducing a fictitious absorption to the system. When the results of a calculation are transformed back to the time representation, they should not depend on the fictitious absorption. This means that one can perform the calculations in the frequency domain assuming $\epsilon'' > 0$ and obtain the final results by analytic continuation to $\epsilon'' < 0$.

The functional-integral representation of the partition function becomes

$$Z = \mathcal{N}\int D[\hat{A},\hat{A}^\dagger]\,e^{iS[\hat{A},\hat{A}^\dagger]}, \tag{16}$$

where $\mathcal{N}$ is the nonessential normalization constant that ensures $Z = 1$ and the measure is defined by

$$D[\hat{A},\hat{A}^\dagger] = \prod_{\substack{\mathbf{r},\omega,\\ j\,=\,\mathrm{cl,q}}} \frac{d\big[\mathrm{Re}\,A^j_\omega(\mathbf{r})\big]\,d\big[\mathrm{Im}\,A^j_\omega(\mathbf{r})\big]}{\pi}. \tag{17}$$

Here and below we set to unity the step size for the grid used to discretize the integral.

We note that the Keldysh formalism is especially appropriate for the description of systems with absorption or gain as it naturally takes into account the finite $\epsilon''(\mathbf{r},\omega)$ in the causality structure of the inverse Green's function, Eqs. (11), (14), and (13).

B. Disorder average

We study the effect of disorder in the refractive index and assume the absorption or gain in the system to be spatially uniform. Hence, we represent the dielectric constant in the form

$$\epsilon(\mathbf{r},\omega) = \epsilon'(\omega) + \Delta\epsilon'(\mathbf{r},\omega) + i\epsilon''(\omega), \tag{18}$$

with the averages over disorder realizations $\langle \Delta\epsilon'(\mathbf{r},\omega)\rangle = 0$ and $\langle \Delta\epsilon'(\mathbf{r},\omega)\,\Delta\epsilon'(\mathbf{r}',\omega)\rangle \propto \delta(\mathbf{r}-\mathbf{r}')$.

In order to define the scattering time τ, let us, for a moment, neglect $\epsilon''(\omega)$. The wave equation (1) can be interpreted as a time-independent Schrödinger equation with the energy $\mathcal{E}(\omega) = \epsilon'(\omega)\,\omega^2$ and the potential energy $V(\mathbf{r},\omega) = -\Delta\epsilon'(\mathbf{r},\omega)\,\omega^2$. The scattering time τ and other characteristic time scales of the system are assumed to be much larger than ω_0^{-1}, where ω_0 is the typical optical frequency. The slowly varying amplitude $\widetilde{A}(\mathbf{r},t) = A(\mathbf{r},t)\exp(i\omega_0 t)$ satisfy the approximate equation

$$i\widetilde{\hbar}\frac{\partial \widetilde{A}}{\partial t} = [-\nabla^2 + V(\mathbf{r},\omega_0) - \mathcal{E}(\omega_0)]\widetilde{A}, \tag{19}$$

which is the time-dependent Schrödinger equation with the "optical Planck constant"

$$\widetilde{\hbar} = \frac{d\mathcal{E}(\omega_0)}{d\omega_0}. \tag{20}$$

The scattering time can now be defined by analogy with the quantum scattering time via the correlation function [35]

$$\langle V(\mathbf{r},\omega_0)\,V(\mathbf{r}',\omega_0)\rangle = \frac{\widetilde{\hbar}}{2\pi\nu\tau}\,\delta(\mathbf{r}-\mathbf{r}'), \tag{21}$$

where $\nu = dn/d\mathcal{E}$ is the quantum density of states (per unit volume). Note that $\nu = \nu_0/\widetilde{\hbar}$, where $\nu_0 = dn/d\omega$ is the standard optical density of modes.

A disorder average of the partition function can be obtained by evaluating the functional integral

$$\langle Z\rangle = \int D[V]\,Z\,\exp\left[-\frac{\pi\nu\tau}{\widetilde{\hbar}}\int d\mathbf{r}\,V^2(\mathbf{r},\omega_0)\right], \tag{22}$$

$$D[V] = \prod_{\mathbf{r}}\sqrt{\frac{\nu\tau}{\widetilde{\hbar}}}\,dV(\mathbf{r},\omega_0). \tag{23}$$

The disorder-dependent part of the action is

$$\Delta S[\hat{A},\hat{A}^\dagger,V] = -\frac{1}{16\pi}\hat{A}^\dagger V\hat{\gamma}\hat{A}, \quad \hat{\gamma} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{24}$$

where $V(\mathbf{r},\omega_0)$ appears as an operator V diagonal in $\mathbf{r}$; F is assumed to be diagonal in $\mathbf{r}$, as well. By completing the square, we obtain the disorder contribution to the partition function

$$\langle e^{i\Delta S}\rangle = \exp\left[-\frac{\widetilde{\hbar}}{4\pi\nu\tau}\int d\mathbf{r}\left(\frac{1}{16\pi}\hat{A}^\dagger(\mathbf{r})\hat{\gamma}\hat{A}(\mathbf{r})\right)^2\right]. \tag{25}$$

The shorthand notation $\hat{A}(\mathbf{r})$ is used for the Keldysh space vector $\hat{A}$ with the fixed index $\mathbf{r}$; that is, it is a vector in the space with the reduced dimensionality. In this notation, $\hat{A}^\dagger(\mathbf{r})\hat{A}(\mathbf{r})$ involves a summation over the remaining indices, for example, ω and the Keldysh index.

The negative sign in the exponent (25) is essential for the properties of nonlinear σ model in the optical medium. In contrast to a fermionic system, the sign cannot be changed by commuting the fields.

