Probability interpretation, an equivalence relation, and a lower bound on the convex-roof extension of negativity

Abbass Sabour* and Mojtaba Jafarpour†
Physics Department, Shahid Chamran University, Ahvaz, Iran
(Received 21 February 2012; published 24 April 2012)

We present a measure of entanglement for pure two-qutrit states which constitutes a probability interpretation for negativity. We also obtain an equivalence relation for the latter, which is used to find a lower bound on its convex-roof extension, for systems with dimensions up to 6. The possibility of the obtained results being valid for higher-dimensional systems is also discussed.

DOI: 10.1103/PhysRevA.85.042323 PACS number(s): 03.67.Mn, 03.65.Ta

I. INTRODUCTION

There has been an extensive effort to formulate suitable entanglement measures over the last decade [1,2]. Entanglement entropy was defined in 1996 for the pure state of a bipartite system of two qubits which has an operational and quite comprehensible interpretation [3]. Entanglement entropy was then extended to the mixed states of two-qubit systems, through the convex-roof method, known as entanglement of formation [4]. Another entanglement measure which is also related to the entanglement of formation and has been used extensively is the well-known concurrence, which was put forward in 1998 [5]. A probabilistic interpretation of this measure for the pure bipartite states has been established recently [6]. Several more measures, which are mostly extensions of concurrence in some manner, were introduced later; however, they have been difficult to evaluate for the mixed states [7–9]. Conditions required of an acceptable measure were put forward in 1998 [10], but it was emphasized later that the only necessary condition is that it should not change under local operations and classical communications [11]. Negativity is a measure that is the simplest to calculate for mixed states and was developed on the basis of the partial positive transpose criterion [12] in 2002 [13]. It satisfies the essential condition for a legitimate measure, but it is not necessarily faithful when applied beyond qubit-qubit and qubit-qutrit systems. In fact, there are some states, known as bound states, which are entangled, although they have zero negativity [14]. It has been revealed that negativity has some relevance to the distillation of entanglement; namely, a nil result for the negativity of a bound state implies that it cannot be distilled for entanglement extraction [15]. In spite of its limitations, negativity is being used extensively due to its easy evaluation. Negativity for the pure state has also been extended to the mixed ones, through the convex-roof method, in 2003 [16]; it has also been shown that it is free from lack of faithfulness when applied to bound states. There have been variations in the definition of negativity with regard to its numerical coefficient in the course of time; but we adhere to the one whose maximum is unity, as follows:

$$N_d := \frac{\|\rho^{T_A}\| - 1}{d-1}, \tag{1}$$

where ρ^{T_A} is the partially transposed density matrix of the bipartite system with respect to the subsystem A and $d := \min\{\dim(A), \dim(B)\}$.

Now we set up an expression for negativity in terms of the Schmidt coefficients. Let us consider the following pure bipartite state:

$$|\psi^{AB}\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} c_{ij}|i^A, j^B\rangle; \tag{2}$$

it may be written in the Schmidt form in the following manner [17]:

$$|\psi^{AB}\rangle = \sum_{i=1}^{d}\sqrt{p_i}|\alpha_i^A,\beta_i^B\rangle, \tag{3}$$

where $|\alpha_i^A,\beta_i^B\rangle$ are called Schmidt bases, and $\{\sqrt{p_i}\}$ are the Schmidt coefficients. It has been shown that $\{p_i\}$ are the eigenvalues of the reduced density matrix $\rho^A = \mathrm{Tr}_B|\psi^{AB}\rangle\langle\psi^{AB}|$ [18]. Now we apply the local unitary transformation U_S to $|\psi^{AB}\rangle$ as expressed in Eq. (3), to write its expansion in the computational basis

$$|\psi^{AB}\rangle \xrightarrow{U_S} |\psi_S^{AB}\rangle := U_S|\psi^{AB}\rangle = \sum_{i=1}^{d}\sqrt{p_i}|i^A,i^B\rangle. \tag{4}$$

Moreover, using Eq. (4) we may express negativity in terms of the Schmidt coefficients as follows [16]:

$$N_d = \frac{1}{d-1}\left\{\left(\sum_{i=1}^{d}\sqrt{p_i}\right)^2 - 1\right\} = \frac{1}{d-1}\sum_{j>i=1}^{d}(2\sqrt{p_i}\sqrt{p_j}). \tag{5}$$

II. A PROBABILISTIC ENTANGLEMENT MEASURE FOR TWO-QUTRIT SYSTEMS

We first consider the following theorem.

