Solvability of the two-photon Rabi Hamiltonian

I. Travěnec
Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia
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The spectrum of the two-photon Rabi Hamiltonian is found, proceeding in full analogy with the solution of the standard (one-photon) Rabi Hamiltonian, published by D. Braak [Phys. Rev. Lett. **107**, 100401 (2011)]. The Hamiltonian is rewritten as a set of two differential equations. Symmetries that get hidden after further treatment are found. One can plainly see how the Hilbert space splits into four disjunct subspaces, categorized by four values of the symmetry parameter $c = \pm 1, \pm i$. There were only two values, ± 1, for the standard Rabi model. The values of c are in fact the eigenvalues of the respective parity operators. Four analytic functions are introduced by a recurrence scheme for the coefficients of their series expansion. All their roots yield the complete spectrum of the Hamiltonian. Eigenstates in Bargmann space can also be found.

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I. INTRODUCTION

There were many trials to solve the model called (in quantum optics) the Rabi Hamiltonian exactly, until Braak [1] recently presented a new extraordinarily effective approach for this task. The same model is known under several pseudonyms, e.g., the single-mode spin-boson system and the Jaynes-Cummings model without rotating-wave approximation; a brief survey can be found in [2]. Let us introduce a more general form of Rabi Hamiltonians [3], describing the interaction between a bosonic mode with energy ω and a two-level atom with level spacing ω_0:

$$\hat{H}^{(m)} = \frac{\omega_0}{2}\sigma^z + \omega b^\dagger b + g(\sigma^+ + \sigma^-)[(b^\dagger)^m + b^m], \quad (1)$$

where $m = 1,2,\ldots$, g is the interaction constant, σ^z and $\sigma^\pm$ are the Pauli matrices, and $b^\dagger$ and b are boson creation and annihilation operators, respectively. We set $\hbar = 1$. For the most studied $m = 1$ case Braak [1] recently presented an algebraic solution without large matrix diagonalization; we are going to solve the $m = 2$ model in full analogy.

Before Braak's general solution, some special points, also called the Juddian points, were known to be exactly solvable. This means that for some constraints on the Hamiltonian parameters, one can find both an eigenfunction and its eigenenergy. It does not give the complete spectrum, just one excited eigenstate. This was shown first for the standard $m = 1$ Rabi Hamiltonian [4]. Later the same picture, but of course with different constraints and eigenvalues, was shown for $m = 2$, i.e., the two-photon Rabi Hamiltonian [5]. These special points manifest themselves as cross sections of general solutions, and they will serve as a check of our results.

It is also known that the parameters of the two-photon Rabi model are restricted by $|4g| < \omega$; otherwise, the eigenfunctions are not normalizable [5,6].

II. THE ALGEBRAIC SOLUTION

The first step of the solution is to go over to Bargmann space [7], introducing complex variable z, where the bosonic operators simplify to

$$b \to \frac{\partial}{\partial z}, \qquad b^\dagger \to z. \quad (2)$$

The eigenfunctions have to be analytic in the whole complex plane. Let us suppose that the solution of the stationary Schrödinger equation is a vector composed of two analytic functions $[\psi_1(z),\psi_2(z)]^T$. We can insert Eq. (2) into the Hamiltonian (1), case $m = 2$, and using the well-known 2×2 Pauli matrices, we obtain a coupled system of second-order differential equations:

$$\begin{aligned} 2g\psi_2'' + \omega z\psi_1' + 2gz^2\psi_2 + \left(\frac{\omega_0}{2} - E\right)\psi_1 = 0, \\ 2g\psi_1'' + \omega z\psi_2' + 2gz^2\psi_1 - \left(\frac{\omega_0}{2} + E\right)\psi_2 = 0, \end{aligned} \quad (3)$$

where the prime indicates the derivative with respect to z and E is the energy from the Hamiltonian's spectrum. For better symmetrization it is useful to go over to a linear combination of the considered functions, namely, $\phi_1(z) = \psi_1(z) + \psi_2(z)$ and $\phi_2(z) = \psi_1(z) - \psi_2(z)$. Thus we get the set

$$\begin{aligned} 2g\phi_1'' + \omega z\phi_1' + (2gz^2 - E)\phi_1 + \frac{\omega_0}{2}\phi_2 = 0, \\ -2g\phi_2'' + \omega z\phi_2' - (2gz^2 + E)\phi_2 + \frac{\omega_0}{2}\phi_1 = 0. \end{aligned} \quad (4)$$