III. NONLINEAR σ MODEL

A. Hubbard-Stratonovich transformation. Saddle point

The term of the fourth order in the fields in Eq. (25) can be converted to a second-order term with the help of the Hubbard-Stratonovich transformation, yielding

$$\exp\left[-\frac{\widetilde{\hbar}}{4\pi\nu\tau}\int d\mathbf{r}\left(\frac{1}{16\pi}\hat{A}^\dagger(\mathbf{r})\hat{\gamma}\hat{A}(\mathbf{r})\right)^2\right] = \mathcal{N}_Q\int D[\hat{Q}]\exp\left[-\frac{\pi\nu\widetilde{\hbar}}{4\tau}\mathrm{Tr}\,\hat{Q}^2 + i\frac{\widetilde{\hbar}}{32\pi\tau}\hat{A}^\dagger\hat{\gamma}\hat{Q}\hat{A}\right], \tag{26}$$

$$\mathrm{Tr}\,\hat{f} \equiv \sum_{j=\mathrm{cl,q}}\int d\mathbf{r}\,\frac{d\omega}{2\pi}\,f^{jj}_{\omega\omega}(\mathbf{r}). \tag{27}$$

The auxiliary field $\hat{Q}$ is the Hermitian operator diagonal in $\mathbf{r}$. The measure $D[\hat{Q}]$ is defined over the independent matrix elements by analogy to Eq. (17). The normalization constant $\mathcal{N}_Q$ is determined by setting $\hat{A} = 0$ and $\hat{A}^\dagger = 0$. The negative coefficient in front of $\mathrm{Tr}\,\hat{Q}^2$ determines the scale of $\hat{Q}$ and can be chosen freely. The present choice leads to the simple form of matrix $\hat{\Lambda}$ introduced in Eq. (34). In order to prove Eq. (26) more easily, one can define the matrix $\hat{\mathcal{A}}(\mathbf{r}) = \hat{A}(\mathbf{r})\otimes\hat{A}^\dagger(\mathbf{r})$ where the tensor product applies to the Keldysh and ω subspaces. Then one represents

$$\hat{A}^\dagger\hat{\gamma}\hat{Q}\hat{A} = \mathrm{Tr}\,(Q\,\hat{\mathcal{A}}\,\hat{\gamma}). \tag{28}$$

Now the field $\hat{Q}$ can be integrated out after completing the square.

Using Eqs. (25) and (26) in Eq. (16) and integrating out the fields $\hat{A}$ and $\hat{A}^\dagger$, we obtain the disorder-averaged partition function

$$\langle Z\rangle = \widetilde{\mathcal{N}}_Q\int D[\hat{Q}]\,e^{iS[\hat{Q}]}, \tag{29}$$

$$iS[\hat{Q}] \equiv -\mathrm{Tr}\left[\frac{\pi\nu\widetilde{\hbar}}{4\tau}\hat{Q}^2 + \ln\left(\hat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{Q}\right)\right], \tag{30}$$

where $\hat{G}_0^{-1}$ is the inverse Green's function operator that does not include the disordered part of the dielectric constant and all $\hat{Q}$-independent factors are included in the normalization constant $\widetilde{\mathcal{N}}_Q$.

In the limit of large scattering time, the main contribution to $\langle Z\rangle$ comes from the neighborhood of a saddle point. The saddle-point equation

$$\hat{Q}(\mathbf{r})\hat{\gamma} = -\frac{1}{\pi\nu}\left(\hat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{Q}\right)^{-1}_{\mathbf{rr}} \tag{31}$$

follows from the condition of stationary variation of $S[\hat{Q}]$ with respect to $\hat{\gamma}\,\hat{Q}$. In the $(\mathbf{k},\omega)$ representation,

$$\left(G_0^{-1}\right)_\omega^{\mathrm{R,A}}(\mathbf{k}) = \mathcal{E}(\omega) - k^2 \pm i\epsilon''(\omega)\,\omega^2 \tag{32}$$

is diagonal. We look for the solutions $Q_\omega^{\mathrm{R,A}}$ in the (cl,cl) and (q,q) blocks of $\hat{Q}$, respectively, which are uniform in $\mathbf{r}$ and diagonal in ω. Equation (31) yields for these blocks

$$Q_\omega^{\mathrm{R,A}} = -\frac{1}{\pi\nu}\sum_{\mathbf{k}} \frac{1}{\mathcal{E}(\omega) - k^2 \pm i\epsilon''(\omega)\,\omega^2 + \frac{\widetilde{\hbar}}{2\tau}Q_\omega^{\mathrm{R,A}}}. \tag{33}$$

The sum over the modes can be converted into an integral over $\nu\,d\mathcal{E}$, where $\mathcal{E} = k^2$. In the limit $\omega\tau \gg 1$ and $\epsilon'' \ll \epsilon'$, the lower integration limit can be extended to $-\infty$. Then $(Q_0^{\mathrm{R,A}})_\omega = \pm i$ is the solution. The full matrix can be written in the form

$$\hat{Q}_0 = i\hat{\Lambda}, \quad \hat{\Lambda} = \begin{pmatrix} 1^{\mathrm{R}} & 2F \\ 0 & -1^{\mathrm{A}} \end{pmatrix}, \tag{34}$$

which includes the regularization in $1_\omega^{\mathrm{R,A}} = e^{\pm i\omega 0^+}$ and the Keldysh block. The regularization leads to an important property $\mathrm{Tr}\,\hat{Q}_0^2 = 0$.

We note that the saddle point $\hat{Q}_0$ lies outside of the manifold of Hermitian matrices $\hat{Q}$. The diagonal part of $\hat{Q}_0$ is anti-Hermitian; this property can be traced back to the negative sign in the exponent in Eq. (25). The $\hat{Q}$ manifold can be continuously deformed to pass through the saddle point by making the transformation $\hat{Q} \mapsto e^{i\phi}\hat{Q}$ in the neighborhood of $\hat{Q} = \hat{\Lambda}$. As ϕ changes from 0 to $\pi/2$, the logarithm argument in Eq. (30) has no zero eigenvalues if $\epsilon'' > 0$. Thus, no singularities are crossed by $\exp(i\,S[\hat{Q}])$ during the deformation.