Theorem 1. Let us consider a two-qutrit system whose state in the computational bases is given by Eq. (2) for $d = d_1 = d_2 = 3$. There is at least a local unitary transformation under which the expansion coefficients $|c_{i,i}^U|$ and also the expansion coefficients $|c_{i,j\neq i}^U|$ are equal, simultaneously.

Proof. Assume that the unitary transformation $U = \hat{U}U_S$ suits the job, where U_S transforms the state to the Schmidt form in the computational bases. This implies that it suffices

*abas.sabor@yahoo.com
†mojtaba_jafarpour@hotmail.com

to find a unitary matrix $\hat{U}$ that transforms the Schmidt form in the computational bases to the required one. Of course, the unitary matrix $\hat{U}$, and thus U, is not unique; we introduce the ones which not only prove the theorem but also lead to a probability interpretation for negativity. The proposed $\hat{U}$ is defined by

$$\hat{U} = u_+ \otimes u_-, \tag{6}$$

where

$$u_\pm := \frac{1}{2}\begin{bmatrix}1 & \pm i & 0\\ \pm i & 1 & 0\\ 0 & 0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{bmatrix} \times \begin{bmatrix}\sqrt{2} & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{bmatrix}, \tag{7}$$

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right). \tag{8}$$

Applying $\hat{U}$ to the Schmidt form (4) for $d = d_1 = d_2 = 3$, we obtain

$$|\psi_U^{AB}\rangle = \hat{U}\sum_{i=1}^{3}\sqrt{p_i}|i^A,i^B\rangle = \sum_{i,j=1}^{3} c_{ij}^U|i^A,j^B\rangle, \tag{9}$$

where the new expansion coefficients in the computational bases are given by

$$|c_{ij}^U|_{i=j} = \frac{1}{3}\sum_{i=1}^{3}\sqrt{p_i} =: c \quad \text{and} \quad |c_{ij}^U|_{i\neq j} = \sqrt{\frac{1-3c^2}{6}}. \tag{10}$$

■

Considering the above result, we conclude that any bipartite pure two-qutrit state may be expressed in the computational bases as follows:

$$|\psi_U^{AB}\rangle = c\sum_{i=1}^{3} e^{i\theta_{ii}}|i^A,i^B\rangle + \sqrt{\frac{1-3c^2}{6}}\sum_{i\neq j}^{3} e^{i\theta_{ij}}|i^A,j^B\rangle, \tag{11}$$

where θ_{ij} are the phases. Now, we define the following orthonormal states expressed in the computational bases:

$$|\varphi_1^{AB}\rangle := \frac{1}{\sqrt{3}}(e^{i\theta_{11}}|1^A,1^B\rangle + e^{i\theta_{22}}|2^A,2^B\rangle + e^{i\theta_{33}}|3^A,3^B\rangle), \tag{12}$$

$$|\varphi_2^{AB}\rangle := \frac{1}{\sqrt{3}}(e^{i\theta_{12}}|1^A,2^B\rangle + e^{i\theta_{23}}|2^A,3^B\rangle + e^{i\theta_{31}}|3^A,1^B\rangle), \tag{13}$$

$$|\varphi_3^{AB}\rangle := \frac{1}{\sqrt{3}}(e^{i\theta_{13}}|1^A,3^B\rangle + e^{i\theta_{21}}|2^A,1^B\rangle + e^{i\theta_{32}}|3^A,2^B\rangle). \tag{14}$$