The next step is to find important symmetries present in the set (4) before they become less clear after some transformation followed by a series expansion. Here we prefer a somewhat different way than Braak [1], but a very obvious one. We will perform two transformations of the variable z. The first one is $z \to -z$. One can easily see that it leaves the set (4) unchanged. Thus, if $\{\phi_1(z),\phi_2(z)\}$ are a solution, then $\{\phi_1(-z),\phi_2(-z)\}$ are also a solution. Hence, as the system is linear, one may work with solutions that are either symmetric or antisymmetric under inversion: $\phi_S(z) = \phi(z) + \phi(-z)$, $\phi_A(z) = \phi(z) - \phi(-z)$. It is then that symmetric (antisymmetric) $\phi_1(z)$ goes with symmetric (antisymmetric) $\phi_2(z)$. Plainly speaking, such solutions must both be either even,

$$\begin{aligned} \phi_1(-z) = \phi_1(z), \\ \phi_2(-z) = \phi_2(z), \end{aligned} \quad (5)$$

or, alternatively, odd,

$$\begin{aligned}\phi_1(-z) &= -\phi_1(z),\\ \phi_2(-z) &= -\phi_2(z).\end{aligned} \tag{6}$$

The second applied transformation is $y = iz$. Set (4) becomes

$$\begin{aligned}-2g\frac{d^2\phi_1}{dy^2} + \omega y\frac{d\phi_1}{dy} - (2gy^2 + E)\phi_1 + \frac{\omega_0}{2}\phi_2 = 0,\\ 2g\frac{d^2\phi_2}{dy^2} + \omega y\frac{d\phi_2}{dy} + (2gy^2 - E)\phi_2 + \frac{\omega_0}{2}\phi_1 = 0.\end{aligned} \tag{7}$$

Now functions ϕ_1 and ϕ_2 have evidently swapped their places, and thus up to some common multiplicative constant,

$$\begin{aligned}\phi_1(iz) &= c\,\phi_2(z),\\ \phi_2(iz) &= c\,\phi_1(z).\end{aligned} \tag{8}$$

Possible values of parameter c can be found by inserting iz instead of z, and we get $\phi_1(-z) = c\ \phi_2(iz) = c^2\ \phi_1(z)$ and, analogously, $\phi_2(-z) = c^2\ \phi_2(z)$. Hence $c^2 = 1$ from Eq. (5) or $c^2 = -1$ from Eq. (6). The symmetry parameter can acquire four values: $c = \pm 1, \pm i$. In the standard $m = 1$ Rabi model, such a parameter has only two values, ± 1, and the transformation $z \to -z$ is sufficient. The values of c are, in fact, the eigenvalues of the parity operator $\Pi_m = \exp[i\pi(b^\dagger b + \sigma_z/2 + 1/2)/m], m = 1,2$ (see, e.g., [5]), which commutes with the Hamiltonian (1), thus becoming one integral of motion.

Finally, let us introduce four functions $G_c(z,E)$ whose roots with respect to E will be used to fulfill Eq. (8):

$$\begin{aligned}G_+(z) &= \phi_2(iz) - \phi_1(z),\\ G_-(z) &= \phi_2(iz) + \phi_1(z),\\ G_i(z) &= i\phi_2(iz) + \phi_1(z),\\ G_{-i}(z) &= i\phi_2(iz) - \phi_1(z).\end{aligned} \tag{9}$$

In the last two cases the second expression in Eqs. (8) was multiplied by i in order to keep the G_c functions real. They share the common parity of their $\phi_{1,2}$ functions. The first expression in Eqs. (8) is then no longer independent. The complete discrete spectra will be given by all roots of all $G_c(z,E)$ functions, again in full analogy with the standard Rabi model [1]. Of course, the roots are meant with respect to E, and they are independent of any chosen z.