B. Effective action

The main contribution to $\langle Z\rangle$ arises from the fluctuations about the saddle point that satisfy the condition $\mathrm{Tr}\,\hat{Q}^2 = 0$. Such fluctuations produce weak variations of the action $S[\hat{Q}]$ (30). The matrices $\hat{Q}$ having the above property can be represented in the general form

$$\hat{Q}(\mathbf{r}) = i\,\hat{R}^{-1}(\mathbf{r})\,\hat{\Lambda}\,\hat{R}(\mathbf{r}), \tag{35}$$

where $\hat{R}$ is diagonal in the $\mathbf{r}$ representation.

In what follows we present the results of the calculation and refer the reader to the appendix for details. After substituting the parametrization (35) in Eq. (30) and omitting the $\hat{Q}$-independent contribution, we arrive at

$$\begin{aligned} i\,S[\hat{Q}] &= -\mathrm{Tr}\,\ln\left(\hat{1} + \hat{\mathcal{G}}\hat{\gamma}\hat{R}\left[\hat{\gamma}\hat{G}_0^{-1}, \hat{R}^{-1}\right]\right) \\ &\approx -\mathrm{Tr}\left(\hat{\mathcal{G}}\hat{\gamma}\hat{R}\left[\hat{\gamma}\hat{G}_0^{-1}, \hat{R}^{-1}\right]\right) + \tfrac{1}{2}\mathrm{Tr}\left(\hat{\mathcal{G}}\hat{\gamma}\hat{R}\left[\hat{\gamma}\hat{G}_0^{-1}, \hat{R}^{-1}\right]\right)^2, \end{aligned} \tag{36}$$

where

$$\hat{\mathcal{G}} = \left(\hat{G}_0^{-1} + i\frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{\Lambda}\right)^{-1} \tag{37}$$

is the disorder-dependent Green's function operator [see Sec. IV B]. The action is expanded in the fluctuations about the saddle point, which are described by the commutator $[\hat{\gamma}\hat{G}_0^{-1}, \hat{R}^{-1}]$; at the saddle point $\hat{R} = \hat{1}$ the commutator vanishes. The disorder-free inverse Green's function consists of the conservative and nonconservative parts:

$$\hat{\gamma}\left(\hat{G}_0^{-1}\right)_\omega(\mathbf{k}) = [\mathcal{E}(\omega) - k^2]\hat{1} + i\epsilon''(\omega)\,\omega^2\hat{\Lambda}. \tag{38}$$

The latter results in a nontrivial contribution to the commutator due to the Keldysh structure of $\hat{\Lambda}$.

There are three leading-order contributions to $S[\hat{Q}]$. Using the $\mathcal{E}(\omega)$ part of $\hat{\gamma}\hat{G}_0^{-1}$ in the linear term in Eq. (36) we arrive at

$$i\,S_1[\hat{Q}] \simeq i\pi\nu\widetilde{\hbar}\,\mathrm{Tr}\,(\partial_t\hat{Q}), \quad (\partial_t\hat{Q})_{tt} \equiv (\partial_t\hat{Q}_{tt'})_{t'=t}. \tag{39}$$

The contribution of the k^2 part of $\hat{\gamma}\hat{G}_0^{-1}$ to the linear term of Eq. (36) is neglected compared to its contribution to the second-order term, which gives

$$i\,S_2[\hat{Q}] \simeq -\frac{\pi}{4}\nu\widetilde{\hbar}\bar{D}\,\mathrm{Tr}\,(\partial_{\mathbf{r}}\hat{Q})^2. \tag{40}$$

To derive this result, we used the property [35]

$$\frac{\widetilde{\hbar}}{2\pi\nu}\sum_{\mathbf{k}}\mathcal{G}^{\mathrm{R}}_{\omega_0}(\mathbf{k})\,\mathcal{G}^{\mathrm{A}}_{\omega_0}(\mathbf{k}) \simeq \left(\frac{1}{\tau} + \frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}}\right)^{-1} \equiv \bar{\tau}, \tag{41}$$

yielding the effective scattering time $\bar{\tau}$, and defined the effective diffusion coefficient in two dimensions,

$$\bar{D} = \frac{1}{2}v^2\bar{\tau} = \frac{2\epsilon'\omega_0^2}{\widetilde{\hbar}^2}\bar{\tau}, \tag{42}$$

where v is the group velocity of light in the medium. In Sec. IV we show that $\bar{\tau}$ and $\bar{D}$ are the relevant parameters to describe the diffusive light propagation [see Eq. (75)]. Finally, the nonconservative part of $\hat{\gamma}\hat{G}_0^{-1}$ yields, in the linear order in the commutator,

$$i\,S_3[\hat{Q}] = \pi\nu\epsilon''\omega_0^2\,\mathrm{Tr}\,(i\hat{\Lambda}\hat{Q} + \hat{\Lambda}^2). \tag{43}$$

The contributions $S_{1,2,3}[\hat{Q}]$ sum up to yield the NLSM effective action

$$\begin{aligned} &i\,S[\hat{Q}] \\ &\quad = -\pi\nu_0\mathrm{Tr}\left[-i\,\partial_t\hat{Q} + \frac{\bar{D}}{4}(\partial_{\mathbf{r}}\hat{Q})^2 - \frac{\epsilon''\omega_0^2}{\widetilde{\hbar}}(i\hat{\Lambda}\hat{Q} + \hat{\Lambda}^2)\right]. \end{aligned} \tag{44}$$