They are similar to the Bell states and display the maximum of entanglement as do the latter. Suppose that the system is in one of the states given by Eqs. (12)–(14), which we refer to by $|\varphi_n^{AB}\rangle$; if a projective von Neumann local measurement (PM) is performed on this state in the computational bases and the value i^A is obtained for A measurement, then a simple analysis shows that B measurement leads to

$$j^B(d;n;i^A) := i^A + n - d\left[\frac{i^A+n-2}{d}\right] - 1, \tag{15}$$

with $d = 3$, where the square brackets denote the integer part. That is, the unknown state may be determined by the latter measurement. We may also construct a set of nine qutrit Bell-type states in the $H_3 \otimes H_3$ space, like the four genuine Bell states that are set up in the $H_2 \otimes H_2$ space, as follows:

$$\begin{aligned}|\varphi_{nm}^{AB}\rangle &= \frac{1}{\sqrt{3}}\sum_{i=1}^{3} e^{i\omega_{mi}}|i^A,j^B(3;n;i)\rangle, \quad n,m = 1,2,3,\\ \omega_{mi} &= \frac{2\pi}{3}(1-\delta_{m1})(1-\delta_{i1})(5-2m)(5-2i).\end{aligned} \tag{16}$$

Now, using Eqs. (12), (13), and (14) in Eq. (11), we have

$$\begin{aligned}|\psi_U^{AB}\rangle &= \sqrt{3}c|\varphi_1^{AB}\rangle + \sqrt{\frac{1-3c^2}{2}}|\varphi_2^{AB}\rangle + \sqrt{\frac{1-3c^2}{2}}|\varphi_3^{AB}\rangle\\ &= \sqrt{p_1}|\varphi_1^{AB}\rangle + \sqrt{p_2}|\varphi_2^{AB}\rangle + \sqrt{p_3}|\varphi_3^{AB}\rangle.\end{aligned} \tag{17}$$

Two interpretations may be associated with p_i: First, Eq. (17) reveals that it is the probability of realization of $|\varphi_i^{AB}\rangle$ in $|\psi_U^{AB}\rangle$ and, second, considering Eq. (11), it is the probability that B obtains i^B in his measurement while A has obtained i^A in hers. Let us choose $c = 1/3$ to obtain

$$\begin{aligned}|\psi_U^{AB}\rangle|_{c=\frac{1}{3}} &= \frac{1}{\sqrt{3}}(|\varphi_1^{AB}\rangle + |\varphi_2^{AB}\rangle + |\varphi_3^{AB}\rangle)\\ &= \frac{1}{3}\sum_{i,j=1}^{3} e^{i\theta_{ij}}|i^A,j^B\rangle;\end{aligned} \tag{18}$$

then, a PM is performed on the above state in the computational bases. It is clear that B's result does not depend on A's result and vice versa; that is, no correlation is observed. However, $c = 1/\sqrt{3}$ yields $|\varphi_1^{AB}\rangle$; in this case A's and B's results are perfectly correlated as may be deduced from Eq. (12). Thus, in light of the above observations, the difference between the probabilities p_1 and $p_2 = p_3$ may be considered as a measure of the quantum correlation between the two qutrits,

$$\Delta P_U = (\sqrt{3}c)^2 - \left(\sqrt{\frac{1-3c^2}{2}}\right)^2 = \frac{9c^2-1}{2}, \tag{19}$$

which may be expressed in the following form, using Eq. (10):

$$\Delta P_U = \frac{1}{3-1}\left\{\left(\sum_{i=1}^{3}\sqrt{p_i}\right)^2 - 1\right\}. \tag{20}$$

This is an interesting result; it is exactly equal to the negativity as expressed by Eq. (5); however, it may not necessarily be unique and depends on the unitary transformation U that has been applied in Eq. (11). Thus, we maximize Eq. (20) over all the unitary transformations U, to define a unique measure as follows:

$$\Delta P := \max_U \Delta P_U. \tag{21}$$

Now, considering Eq. (5); we also obtain

$$\Delta P \geqslant N_3; \tag{22}$$

that is, the derived measure also provides an upper bound on negativity. But still we can achieve even better than this: Our extensive numerical computations have revealed that actually the inequality in Eq. (22) is an equality

$$\Delta P = N_3; \tag{23}$$

thus, we conclude that ΔP is a probabilistic measure of two-qutrit entanglement, which is equal to the negativity and provides a probabilistic interpretation for the latter; its value ranges from zero to 1.