Having categorized the symmetries, we return to set (4) and perform the transformation $\phi_{1,2} = e^{-\kappa z^2}\bar{\psi}_{1,2}$. We get

$$\begin{aligned}2g\bar{\psi}_1'' + (\omega - 8g\kappa)z\bar{\psi}_1' - (4g\kappa + E)\bar{\psi}_1 + \frac{\omega_0}{2}\bar{\psi}_2 = 0,\\ -2g\bar{\psi}_2'' + (\omega + 8g\kappa)z\bar{\psi}_2' - (4\omega\kappa z^2 - 4g\kappa + E)\bar{\psi}_2\\ + \frac{\omega_0}{2}\bar{\psi}_1 = 0,\end{aligned} \tag{10}$$

where we used the parameter κ to simplify the first equation by removing the term $(8g\kappa^2 - 2\omega\kappa + 2g)z^2\bar{\psi}_1$, thus specifying its value

$$\kappa = \frac{\omega - \sqrt{\omega^2 - 16g^2}}{8g}. \tag{11}$$

In fact there should be a $\pm$ sign in front of the square root, but the plus sign would make κ divergent in the limit $g \to 0$, which is physically not reasonable. Notice that κ remains real under the above-mentioned restriction $4|g| < \omega$. There is also a further analogy with the standard Rabi model, where the special case $\omega_0 = 0$ is exactly solved with the help of the coherent state $\exp(\pm 2g/\omega\, b^\dagger)|0\rangle$, where $|0\rangle$ is the lowest bosonic state [8,9]. In our notation, Braak performs the transformation $\phi_{1,2} \propto \exp(-2gz/\omega)\bar{\psi}_{1,2}$; recall $b^\dagger \to z$ in (2). The $m = 2$ Rabi Hamiltonian solution at $\omega_0 = 0$ involves the squeezed vacuum [10] term $\exp(\pm\kappa\, b^{\dagger 2})|0\rangle$. We will return to this case later.

In the next step we expand the functions $\bar{\psi}_{1,2}$:

$$\begin{aligned}\bar{\psi}_1(z) &= \sum_{n=-\infty}^{\infty} Q_n(E)z^n,\\ \bar{\psi}_2(z) &= \sum_{n=-\infty}^{\infty} K_n(E)z^n.\end{aligned} \tag{12}$$

To keep the solutions analytic we expect $Q_n(E) = K_n(E) = 0$ for $n < 0$ [1]. Inserting these expressions into Eqs. (10), we get the iteration scheme

$$\begin{aligned}2g(n+2)(n+1)Q_{n+2} + [(\omega - 8g\kappa)n - 4g\kappa - E]Q_n\\ + \frac{\omega_0}{2}K_n = 0,\\ -2g(n+2)(n+1)K_{n+2} + [(\omega + 8g\kappa)n + 4g\kappa - E]K_n\\ - 4\omega\kappa K_{n-2} + \frac{\omega_0}{2}Q_n = 0.\end{aligned} \tag{13}$$

Notice that the indexes differ by 0, 2, or 4; thus only the coefficients Q_n and K_n with common parity will be nonzero, either for even $n = 0,2,4\ldots$ [Eq. (5)] or odd $n = 1,3,5\ldots$ [Eq. (6)]. The prefactor $e^{-\kappa z^2}$ does not spoil the parity. The starting point of our iteration scheme is either at $n = 0$ or $n = 1$. Let us first have a look on the case $n = 0$ and the symmetry parameter $c = 1$. The top expression in Eqs. (8), $\phi_1(iz) = \phi_2(z)$, at $z = 0$ implies $Q_0 = K_0$, which can itself be a function of Hamiltonian parameters, serving as a normalization constant for the eigenfunctions. But as we are interested only in the roots of $G_+(z,E)$, this constant becomes an unimportant multiplication factor, and we can choose

$$Q_0 = 1, \quad K_0 = 1. \tag{14}$$

The second case is $c = -1$. We have $\phi_1(iz) = -\phi_2(z)$, and again at $z = 0$,

$$Q_0 = 1, \quad K_0 = -1. \tag{15}$$

The third case with $c = i$ is only a bit more complicated. The symmetry condition from Eqs. (8) is now $\phi_1(iz) = i\phi_2(z)$, and its value at $z = 0$ is rather trivial, $Q_0 = K_0 = 0$. But we will compare the first derivatives with respect to z, i.e., $\phi_1'(iz) = i\phi_2'(z)$ at $z = 0$, and we get $iQ_1 = iK_1$; thus now our starting point is

$$Q_1 = 1, \quad K_1 = 1. \tag{16}$$

Finally, for $c = -i$ we get

$$Q_1 = 1, \quad K_1 = -1. \tag{17}$$

Concluding this part we can see that the Hilbert space of eigenfunctions splits into four disjunct subspaces. The corresponding eigenvalues can be found separately as roots of four $G_c(E)$ functions, Eq. (9). We substitute $\phi_{1,2} = e^{-\kappa z^2}\bar{\psi}_{1,2}$, and the coefficients of the expanded $\bar{\psi}_{1,2}$ functions are found by applying the iteration scheme (13) subsequently with four starting points, Eqs. (14)–(17). The coefficients not defined by Eq. (13) are zero because of parity demands. $K_{-2} = K_{-1} = 0$ as well.