The action vanishes at the saddle point, $S[i\,\hat{\Lambda}] = 0$. The key assumption behind the NLSM is the smallness of the action for fluctuations of $\hat{Q}$ restricted to the manifold $\mathrm{Tr}\,\hat{Q}^2 = 0$, compared to the action for arbitrary fluctuations about the saddle point. The terms $S_{1,2}[\hat{Q}]$, which also appear in the NLSM for disordered fermionic systems [33], depend only on the derivatives of $\hat{Q}$. Therefore, the dominant contribution to the partition function comes from the fluctuations $\hat{Q}_{tt'}(\mathbf{r})$ [or $\hat{R}_{tt'}(\mathbf{r})$] that are slowly varying functions of $\mathbf{r}$ and $(t + t')/2$. These "massless modes" are associated with the diffusive light propagation. The assumption of slow variation justifies neglecting of higher-order terms in the expansion (36). The contribution $S_3[\hat{Q}]$ results from the nonconservative nature of the medium. It is, in general, comparable to the to the "massive" $\mathrm{Tr}\,\hat{Q}^2$ term, unless the rate of absorption or gain is smaller than the scattering rate:

$$\frac{|\epsilon''|\omega_0^2}{\widetilde{\hbar}} \ll \frac{1}{\tau}. \tag{45}$$

This condition specifies the regime when the light propagation is diffusive. If this requirement is not fulfilled, the massive fluctuations beyond the NLSM have to be taken into account.

IV. LIGHT DIFFUSION

In this section we calculate the disorder-averaged Green's function correlator. In particular, we consider the contribution that arises from the fluctuations of the field $\hat{Q}$ in the neighborhood of the saddle point. The correlator possesses a diffusion-pole structure modified by a finite absorption or gain rate.

A. Fluctuations about the saddle point

We consider the parametrization

$$\hat{Q} = i\hat{U}e^{-\hat{W}/2}\hat{\sigma}_z e^{\hat{W}/2}\hat{U}^{-1}, \quad \hat{U} = \hat{U}^{-1} = \begin{pmatrix} 1 & F \\ 0 & -1 \end{pmatrix}, \tag{46}$$

where $\hat{\sigma}_z$ is the Pauli matrix. Because $\hat{\Lambda} = \hat{U}\hat{\sigma}_z\hat{U}^{-1}$ (if the regularization of unit operators is neglected), this parametrization is equivalent to Eq. (35) with $\hat{R} = \hat{U}\exp(\hat{W}/2)\,\hat{U}^{-1}$. It can be verified by explicit calculation that the diagonal blocks of $\hat{W}$ do not contribute to $S[\hat{Q}]$ and the Green's function correlator, at least, up to the second order in $\hat{W}$. We, therefore, represent this field in the form

$$\hat{W} = \begin{pmatrix} 0 & w \\ w^\dagger & 0 \end{pmatrix}. \tag{47}$$

The specific choice of $\hat{W}$ as a Hermitian matrix is justified by the requirement of convergence of the functional integral for the partition function (see below). The operator w is diagonal in the $\mathbf{r}$ representation.

By expanding the parametrization (46) in the powers of $\hat{W}$ we find the first- and second-order deviations from the saddle point,

$$\delta\hat{Q}^{(1)} = i\begin{pmatrix} -Fw^\dagger & -w - Fw^\dagger F \\ w^\dagger & w^\dagger F \end{pmatrix}, \tag{48}$$

$$\delta\hat{Q}^{(2)} = \frac{i}{2}\begin{pmatrix} ww^\dagger & ww^\dagger F + Fww^\dagger \\ 0 & -ww^\dagger \end{pmatrix}. \tag{49}$$

We note that only the latter matrix has the causality structure; however, the fluctuations of $\hat{Q}$ are not required to obey causality. By using $\delta\hat{Q}^{(1,2)}$ in Eq. (44) we can calculate fluctuations of the effective action.

The first-order variation of $S[\hat{Q}]$ depends on the derivatives [36] of the distribution function F generated by the first two terms in Eq. (44); the third term yields an identically vanishing first-order contribution. The saddle-point equation (33) determines the retarded and advanced sectors of $\hat{Q}$, but not the function F. By setting to zero the variation of the effective action near the saddle point, we obtain the Usadel equation

$$\left(-\partial_{\bar{t}} + \bar{D}\partial_{\mathbf{r}}^2\right) F_{\omega_0}(\mathbf{r},\bar{t}) = 0 \tag{50}$$

for $F_{tt'}(\mathbf{r})$ in the mixed representation of the slow time variable $\bar{t} = (t+t')/2$ and the large frequency ω, conjugate to $t - t'$.

The second-order variation is

$$\begin{aligned} &i\delta S^{(2)}[w,w^\dagger] \\ &\quad = -\pi\nu_0 \mathrm{Tr}\Big[-i\partial_t\,\delta\hat{Q}^{(2)} + \frac{\bar{D}}{4}(\partial_{\mathbf{r}}\,\delta\hat{Q}^{(1)})^2 \\ &\qquad + i\frac{\bar{D}}{2}(\partial_{\mathbf{r}}\hat{\Lambda})(\partial_{\mathbf{r}}\,\delta\hat{Q}^{(2)}) - i\frac{\epsilon''\omega_0^2}{\widetilde{\hbar}}\hat{\Lambda}\delta\hat{Q}^{(2)}\Big] \\ &\quad = -\frac{\pi\nu_0}{2}\sum_{\omega\omega'\mathbf{k}}|w_{\omega\omega'}(\mathbf{k})|^2\left[-i(\omega-\omega') + \bar{D}k^2 + \frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}}\right], \end{aligned} \tag{51}$$

where $w(\mathbf{k})$ is the Fourier transform of $w(\mathbf{r})$. Of the two terms with spatial gradients, the second term has a zero trace. The first term yields the $\bar{D}k^2$ contribution to $\delta S^{(2)}$, as well as the additional correction

$$i\delta S_F^{(2)}[w^\dagger] = -\frac{\pi\nu_0}{2}\bar{D}\,\mathrm{tr}[w^\dagger(\partial_{\mathbf{r}}F)]^2, \tag{52}$$

where "tr" denotes the trace of operators that do not have the Keldysh matrix structure. This correction vanishes when $F(\mathbf{r})$ is uniform, which we will assume. With the help of Eq. (51), the disorder-averaged partition function can be approximated by the functional integral

$$\langle Z\rangle \approx \mathcal{N}_w \int D[w,w^\dagger]\, e^{i\delta S^{(2)}[w,w^\dagger]}, \tag{53}$$

where $\mathcal{N}_w$ is a normalization constant. For a medium with gain, the divergence of the integral for the modes with

$$k < k_{\min} \equiv \sqrt{-\frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}\bar{D}}} \tag{54}$$

indicates that the long-scale fluctuations become unstable due to onset of lasing (see Sec. IV C). Thus, in the long-wavelength limit the linear-gain theory breaks down and the nonlinear effects have to be taken into account [15,16].