One may wonder if it is possible to generalize the measure expressed in Eq. (21) to higher-dimensional systems. In fact, we have tried the four-dimensional case too. In this case we need to transform the state in Eq. (2), for $d = d_1 = d_2 = 4$, to the required form; considering Eq. (15) we may write

$$\begin{aligned} \left|\psi_U^{AB}\right\rangle = \sqrt{p_1}\left|\psi_1^{AB}\right\rangle + \sqrt{p_2}\left|\psi_2^{AB}\right\rangle + \sqrt{p_3}\left|\psi_3^{AB}\right\rangle \\ + \sqrt{p_4}\left|\psi_4^{AB}\right\rangle, \end{aligned} \tag{24}$$

$$\left|\psi_n^{AB}\right\rangle := \frac{1}{2}\sum_{i=1}^{4} e^{i\theta_{i,j(4;n;i)}}\left|i^A, j^B(4;n;i^A)\right\rangle, \tag{25}$$

where $p_2 = p_3 = p_4$, and the same definition as in Eq. (22) is applicable again. However, we have not been able to find an appropriate U to do the job, analytically; thus we have resorted to relatively tedious numerical computations. We note that any bipartite state may be transformed to the Schmidt form; thus it suffices to consider the latter type of state for examination. We have tried the four-dimensional bipartite states with unnormalized Schmidt coefficients {1,1,1,2}, {1,1,1,5}, {1,1,1,10}, {1,1,1,20}, {1,1,5,5}, {1,2,3,4}, {2,3,4,5}, {3,4,5,6}, {4,5,6,7}, {1,3,5,7}, {1,4,7,10}, and {1,5,9,13} and have ascertained that the negativity and our defined measure are equal in all cases. Considering the trend, one may conjecture that our results may even be valid for $d \geqslant 4$; however, we believe that more investigations are called for, to check this conjecture.

III. AN EQUIVALENCE RELATION FOR PURE-STATE NEGATIVITY

An equivalence relation for the pure-state negativity is proved in this section. We start with the following theorem.

Theorem 2. A unitarily transformed expansion of the bipartite state (2) is given by

$$|\psi^{AB}\rangle \xrightarrow{U} U|\psi^{AB}\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} c_{ij}^U |i^A, j^B\rangle. \tag{26}$$

Now, defining

$$\gamma_{ij}^U := 2\left(c_{ii}^U c_{jj}^U - c_{ij}^U c_{ji}^U\right), \tag{27}$$

$$N_d^U := \frac{1}{d-1}\sum_{j>i=1}^{d} \left|\gamma_{ij}^U\right|, \tag{28}$$

$$N_d' := \max_U N_d^U, \tag{29}$$

where $d = \min\{d_1,d_2\}$; then $N_d' = N_d$.

Proof. First, assuming $U = U_S$, $|\psi^{AB}\rangle$ is transformed to the Schmidt form $|\psi_S^{AB}\rangle = U_S|\psi^{AB}\rangle$ in the computational bases, whose corresponding negativity is given by Eq. (5). Considering the latter and Eqs. (28) and (29), we note that for $c_{ij}^U = \sqrt{p_i}\delta_{ij}$, N_d^U reduces to N_d; thus we may write

$$N_d' \geqslant N_d; \tag{30}$$

that is, N_d is a lower bound on N_d'. Now we demonstrate that the inequality in Eq. (30) is actually an equality. To show this, we follow the method of Ref. [19] which we have recently developed to prove an equality for the entropic uncertainty relation. First we note that in the $(d_1 \times d_2)$-dimensional space of the bipartite system, the negativity expressed in the Schmidt form requires $n_1 = \min\{d_1,d_2\} - 1$ independent variables. Application of a general local unitary transformation to this state also requires $n_2 = d_1^2 + d_2^2$ more variables. Thus, we define the following function of $n = n_1 + n_2$ variables:

$$\Delta_d\left(U; \left|\psi_S^{AB}\right\rangle\right) := N_d^U\left(U; \left|\psi_S^{AB}\right\rangle\right) - N_d\left(\left|\psi_S^{AB}\right\rangle\right), \tag{31}$$