Before proceeding to numerical calculations, we ought to mention several special cases, where the exact solution was already known. They will serve as a check of our general solution.

III. SOME EXACTLY KNOWN CASES

A. Case $g = 0$

If the interaction constant is zero, the system separates into independent two-state atoms with energy levels $\pm\omega_0/2$ and a phonon with the mode $N\omega$, where $N = 0,1,2,\ldots$. Thus the overall energy is $\pm\omega_0/2 + N\omega$. It is instructive to see how these values split into four groups as roots of four G_c functions. Therefore we will solve this simple case explicitly. We return to the original functions $\psi_{1,2}$, as the set of equations (3) decouples for $g = 0$, and the two independent solutions are found easily:

$$\begin{aligned}\psi_1 &= C_1 z^{\frac{2E-\omega_0}{2\omega}} = C_1 z^{k'},\\ \psi_2 &= C_2 z^{\frac{2E+\omega_0}{2\omega}} = C_2 z^{k},\end{aligned} \tag{18}$$

where we denoted the exponents by k' and k, as the parity conditions (5) and (6) are common for the $\phi_{1,2}$ and $\psi_{1,2}$ functions. They force k' and k to be integers, and the analyticity demands make them non-negative. The energies $E = -\omega_0/2 + k\omega$ and $E = \omega_0/2 + k'\omega$ should be common, but they are generally different; thus the overall solutions will have either $C_1 = 0$ or $C_2 = 0$.

Let us first analyze the case $(0,\psi_2)^T$ with energy $E = -\omega_0/2 + k\omega$. The linear combinations are $\phi_1 = \psi_1 + \psi_2 = C_2 z^k$ and $\phi_2 = \psi_1 - \psi_2 = -C_2 z^k$. The solutions for the functions $G_c = 0$ finally yield

$$\begin{aligned}G_- &= -C_2(iz)^k + C_2 z^k = 0 \Rightarrow\ k = 0,4,8,\ldots,\\ G_+ &= -C_2(iz)^k - C_2 z^k = 0 \Rightarrow\ k = 2,6,10,\ldots,\\ G_{-i} &= -iC_2(iz)^k + C_2 z^k = 0 \Rightarrow\ k = 3,7,11,\ldots,\\ G_i &= -iC_2(iz)^k - C_2 z^k = 0 \Rightarrow\ k = 1,5,9,\ldots\end{aligned} \tag{19}$$

The second solution $(\psi_1,0)^T$ with $\phi_1 = \phi_2 = C_1 z^{k'}$ and eigenenergies $E = \omega_0/2 + k'\omega$ gives

$$\begin{aligned}G_- &= C_1(iz)^{k'} + C_1 z^{k'} = 0 \Rightarrow\ k' = 2,6,10,\ldots,\\ G_+ &= C_1(iz)^{k'} - C_1 z^{k'} = 0 \Rightarrow\ k' = 0,4,8,\ldots,\\ G_{-i} &= iC_1(iz)^{k'} + C_1 z^{k'} = 0 \Rightarrow\ k' = 1,5,9,\ldots,\\ G_i &= iC_1(iz)^{k'} - C_1 z^{k'} = 0 \Rightarrow\ k' = 3,7,11,\ldots\end{aligned} \tag{20}$$

We will later analyze mainly the cases when ω_0 and ω are comparable, and the eigenenergies as roots of G_c functions reorganize as follows:

$$\begin{aligned}G_-&: -\frac{\omega_0}{2},\quad \frac{\omega_0}{2}+2\omega,\quad -\frac{\omega_0}{2}+4\omega,\ldots,\\ G_+&: \frac{\omega_0}{2},\quad -\frac{\omega_0}{2}+2\omega,\quad \frac{\omega_0}{2}+4\omega,\ldots,\\ G_{-i}&: -\frac{\omega_0}{2}+\omega,\quad \frac{\omega_0}{2}+3\omega,\quad -\frac{\omega_0}{2}+5\omega,\ldots,\\ G_i&: \frac{\omega_0}{2}+\omega,\quad -\frac{\omega_0}{2}+3\omega,\quad \frac{\omega_0}{2}+5\omega,\ldots\end{aligned} \tag{21}$$

We can see that real values of the symmetry parameter c are connected with an even number of phonon excitations N, whereas the imaginary c is coupled to odd N. The global ground state is always given by the lowest root of G_-, which is true also for nonzero g.