B. Disorder-averaged correlator

Green's functions and their combinations can be expressed in terms of derivatives of the partition function with respect to the source fields:

$$G^{jk}(1,2) = -\frac{i}{16\pi}\left.\frac{\delta^2 Z[\hat{J},\hat{J}^\dagger]}{\delta[J^j(1)]^*\,\delta J^k(2)}\right|_{\hat{J}=\hat{J}^\dagger=0}, \tag{55}$$

$$\begin{aligned} &G^{jk}(1,2)\,G^{lm}(3,4) + G^{jm}(1,4)\,G^{lk}(3,2) \\ &\quad = -\frac{1}{(16\pi)^2}\left.\frac{\delta^4 Z[\hat{J},\hat{J}^\dagger]}{\delta[J^j(1)]^*\,\delta J^k(2)\,\delta[J^l(3)]^*\,\delta J^m(4)}\right|_{\hat{J}=\hat{J}^\dagger=0}, \end{aligned} \tag{56}$$

$$Z[\hat{J},\hat{J}^\dagger] = \mathcal{N}\int D[\hat{A},\hat{A}^\dagger]\, e^{iS[\hat{A},\hat{A}^\dagger]+\hat{J}^\dagger\hat{A}+\hat{A}^\dagger\hat{J}}, \tag{57}$$

where $j,k,l,m = \mathrm{cl,q}$ and $1,2,\ldots$ are full sets of coordinates in some representation, for example, $1 = (\mathbf{k}_1,\omega_1)$, etc. By inverting the matrix (11), we identify the sectors of the Green's function as $G^{\mathrm{cl,q}} = G^{\mathrm{R}}$, $G^{\mathrm{q,cl}} = G^{\mathrm{A}}$, $G^{\mathrm{cl,cl}} = G^{\mathrm{K}} \neq [(G^{-1})^{\mathrm{K}}]^{-1}$, and $G^{\mathrm{q,q}} = 0$.

The disorder-averaged Green's functions and correlators are obtained by using the above expressions with the disorder-averaged partition function [37]

$$\langle Z[\hat{J},\hat{J}^\dagger]\rangle = \widetilde{\mathcal{N}}_Q\int D[\hat{Q}]\,\exp(iS[\hat{Q}] + 16\pi i\,\hat{J}^\dagger\hat{\mathcal{G}}_{\hat{Q}}\hat{J}), \tag{58}$$

$$\hat{\mathcal{G}}_{\hat{Q}} \equiv \left(\hat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{Q}\right)^{-1}. \tag{59}$$

We find, in particular,

$$\langle G^{\mathrm{R,A,K}}(1,2)\rangle = \langle \mathcal{G}^{\mathrm{R,A,K}}_{\hat{Q}}(1,2)\rangle_{\hat{Q}}, \tag{60}$$

$$\langle G^{\mathrm{R}}(1,2)\, G^{\mathrm{A}}(3,4)\rangle = \langle \mathcal{G}^{\mathrm{R}}_{\hat{Q}}(1,2)\, \mathcal{G}^{\mathrm{A}}_{\hat{Q}}(3,4) + \mathcal{G}^{\mathrm{K}}_{\hat{Q}}(1,4)\, \mathcal{G}^{\mathrm{Q}}_{\hat{Q}}(3,2)\rangle_{\hat{Q}}, \tag{61}$$

where the average $\langle\cdots\rangle_{\hat{Q}}$ over $\hat{Q}$ is performed with the exponential weight $\exp(iS[\hat{Q}])$. Equation (60) shows that $\hat{\mathcal{G}} = \hat{\mathcal{G}}_{i\hat{\Lambda}}$ [Eq. (37)] is the disorder-averaged Green's function in the lowest-order saddle-point approximation. The component $\mathcal{G}^{\mathrm{Q}}_{\hat{Q}} \equiv \mathcal{G}^{\mathrm{q,q}}_{\hat{Q}}$ in Eq. (61) is, in general, nonzero when $\hat{Q}$ does not have the causality structure. This observation is essential for the following calculation.

We calculate the Green's function correlator

$$\mathcal{R}(1,2,3,4) \equiv \langle G^{\mathrm{R}}(1,2)\, G^{\mathrm{A}}(3,4)\rangle - \langle G^{\mathrm{R}}(1,2)\rangle\langle G^{\mathrm{A}}(3,4)\rangle \tag{62}$$

by expansion about the saddle point. The lowest-order correction to the Green's function (59) is

$$\hat{\mathcal{G}}_{\hat{Q}} - \hat{\mathcal{G}} \simeq -\frac{\widetilde{\hbar}}{2\tau}\hat{\mathcal{G}}\hat{\gamma}\delta\hat{Q}^{(1)}\hat{\mathcal{G}} = i\frac{\widetilde{\hbar}}{2\tau}\begin{pmatrix} \mathcal{G}^{\mathrm{R}} w \mathcal{G}^{\mathrm{A}} + F\mathcal{G}^{\mathrm{A}} w^{\dagger}\mathcal{G}^{\mathrm{R}} F & F\mathcal{G}^{\mathrm{A}} w^{\dagger}\mathcal{G}^{\mathrm{R}} \\ -\mathcal{G}^{\mathrm{A}} w^{\dagger}\mathcal{G}^{\mathrm{R}} F & -\mathcal{G}^{\mathrm{A}} w^{\dagger}\mathcal{G}^{\mathrm{R}} \end{pmatrix}. \tag{63}$$