where $|\psi_S^{AB}\rangle$ is the general state expressed in the Schmidt form. N_d is independent of the parameters which define the unitary transformation U; thus, maximizing Eq. (31) with respect to all the n parameters is just equivalent to maximizing the difference $N_d' - N_d$. Carrying out numerical computations up to d_1 and d_2 equal to 6, we have found zero for the latter in all cases, without fail. Thus, we find an equivalence relation for the negativity as follows:

$$N_d = N_d' = \max_U \frac{1}{d-1}\sum_{j>i=1}^{d} \left|\gamma_{ij}^U\right| \quad \text{for} \quad \max\{d_1,d_2\} \leqslant 6. \tag{32}$$

■

This is an interesting equivalence result; it is used in the sequence to derive a lower bound on the convex-roof extended negativity (CREN) and potentially may be useful in other entanglement investigations. We also conclude that for any arbitrary unitary transformation with $\max\{d_1,d_2\} \leqslant 6$, N_d^U provides a lower bound on the negativity,

$$N_d \geqslant N_d^U. \tag{33}$$

IV. A LOWER BOUND ON CONVEX-ROOF EXTENDED NEGATIVITY

The convex-roof method [20] has been used to extend the pure-state negativity to mixed states [16]; however, it is very difficult to evaluate the CREN for the general case. In fact, only highly symmetrical states, for example the isotropic ones [21], have rendered this evaluation possible [16].

Here we intend to derive a lower bound on the CREN, which turns out to be specifically useful in the estimation of entanglement in nonsymmetrical mixed states, whereas the lack of symmetry hinders an easy evaluation of the CREN itself. Our method of approach is similar to the one we took recently to obtain a lower bound on the concurrence [22]. We assume that the ensemble $\varepsilon \equiv \{|\psi_i\rangle; p_i\}$ represents the state of the system; that is, its density operator is given by $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. Thus, the ensemble average of the negativity N_d corresponding to this density operator is expressed by

$$N_d(\varepsilon) := [N_d]_{\varepsilon \equiv \{|\psi_i\rangle; p_i\}} = \sum_i p_i N_d(|\psi_i\rangle). \tag{34}$$

As there may be several decompositions of the density operator [23], the CREN ($\mathcal{C}$) is defined as the infimum of the average negativity with respect to the decomposition ensembles; thus, it is decomposition independent and depends only on the ρ itself:

$$\mathcal{C}_d(\rho) := \min_{\{|\psi_i\rangle;p_i\}} \sum_i p_i N_d(|\psi_i\rangle)\Big|_{\sum_i p_i|\psi_i\rangle\langle\psi_i|=\rho}. \tag{35}$$

Now, considering Eqs. (18)–(20), we may write

$$\begin{aligned} N_d(\varepsilon) &= \sum_i p_i N_d(|\psi_i\rangle) = \sum_i p_i N_d'(|\psi_i\rangle) \\ &= \sum_i p_i N_d'(U_i|\psi_i\rangle) = \sum_i p_i \left(\max_{U_i} \frac{1}{d-1} \sum_{k>j=1}^{d} |\gamma_{i;jk}^{U_i}|\right) \\ &\geqslant \frac{1}{d-1} \max_U \sum_{k>j=1}^{d} \sum_i p_i |\gamma_{i;jk}^{U}|, \end{aligned} \tag{36}$$

where we have assumed

$$U_i\,|\psi_i\rangle = \sum_{j=1}^{d_1} \sum_{k=1}^{d_2} c_{i;jk}^{U_i}\,|j,k\rangle, \tag{37}$$

$$\gamma_{i;jk}^{U_i} := 2\big(c_{i;jj}^{U_i} c_{i;kk}^{U_i} - c_{i;jk}^{U_i} c_{i;kj}^{U_i}\big), \tag{38}$$

and the following inequality has been used:

$$\max_{U_1} f_1(U_1) + \max_{U_2} f_2(U_2) + \cdots \geqslant \max_U [f_1(U) + f_2(U) + \cdots]. \tag{39}$$