B. Case $\omega_0 = 0$

If the gap ω_0 between atomic levels disappears, all eigenenergies become twice degenerate. They are exactly known, and now the complete spectrum is given by [5]

$$E = -\frac{\omega}{2} + \left(n+\frac{1}{2}\right)\Omega\omega \quad n = 0,1,2,\ldots, \tag{22}$$

where another dimensionless quantity was introduced:

$$\Omega = \sqrt{1-\frac{16g^2}{\omega^2}} = 1 - \frac{8g\kappa}{\omega}. \tag{23}$$

Let us reproduce this result. We return to the $\phi_{1,2}$ functions because now the system of equations (4) decouples for $\omega_0 = 0$, and the general solutions are

$$\begin{aligned}\phi_1 &= \exp\left[\left(\kappa-\frac{\omega}{4g}\right)z^2\right]\left[C_{11}H_{-n-1}\left(\sqrt{\frac{\omega\Omega}{4g}}z\right)\right.\\ &\quad\left.+C_{21}\,{}_1F_1\left(\frac{n+1}{2},\frac{1}{2},\frac{\omega\Omega}{4g}z^2\right)\right],\\ \phi_2 &= \exp(\kappa z^2)\\ &\quad\times\left[C_{12}H_n\left(\sqrt{\frac{\omega\Omega}{4g}}z\right)+C_{22}\,{}_1F_1\left(-\frac{n}{2},\frac{1}{2},\frac{\omega\Omega}{4g}z^2\right)\right],\end{aligned} \tag{24}$$

exploiting the well-known Hermitian polynomials H_n and the hypergeometric function ${}_1F_1$. Further we introduced the quantity $n = (\omega+2E)/(2\Omega\omega) - 1/2$, which is nothing but Eq. (22) reversed. Thus, if the parity demands force n to be a non-negative integer again, the spectrum is reproduced. Let us show it in detail at least for the simpler case of even n. We make use of a formula relating the Hermitian polynomials and the hypergeometric function, valid for $n = 0,2,4,\ldots$,

$$\frac{H_n(iqz)}{2^{n/2}(n-1)!!} = i^n e^{-q^2z^2}\,{}_1F_1\left(\frac{n+1}{2},\frac{1}{2},q^2z^2\right), \tag{25}$$

whereas for any n, including the noninteger one, the Kummer transformation [11] gives

$${}_1F_1\left(-\frac{n}{2},\frac{1}{2},-y\right) = e^{-y}\,{}_1F_1\left(\frac{n+1}{2},\frac{1}{2},y\right). \tag{26}$$

First of all we set $C_{11} = 0$ because the Hermitian polynomial with a negative index has no parity, whereas the rest of the

solutions (24) are even functions for even n. We define $q^2 = \omega\Omega/(4g)$, $y = q^2z^2$, and require $G_\pm = 0$ using Eqs. (23)–(26). We get

$$(-2)^{n/2}(n-1)!!C_{12} + C_{22} \mp C_{21} = 0. \tag{27}$$

For a noninteger n (or odd integer), the term with the Hermitian polynomial would become complex, but our functions $\phi_2(iz)$ and $G_\pm$ are real; hence we set also $C_{12} = 0$. We already mentioned that one parameter can be chosen, say the integration constant $C_{21} = \phi_1(0) = 1$. Thus $C_{22} = \phi_2(0) = \pm 1$, and the two G_c functions are identical, $G_+ = G_-$. One can compare the series expansion of these $\phi_{1,2}(z)$ solutions with those given by scheme (13) with starting points (14) or (15) and see that in fact we managed to perform the complete sum of Eqs. (12).