The Gaussian averages with the action (51) are as follows:

$$\langle w\rangle_w = \langle w^{\dagger}\rangle_w = 0, \quad \langle w^{\dagger}(1,2)\, w^{\dagger}(3,4)\rangle_w = 0, \tag{64}$$

$$\langle w(1,2)\, w^{\dagger}(3,4)\rangle_w = \frac{2}{\pi\nu_0}\frac{\delta_{\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_4-\mathbf{k}_3}\delta_{\omega_1,\omega_4}\delta_{\omega_2,\omega_3}}{\bar{D}(\mathbf{k}_1-\mathbf{k}_2)^2 - i(\omega_1-\omega_2) + \frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}}}. \tag{65}$$

Therefore, the leading contribution to the correlator comes from the $\langle G^{\mathrm{K}}_{\hat{Q}}(1,4)\, \mathcal{G}^{\mathrm{Q}}_{\hat{Q}}(3,2)\rangle_{\hat{Q}}$ term in Eq. (61), which is given by the product of diagonal blocks in Eq. (63). We find

$$\mathcal{R}(1,2,3,4) = \frac{\widetilde{\hbar}}{2\pi\nu\tau^2}\mathcal{G}^{\mathrm{R}}(1)\,\mathcal{G}^{\mathrm{R}}(2)\,\mathcal{G}^{\mathrm{A}}(3)\,\mathcal{G}^{\mathrm{A}}(4) \times \frac{\delta_{\mathbf{k}_1-\mathbf{k}_4,\mathbf{k}_2-\mathbf{k}_3}\delta_{\omega_1,\omega_2}\delta_{\omega_4,\omega_3}}{\bar{D}(\mathbf{k}_1-\mathbf{k}_4)^2 - i(\omega_1-\omega_4) + \frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}}}. \tag{66}$$

The correlator has a diffusion pole with the diffusion coefficient $\bar{D}$. The pole is modified by the ϵ'' term that arises from the corresponding contribution in the effective action (44). This term defines the absorption rate

$$\frac{1}{\tau_{\mathrm{a}}} \equiv \frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}}, \tag{67}$$

negative for gain.

C. Discussion

The pole structure of the correlator implies that the light intensity I in the medium satisfies the diffusion equation with a nonconservative term:

$$\left(\partial_t - \bar{D}\nabla^2 + \tau_{\mathrm{a}}^{-1}\right) I = 0, \tag{68}$$

$$\bar{D} = \frac{1}{2}v^2\bar{\tau} = \frac{1}{2}v^2\left(\frac{1}{\tau} + \frac{1}{\tau_{\mathrm{a}}}\right)^{-1}. \tag{69}$$

We compare this equation with

$$\left[\frac{\tau}{1+2\tau/\tau_{\mathrm{a}}}\partial_t^2 + \partial_t - D'\nabla^2 + \tau_{\mathrm{a}}^{-1}\frac{1+\tau/\tau_{\mathrm{a}}}{1+2\tau/\tau_{\mathrm{a}}}\right] I = 0, \tag{70}$$

$$D' \equiv \frac{1}{2}v^2\left(\frac{1}{\tau} + \frac{2}{\tau_{\mathrm{a}}}\right)^{-1}, \tag{71}$$

which follows from the photon transport equation (see Eq. (15) of Ref. [17]). According to Ref. [17], the light propagation is diffusive if the second derivative with respect to time in Eq. (70) can be neglected. This is the case when

$$\tau \ll \Delta t, \tag{72}$$

where Δt is the characteristic time scale of intensity variation. The reaction of the medium on a fluctuation of intensity will be determined by the shortest time scale, so that $\Delta t \lesssim |\tau_{\mathrm{a}}|$ can be assumed. Therefore, when neglecting the corrections of the order of $\tau/\Delta t$ in Eq. (70), we also have to neglect the contributions of the order of τ/τ_{a}. In particular, it is consistent with the diffusion approximation to set

$$D' \simeq D \equiv \tfrac{1}{2}v^2\tau. \tag{73}$$

The independence of absorption for the diffusion coefficient was also supported by the numerical evidence in Ref. [17]. It is worth commenting on the claim [18,19] that the diffusion coefficient in the medium with absorption must be equal to D even for $\tau/\tau_{\mathrm{a}} \sim 1$. A closer look at the derivation of the diffusion coefficient from the transport equation in Ref. [18] reveals that the time-derivative terms neglected in Eqs. (A9) and (A11) of that article would yield the diffusion coefficient

$$D'' = \frac{1}{2}v^2\tau\left(1 - 2\frac{\tau}{\tau_{\mathrm{a}}}\right) \simeq D' \tag{74}$$

were they taken into account. Thus, the (approximate) independence of the diffusion coefficient of absorption is a consequence of the self-consistent application of the diffusion-approximation conditions (45) and (72).

The NLSM effective action (44) is derived under the condition (72) as well. This condition guarantees the slow variation of $\hat{Q}$ and makes it possible to neglect the contribution of $\mathcal{E}(\omega)$ part of $\hat{\gamma}\hat{G}_0^{-1}$ [Eq. (38)] to the second-order term in Eq. (36). This contribution would result in a second-time-derivative term in the effective action. Again, the diffusion approximation requires that we set

$$\bar{D} \simeq D, \quad \bar{\tau} \simeq \tau \tag{75}$$

in the NLSM expressions. Thus, the NLSM and the theory of transport equation agree in the diffusive regime.