To simplify the notation we combine the two indices j and k into the single index $I(j,k) = j + d_1(k-1)$; thus we write

$$|\psi_i\rangle = \sum_{I=1}^{d_1\times d_2} c_{i;I}|I\rangle, \tag{40}$$

$$\begin{aligned} \rho &= \sum_i p_i|\psi_i\rangle\langle\psi_i| = \sum_{I,I'=1}^{d_1\times d_2} \left(\sum_i p_i c_{i;I} c_{i;I'}^*\right)|I\rangle\langle I'| \\ &= \sum_{I,I'=1}^{d_1\times d_2} \rho_{I,I'}|I\rangle\langle I'|, \end{aligned} \tag{41}$$

where $\rho_{I,I'} = \sum_i p_i c_{i;I} c_{i;I'}^*$ is the matrix representation of the density operator ρ.

Now, using Eq. (38), Eq. (15) in Ref. [22], and the fact that the mixed-state concurrence is obtained through the convex-roof method from the pure-state concurrence, we obtain

$$\sum_i p_i \left|\gamma_{i;jk}^{U}\right| = 2\sum_i p_i \left|c_{i;I(j,j)}^{U} c_{i;I(k,k)}^{U} - c_{i;I(j,k)}^{U} c_{i;I(k,j)}^{U}\right| \geqslant C\left(\sigma_{jk}^{U}\right), \tag{42}$$

where $C(\sigma_{jk}^U)$ is the mixed two-qubit concurrence and the unnormalized density matrix σ_{jk}^U is expressed by

$$\sigma_{jk}^U \equiv \begin{bmatrix} \rho_{I(j,j),I(j,j)}^U & \rho_{I(j,j),I(j,k)}^U & \rho_{I(j,j),I(k,j)}^U & \rho_{I(j,j),I(k,k)}^U \\ \rho_{I(j,k),I(j,j)}^U & \rho_{I(j,k),I(j,k)}^U & \rho_{I(j,k),I(k,j)}^U & \rho_{I(j,k),I(k,k)}^U \\ \rho_{I(k,j),I(j,j)}^U & \rho_{I(k,j),I(j,k)}^U & \rho_{I(k,j),I(k,j)}^U & \rho_{I(k,j),I(k,k)}^U \\ \rho_{I(k,k),I(j,j)}^U & \rho_{I(k,k),I(j,k)}^U & \rho_{I(k,k),I(k,j)}^U & \rho_{I(k,k),I(k,k)}^U \end{bmatrix}, \tag{43}$$

where $\rho^U = U\rho U^\dagger$. Now, defining

$$\tilde{N}_d^U(\rho) := \frac{1}{d-1} \sum_{k>j=1}^{d} C\left(\sigma_{jk}^U\right), \tag{44}$$

Eq. (36) may be expressed by

$$N_d(\varepsilon) \geqslant \max_U \tilde{N}_d^U(\rho). \tag{45}$$

We note that the right-hand side (RHS) of the above inequality is independent of the ensemble decomposition; thus we minimize the LHS with respect to the latter, which does not affect the faithfulness of the inequality. Then, considering (35) we finally obtain the lower bound on the CREN (the LCREN $\mathcal{L}$) as follows:

$$\mathcal{C}_d(\rho) \geqslant \max_U \tilde{N}_d^U(\rho) =: \mathcal{L}_d(\rho). \tag{46}$$

Obviously, $\tilde{N}_d^U(\rho)$ will also be a lower bound on $\mathcal{C}_d(\rho)$ for any U, but not necessarily as strong as in the case when the latter is the maximizing transformation yielding $\mathcal{L}_d(\rho)$.