Having exact formulas for some $\phi_{1,2}$ and $G_c(z,E)$ functions at our disposal allows us to make several observations. In the Supplement Material for Ref. [1], Braak reports some problems with the radius of convergence R of his series in z. R seemed to be finite in some cases, though analyticity in the whole complex plane of z is required. To keep the series analogous to our Eqs. (12) convergent, as a necessary condition, the ratio K_{n+1}/K_n had to go to zero for $n \to \infty$. If it was nonzero, R became finite. For the special case $\omega_0 = 0$ we can calculate this ratio explicitly. The coefficients Q_n with $n = 2k$ become rather simple:

$$Q_{2k} = \frac{1}{(2g)^k(2k)!}(E - \varepsilon_{2k-2})(E - \varepsilon_{2k-4})\dots(E - \varepsilon_0), \tag{28}$$

where ε_n are eigenenergies from Eq. (22). En route we see that the energy $E = \varepsilon_n$ terminates $\bar{\psi}_1(z)$ to a (Hermitian) polynomial. The ratio

$$Q_{2k+2}/Q_{2k} \approx -\frac{\omega\Omega}{2g}\frac{1}{2k} \to 0, \qquad k \to \infty, \tag{29}$$

as required. The equations for K_{2k} are not so simple; thus we resort to the exact solution of ϕ_2 and find

$$K_{2k+2}/K_{2k} \approx \frac{\omega\Omega}{2g}\frac{1}{2k} \to 0, \quad k \to \infty. \tag{30}$$

For odd n and especially for nonzero ω_0 we performed at least numerical study of K_n and Q_n from scheme (13) and found that their ratios are also proportional to $1/n$ in leading term. Hence we experience no problems with analyticity of the expanded functions, even including very large z.

The next remark concerns the practical numerical calculations of the roots $G_c(z,E) = 0$, independent of z. It turns out that such calculations are numerically more stable for large z, which is allowed by the previous notion. One can exploit the large-z asymptote for the $\phi_1(z)$ contribution to $G_\pm$ given by [11]

$${}_1F_1(a,b,y) = \frac{\Gamma(b)}{\Gamma(a)}e^y y^{a-b}\left[1 + O\left(\frac{1}{y}\right)\right], \quad y \gg 0, \tag{31}$$

which is analytic for non-negative integer power $a - b = n/2$, i.e., $n = 0,2,\dots$, as expected. $\phi_2(iz)$ gives the same result after Kummer transformation (26).

Further we return to the complete spectrum in Eq. (22). For $|g| \to \omega/4$ the quantity $\Omega \to 0$, and the energy becomes infinitely many times degenerate. This point is physically unsound, though well defined in the sense of a limit.

Concluding, the eigenvalue of Eq. (22) for $n = 0$ coincides with the mutually equal lowest roots of G_- and G_+; for $n = 1$ it is the lowest root of both G_{-i} and G_i, for $n = 2$ is is the second lowest root of both G_- and G_+, etc.

C. Special cases with nonzero ω_0, ω, g

Let us now recall the result of Emary and Bishop [5], who found a set of isolated solutions for our model. We are not going to rederive it, but the basic fact is that under some constraint on the Hamiltonian's parameters and energy E, at least one of the original eigenfunctions $\psi_{1,2}$ becomes a product of some exponential function and of a polynomial.

The main statement is that there exist exactly known eigenstates with eigenenergies

$$E = -\frac{\omega}{2} + \left(N + \frac{1}{2}\right)\Omega\omega \tag{32}$$

if the following conditions are fulfilled:

$$\begin{aligned} 2 - 6\Omega^2 + \frac{\omega_0^2}{4\omega^2} &= 0, \quad N = 2, \\ 6 - 10\Omega^2 + \frac{\omega_0^2}{4\omega^2} &= 0, \quad N = 3, \\ 8(3 - 30\Omega^2 + 35\Omega^4) + 2(7 - 17\Omega^2)\frac{\omega_0^2}{4\omega^2} + \frac{\omega_0^4}{16\omega^4} &= 0, \\ N &= 4, \end{aligned} \tag{33}$$

etc. We will proceed so that we choose the values of ω_0 and ω, and then we gradually change g. Equations (33) yield values of g, and correspondingly, the energy is derived from Eq. (32). These solutions will manifest themselves as crossing points of appropriate roots of G_- and G_+ in the case of even N or crossings of the G_{-i} and G_i roots in the case of odd N. This is again analogous to the standard $m = 1$ Rabi model, where the Juddian points appeared as crossings of the roots of $G_-^{(1)}$ and $G_+^{(1)}$ [1]. Recall that, contrary to Eq. (22), this is only one known excited state for a chosen set of Hamiltonian parameters, not the complete spectrum.