In the medium with gain, the diffusive relaxation competes with the amplification. Because the long-scale intensity fluctuations disperse slower, they become unstable, and the random lasing sets in. The cutoff wave number $k_{\min}$ (54) determines the critical sample size

$$l = \sqrt{D\,|\tau_{\mathrm{a}}|}, \tag{76}$$

above which the system is lasing and the linear-gain theory does not apply. Alternatively, the above expression yields the lasing-threshold value of $|\tau_{\mathrm{a}}|$ if l is given.

V. CONCLUSIONS

We obtained the functional-integral form of the partition function for an optical medium with linear absorption or gain. The Keldysh technique is particularly suitable for description of nonconservative systems because it provides a natural representation for the action. The *disorder-averaged* partition function is expressed as a functional integral over the auxiliary matrix field $\hat{Q}$. Within the framework of nonlinear σ model, we considered the fluctuations about the saddle point that fulfill the condition $\mathrm{Tr}\,\hat{Q}^2 = 0$. We found that the effective action $S[\hat{Q}]$ for these fluctuations contains an extra term due to absorption or gain.

With the help of the NLSM partition function, we computed the disorder-averaged Green's function correlator. The leading contribution from the vicinity of the saddle point has the diffusion-pole structure modified by a finite absorption or gain rate. The diffusion coefficient is found to be approximately independent of the absorption or gain in agreement with the theory of photon transport equation. In the medium with gain, the linear theory is not applicable in the long-wavelength limit. If the sample size exceeds a certain critical length, the random lasing sets in.

ACKNOWLEDGMENTS

This work was supported by a grant from the Kyung Hee University in 2011 (Grant No. KHU-20110683). We thank Johann Kroha for helpful discussions.

APPENDIX: DERIVATION OF THE EFFECTIVE ACTION

1. Derivation of Eq. (36)

After substitution of the parametrization (35) in Eq. (30) we obtain

$$\begin{aligned} i\,S[\hat{Q}] &= -\mathrm{Tr}\,\ln\left[\hat{R}\hat{\gamma}\hat{R}^{-1}\left(\hat{R}\hat{\gamma}\hat{G}_0^{-1}\hat{R}^{-1} + i\frac{\widetilde{\hbar}}{2\tau}\hat{\Lambda}\right)\right] \\ &= -\mathrm{Tr}\,\ln\hat{\gamma} - \mathrm{Tr}\,\ln\left(\hat{\gamma}\hat{\mathcal{G}}^{-1} + \hat{R}\left[\hat{\gamma}\hat{G}_0^{-1}, \hat{R}^{-1}\right]\right). \end{aligned} \tag{A1}$$

By separating $\mathrm{Tr}\,\ln(\hat{\gamma}\hat{\mathcal{G}}^{-1})$ and dropping the $\hat{Q}$-independent terms we arrive at Eq. (36).

2. Derivation of Eq. (39)

The conservative part of $\hat{\gamma}\hat{G}_0^{-1}$, when substituted in the first trace in Eq. (36), yields

$$\begin{aligned} i\,S_1[\hat{Q}] &= -\mathrm{Tr}\sum_{\substack{\omega,\omega'\\ \mathbf{k},\mathbf{k}'}} \hat{\mathcal{G}}_\omega(\mathbf{k})\hat{\gamma}\,\hat{R}_{\omega\omega'}(\mathbf{k}-\mathbf{k}')\,\hat{R}^{-1}_{\omega'\omega}(\mathbf{k}'-\mathbf{k}) \\ &\quad\times\left[\mathcal{E}(\omega') - \mathcal{E}(\omega) - k'^2 + k^2\right] \\ &\simeq -\mathrm{Tr}\sum_{\substack{\omega,\Delta\omega\\ \mathbf{k},\Delta\mathbf{k}}} \hat{\mathcal{G}}_\omega(\mathbf{k})\hat{\gamma}\,\hat{R}_{\omega,\omega+\Delta\omega}(-\Delta\mathbf{k})\,\hat{R}^{-1}_{\omega+\Delta\omega,\omega}(\Delta\mathbf{k}) \\ &\quad\times\left[\widetilde{\hbar}\Delta\omega - (2\mathbf{k}+\Delta\mathbf{k})\cdot\Delta\mathbf{k}\right], \end{aligned} \tag{A2}$$

where $\Delta\omega = \omega' - \omega$ and $\Delta\mathbf{k} = \mathbf{k}' - \mathbf{k}$. We note that $\hat{R}$ is peaked at small wave vectors in the k representation. The sum

$$\sum_{\mathbf{k}} \hat{\mathcal{G}}_\omega(\mathbf{k})\hat{\gamma} = -i\pi\nu\hat{\Lambda} \tag{A3}$$

follows from the saddle-point condition; furthermore, $\sum_{\mathbf{k}} \hat{\mathcal{G}}_\omega(\mathbf{k})\,\mathbf{k} = 0$ due to the symmetry. After calculating the $\Delta\mathbf{k}$ sum we arrive at

$$\begin{aligned} i\,S_1[\hat{Q}] &= i\pi\nu\,\mathrm{Tr}\sum_{\omega,\Delta\omega}\int d\mathbf{r}\,\hat{\Lambda}\,\hat{R}_{\omega,\omega+\Delta\omega}(\mathbf{r}) \\ &\quad\times\left[\widetilde{\hbar}\Delta\omega + \partial_{\mathbf{r}}^2\right]\hat{R}^{-1}_{\omega+\Delta\omega,\omega}(\mathbf{r}) \\ &= \pi\nu\,\mathrm{Tr}\int d\mathbf{r}\,dt\,dt'\,i\,\hat{\Lambda}\,\hat{R}_{t't}(\mathbf{r})\left[i\widetilde{\hbar}\partial_t + \partial_{\mathbf{r}}^2\right]\hat{R}^{-1}_{tt'}(\mathbf{r}). \end{aligned} \tag{A4}$$

Applying the representation (35) we obtain Eq. (39) from the ∂_t part. The $\partial_{\mathbf{r}}^2$ part is neglected compared to $i\,S_2[\hat{Q}]$; the latter contribution is multiplied by $\widetilde{\hbar}\bar{D} \sim \omega_0\bar{\tau} \gg 1$.