To check our result, we apply (46) to the following isotropic state studied in Ref. [16]:

$$\rho_{\text{iso}}^F = \frac{1-F}{d^2-1}(\mathbf{1}^{AB} - |\varphi^{AB}\rangle\langle\varphi^{AB}|) + F|\varphi^{AB}\rangle\langle\varphi^{AB}|, \tag{47}$$

where

$$F = \langle\varphi^{AB}|\rho_{\text{iso}}^F|\varphi^{AB}\rangle, \quad |\varphi^{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i^A, i^B\rangle. \tag{48}$$

Let us choose the trivial unitary transformation $\tilde{U} = \mathbf{1}^{AB} = \mathbf{1}^{A} \otimes \mathbf{1}^{B}$. Considering that all the $\sigma_{jk}^{\tilde{U}}$ matrices are equal due to the symmetry of the isotropic density operator, we find

$$\sigma_{jk}^{\tilde{U}} = \begin{bmatrix} x & 0 & 0 & z \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ z & 0 & 0 & x \end{bmatrix}, \tag{49}$$

where $x = (F + 1/d)/(d-1)$, $y = (1-F)/(d^2-1)$, and $z = (Fd - 1/d)/(d^2-1)$. The density matrix in Eq. (49) is the familiar X state, whose concurrence is $\max\{0, 2(|z| - |y|)\}$ [24]. Therefore, we finally have

$$\begin{aligned} \tilde{N}_d^{\tilde{U}}(\rho_{\text{iso}}^F) &= \frac{1}{d-1} \sum_{k>j=1}^{d} C(\sigma_{jk}^{\tilde{U}}) = \frac{1}{d-1} \left(\sum_{k>j=1}^{d} 1 \right) C(\sigma_{12}^{\tilde{U}}) = \frac{1}{d-1} \times \frac{d(d-1)}{2} \times \max \left\{ o, 2\left(\left| \frac{Fd - \frac{1}{d}}{d^2-1} \right| - \left| \frac{1-F}{d^2-1} \right| \right) \right\} \\ &= \max \left\{ 0, \frac{Fd-1}{d-1} \right\} = \mathcal{C}_d(\rho_{\text{iso}}^F). \end{aligned} \tag{50}$$

This is an interesting result; the bound derived on the CREN is exactly equal to the CREN itself for the isotropic state. That is, the unitary transformation $\tilde{U}$ was the maximizing U in Eq. (46). However, $\mathcal{L}_d(\rho)$ is just a lower bound in the general case and no judgment regarding its sharpness may be made; it is particularly of use whenever the state is not symmetric enough to render the evaluation of the CREN possible.

V. CONCLUSIONS

We have obtained an upper bound on the negativity of a pure two-qutrit state, which turns out to be equal to the negativity itself; thus providing an operational interpretation for the latter. The possibility that the obtained bound might be valid for higher dimensions has also been discussed. We have also derived an equivalence relation for the pure-state negativity which has proved to be valid for dimensions up to 6, with potential applications in entanglement investigations. As an example, we have used this equivalence relation to obtain a lower bound on the convex-roof extended negativity for mixed states, which is particularly useful in assessing entanglement in nonsymmetric states.

[1] M. B. Plenio and S. Virmani, Quantum Inf. Comput. **7**, 1 (2007).
[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[3] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[4] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
[5] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[6] A. Sabour and M. Jafarpour, Chin. Phys. Lett. **28**, 070301 (2011).
[7] A. Uhlmann, Phys. Rev. A **62**, 032307 (2000).
[8] P. Rungta, V. Buzek, C. M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A **64**, 042315 (2001).
[9] P. Rungta and C. M. Caves, Phys. Rev. A **67**, 012307 (2003).
[10] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[11] G. Vidal, J. Mod. Opt. **47**, 355 (2000).
[12] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[13] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[14] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
[15] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[16] S. Lee, D. P. Chi, S. D. Oh, and J. Kim, Phys. Rev. A **68**, 062304 (2003).
[17] E. Schmidt, Math. Ann. **63**, 433 (1907).
[18] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, New York, 2002), p. 123.
[19] M. Jafarpour and A. Sabour, Phys. Rev. A **84**, 032313 (2011).
[20] A. Uhlmann, Open Syst. Inf. Dyn. **5**, 209 (1998).
[21] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
[22] M. Jafarpour and A. Sabour, Quantum Inf. Process., doi: 10.1007/s11128-011-0288-0.
[23] N. Grossman, Philos. Sci. **41**, 333 (1974).
[24] T. Yu and J. H. Eberly, Quantum Inf. Comput. **7**, 459 (2007).