IV. NUMERICAL RESULTS

The resulting energies as roots of G_c functions should not depend on z; this would be true after summing an infinite number of summands in Eq. (12). In numerical calculations we truncate the sum and hope that the higher powers are not significant. We cannot use very small z because of numerical instability. Somewhat surprisingly, we can use values as large as $z = 1000$ or even $z = 10^4$ without a real change of the roots. This fact was already noted for exactly solvable cases; nevertheless, one even does not have to use compromise medium values despite the truncated expansions. Using the symbolic program MATHEMATICA, we could sum up to the z^L term with $L = 34$ for even solutions or $L = 35$ for the odd ones. If the values of a root with smaller L converged to the same value, we accept it. It turns out that for smaller values of $|g|$ the convergence is excellent, but it becomes poorer as

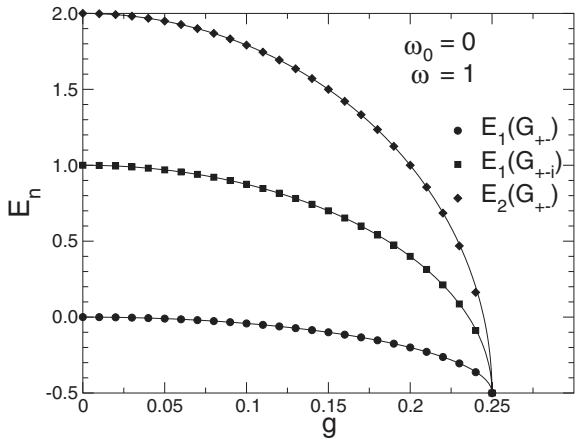

FIG. 1. Eigenenergies as roots of $G_c(E)$ functions for $\omega_0 = 0$ and $\omega = 1$. Solid lines are exact values. All quantities E_n, ω_0, ω, and g are in energy units with $\hbar = 1$.

we approach the maximal possible value, i.e., if $|g| \to \omega/4$. The interval of g in our plots is limited to $0 \leqslant g \leqslant w/4$. It is known that although the eigenfunctions differ after changing the sign of g, the eigenenergies remain the same; i.e., there is a mirror symmetry $E(-g) = E(g)$.

Let us test our calculations at first on the exactly solved case $\omega_0 = 0$. We choose $\omega = 1$. Parts of parabolas with a common top form the exact spectrum from Eq. (22); see solid lines in Fig. 1. It is clear that, in the vicinity of the infinitely many times degenerate point with $g = \omega/4$, the roots of $G_c(E)$ functions become very dense and the functions themselves are quickly oscillating. That is the reason why, even for the orders as large as $L = 34$, the values of appropriate roots did not converge completely, and we have to resort to some fitting procedure, yielding a better guess of the saturation value for $L \to \infty$. The roots are denoted so that the lowest one is $E_1(G_c)$, the second lowest one is $E_2(G_c)$, etc. We can see that the calculated eigenenergies fit the exact values almost perfectly, except for some deviation at g close to $\omega/4$ and for the higher root, in this case $E_2(G_\pm)$.

In Fig. 2 we present the three lowest roots of $G_\pm$ (solid symbols) and two lowest roots of $G_{\pm i}$ (smaller open symbols). The values $\omega_0 = 1$ and $\omega = 2$ are chosen so that the spectrum

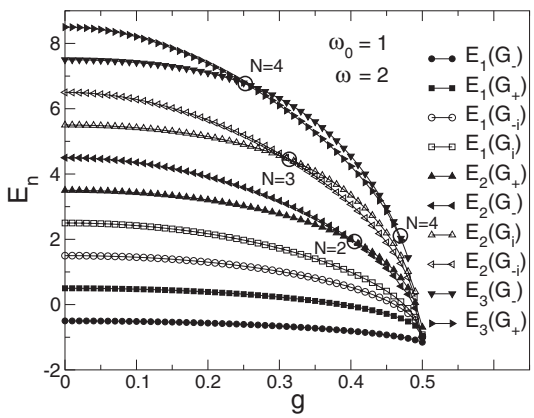

FIG. 2. Eigenenergies as roots of $G_c(E)$ functions for $\omega_0 = 1$ and $\omega = 2$. All quantities are in energy units with $\hbar = 1$.