3. Derivation of Eq. (40)

We substitute the k^2 part of $\hat{\gamma}\hat{G}_0^{-1}$ in the second trace in Eq. (36) to get

$$\begin{aligned} i\,S_2[\hat{Q}] &= \frac{1}{2}\mathrm{Tr}\sum_{\mathbf{k}_1\ldots\mathbf{k}_4}\hat{\mathcal{G}}(\mathbf{k}_1)\hat{\gamma}\,\hat{R}(\mathbf{k}_1-\mathbf{k}_2)\,\hat{R}^{-1}(\mathbf{k}_2-\mathbf{k}_3) \\ &\quad\times\hat{\mathcal{G}}(\mathbf{k}_3)\hat{\gamma}\,\hat{R}(\mathbf{k}_3-\mathbf{k}_4)\hat{R}^{-1}(\mathbf{k}_4-\mathbf{k}_1)\left(k_2^2-k_3^2\right)\left(k_4^2-k_1^2\right) \\ &\simeq 2\mathrm{Tr}\sum_{\substack{\bar{\mathbf{k}}\,\Delta\bar{\mathbf{k}}\\ \Delta\mathbf{k}\,\Delta\mathbf{k}'}}\hat{\mathcal{G}}(\bar{\mathbf{k}})\hat{\gamma}\,\hat{R}\left(\Delta\bar{\mathbf{k}} - \frac{\Delta\mathbf{k}+\Delta\mathbf{k}'}{2}\right)\hat{R}^{-1}(\Delta\mathbf{k}') \\ &\quad\times\hat{\mathcal{G}}(\bar{\mathbf{k}})\hat{\gamma}\,\hat{R}\left(-\Delta\bar{\mathbf{k}} - \frac{\Delta\mathbf{k}+\Delta\mathbf{k}'}{2}\right)\hat{R}^{-1}(\Delta\mathbf{k}) \\ &\quad\times(\bar{\mathbf{k}}\cdot\Delta\mathbf{k})(\bar{\mathbf{k}}\cdot\Delta\mathbf{k}'), \end{aligned} \tag{A5}$$

where $\bar{\mathbf{k}} = \sum_{i=1}^4 \mathbf{k}_i/4$ and we neglected the contributions of higher order in $\Delta\bar{\mathbf{k}} = (\mathbf{k}_1 + \mathbf{k}_4 - \mathbf{k}_2 - \mathbf{k}_3)/2$, $\Delta\mathbf{k} = \mathbf{k}_4 - \mathbf{k}_1$, and $\Delta\mathbf{k}' = \mathbf{k}_2 - \mathbf{k}_3$. We use the representation

$$\hat{\mathcal{G}}\hat{\gamma} = \tfrac{1}{2}\mathcal{G}^{\mathrm{R}}(\hat{1}+\hat{\Lambda}) + \tfrac{1}{2}(\hat{1}-\hat{\Lambda})\,\mathcal{G}^{\mathrm{A}} \tag{A6}$$

and the well-known relations [see Eqs. (42) and (41)]

$$\sum_{\mathbf{k}}\mathcal{G}^{\mathrm{R}}_\omega(\mathbf{k})\,\mathcal{G}^{\mathrm{A}}_{\omega'}(\mathbf{k})\,k_\alpha k_\beta \simeq \tfrac{1}{2}\pi\nu\widetilde{\hbar}\bar{D}\delta_{\alpha\beta}, \tag{A7}$$

$$\sum_{\mathbf{k}}\mathcal{G}^{\mathrm{R(A)}}_\omega(\mathbf{k})\,\mathcal{G}^{\mathrm{R(A)}}_{\omega'}(\mathbf{k})\,k_\alpha k_\beta \simeq 0, \tag{A8}$$

to find

$$\begin{aligned} i\,S_2[\hat{Q}] &= -\tfrac{1}{2}\pi\nu\widetilde{\hbar}\bar{D}\,\mathrm{Tr}\left[(\hat{1}+\hat{\Lambda})\hat{R}\,(\partial_{\mathbf{r}}\hat{R}^{-1})\right. \\ &\quad\left.\cdot(\hat{1}-\hat{\Lambda})\hat{R}\,(\partial_{\mathbf{r}}\hat{R}^{-1})\right] = \tfrac{1}{4}\pi\nu\widetilde{\hbar}\bar{D}\,\mathrm{Tr}\left[\partial_{\mathbf{r}}(\hat{R}^{-1}\hat{\Lambda}\hat{R})\right]^2, \end{aligned} \tag{A9}$$

from which Eq. (40) follows.

4. Derivation of Eq. (43)

The nonconservative part of $\hat{\gamma}\hat{G}_0^{-1}$, being substituted in the first trace in Eq. (36) yields

$$\begin{aligned} i\,S_3[\hat{Q}] &\simeq -i\epsilon''\omega_0^2\,\mathrm{Tr}\sum_{\mathbf{k}\mathbf{k}'}\hat{\mathcal{G}}(\mathbf{k})\hat{\gamma}\,\hat{R}(\mathbf{k}-\mathbf{k}') \\ &\quad\times\left[\hat{\Lambda}\,\hat{R}^{-1}(\mathbf{k}'-\mathbf{k}) - \hat{R}^{-1}(\mathbf{k}'-\mathbf{k})\,\hat{\Lambda}\right]. \end{aligned} \tag{A10}$$

We change the variable $\mathbf{k}' = \mathbf{k} + \Delta\mathbf{k}$ and apply Eq. (A3). After cyclically moving the operators under the trace we obtain Eq. (43).

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