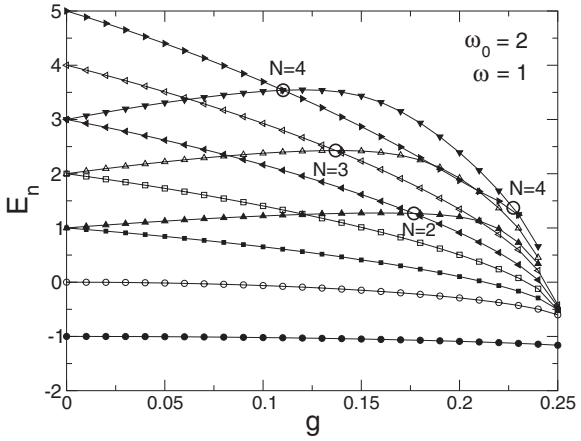

FIG. 3. Eigenenergies as roots of $G_c(E)$ functions for $\omega_0 = 2$ and $\omega = 1$. The description of the symbols is the same as in Figure 2. All quantities are in energy units with $\hbar = 1$.

at $g = 0$ becomes equidistant; see Eq. (21). The large open circles are exact solutions of Eqs. (33) and (32). The value of N is written nearby. We can see an almost perfect match with the crossings of appropriate lines. The bottom four lines with $N = 0$ and $N = 1$ do not cross; lines with $N = 4$ cross twice.

Figure 3 shows the same roots denoted by the same symbols as Fig. 2, except for $\omega_0 = 2$ and $\omega = 1$. Besides the well-fitted exact crossing points, there are other crossings of lines with different N, which are not exactly known. A similar figure with, in fact, the same Hamiltonian parameters was already published in [2], where the authors plot also numerical results from larger matrix diagonalization.

There are a couple of simpler analytic results that can be derived from our approach. We can find, e.g., the small-g expansion of the (global) ground-state energy $E_0 = E_1(G_-)$:

$$E_0 \approx -\frac{\omega_0}{2} - \frac{8g^2}{2\omega + \omega_0} + O(g^4), \quad g \ll \omega, \omega_0, \tag{34}$$

which can be compared with a similar result for the $m = 1$ Rabi model: $E_0^{(1)} \approx -\omega_0/2 - 4g^2/(\omega + \omega_0) + \ldots$ [9].

V. SUMMARY

We have found the complete spectrum of the two-photon Rabi Hamiltonian as roots of four analytic functions $G_\pm$ and $G_{\pm i}$ in the whole parametric space. These functions are given by the recurrence scheme (13) with four starting points, (14)–(17). The unnormalized eigenfunctions in Bargmann space can be found as well, using $\psi_1(z) = [\phi_1(z) + \phi_2(z)]/2$ and $\psi_2(z) = [\phi_1(z) - \phi_2(z)]/2$.

Solano [12] states that Braak [1] managed to enlarge the class of exactly solvable models and that he added the standard Rabi model to the short list of exactly solvable quantum systems. We believe that this paper adds also the two-photon Rabi Hamiltonian to the same list. This list can almost surely be extended further by using Braak's approach on other related models, and another task for the future is finding a deeper understanding of the criteria of its applicability. Nevertheless, there are disputes on whether the term *exact* solvability should be used, if the G_c functions are given only by some Taylor expansions with coefficients coming from

a recurrence scheme. In my opinion, the word *integrability* should be left rather for models where a sufficient number of integrals of motion are known, which is not the case for Rabi models. Possibly, some new name should be introduced for the "solvability" in Braak's sense.

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[1] D. Braak, Phys. Rev. Lett. **107**, 100401 (2011).
[2] V. V. Albert, G. D. Scholes, and P. Brumer, Phys. Rev. A **84**, 042110 (2011).
[3] I. I. Rabi, Phys. Rev. **51**, 652 (1937).
[4] B. R. Judd, J. Phys. C **12**, 1685 (1979).
[5] C. Emary and R. F. Bishop, J. Phys. A **35**, 8231 (2002).
[6] S. N. Dolya, J. Math. Phys. **50**, 033512 (2009).
[7] V. Bargmann, Commun. Pure Appl. Math. **14**, 197 (1961).
[8] R. F. Bishop, N. J. Davidson, R. M. Quick, and D. M. van der Walt, Phys. Lett. A **254**, 215 (1999).
[9] I. Travenec and L. Samaj, Phys. Lett. A **375**, 4104 (2011).
[10] C. Brif, Ann. Phys. (NY) **251**, 180 (1996).
[11] A. D. Polyanin and V. F. Zaitsev, *Exact Solutions for Ordinary Differential Equations* (Chapman and Hall/CRC, New York, 2003).
[12] E. Solano, Physics **4**, 68 (2011).