Entanglement, mixedness, and perfect local discrimination of orthogonal quantum states

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It is shown that local distinguishability of orthogonal mixed states can be completely characterized by the local distinguishability of their supports irrespective of entanglement and mixedness of the states. This leads to two kinds of upper bounds on the number of locally distinguishable orthogonal mixed states. The first one depends only on pure-state entanglement within the supports of the states and therefore may be easy to compute in many instances. The second bound is optimal in the sense that it optimizes the bounding quantities, not necessarily the function of entanglement alone, over all orthogonal mixed-state ensembles (satisfying certain conditions) admissible within the supports of the density matrices.

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I. INTRODUCTION

A characteristic feature of quantum theory is that composite quantum systems, whose parts do not interact, may possess nonlocal properties. For example, entanglement [1] and quantum information both exhibit nonlocality. Entangled states are nonlocal because they give rise to correlations that cannot be explained by local hidden variable theories [2], whereas nonlocality of quantum information is in the sense that a measurement on the whole system sometimes reveals more information about the state than coordinated local measurements on its parts [3–8]. This nonlocal nature of quantum information is generally manifested in the setting of local discrimination of quantum states [3,4,6–19]. One of the principal goals in quantum information theory is to understand and quantify the relationship between entanglement and nonlocality of quantum information.

The difficulty in quantifying the role of entanglement in local state discrimination is evident from some of the early results, which show that the presence of entanglement is neither necessary nor sufficient to ensure whether a given set of orthogonal states is locally indistinguishable. That entanglement is not necessary is evident from the examples of locally indistinguishable sets of orthogonal product states exhibiting “nonlocality without entanglement” or forming an unextendible product basis (UPB) [4,5]. On the other hand, any two orthogonal states can be perfectly distinguished no matter how entangled they are [9], showing that entanglement is not sufficient for local indistinguishability. Nevertheless, entanglement is often the key factor in a typical locally indistinguishable set, as in the case of a complete bipartite orthogonal basis containing one or more entangled states [6,7]; if such a set can be perfectly distinguished locally, then one can create entanglement from product states using local operations and classical communication (LOCC) [6,7], a task known to be impossible.

Significant progress, which also motivated the present work, was reported in Ref. [14], where it was shown that entanglement does guarantee difficulty in local state discrimination. In particular, it was shown that if the states (pure or mixed) $\sigma_1,\sigma_2,\ldots,\sigma_N$ can be perfectly discriminated by LOCC, then the number of states is bounded by

$$N \leqslant D/\overline{d(\sigma_i)} \leqslant D/\overline{r(\sigma_i)} \leqslant D/\overline{2^{E(\sigma_i)}} \leqslant D/\overline{2^{G(\sigma_i)}}, \quad (1)$$

where D is the dimension of the Hilbert space of the composite quantum system; $d(\sigma)$ is a quantity (to be defined later) resembling distance to the nearest separable state; $r(\sigma)=\text{rank}(\rho)[1+R_g(\rho)]$, where ρ is the normalized projector onto the support [20] of σ and $R_g(\rho)$ is the robustness of entanglement [21]; $E(\sigma_i)=E_R(\sigma_i)+S(\sigma_i)$, where $E_R(\sigma)$ is the relative entropy [22] and $S(\sigma)$ is the von Neumann entropy; $G(\sigma)$ is the geometric measure [14,23]; and $\overline{x_i}=\frac{1}{N}\sum_{i=1}^{N}x_i$ denotes the average. If the inequality is violated for any of the bounding quantities, then we can certainly conclude that the given set of states cannot be perfectly locally distinguished. However, if the inequality is satisfied, then no such definite conclusion can be drawn.

For pure states the bounding quantities (from right to left) correspond to well-defined distance-like entanglement measures, namely, geometric measure, relative entropy, and robustness of entanglement, thereby allowing a clear interpretation: the number of pure states that can be perfectly distinguished by LOCC is bounded by the total dimension over average entanglement. This therefore clarifies the matter to a great extent for pure states. For mixed states, however, no such clear conclusion can be drawn, and the role of entanglement still remains unclear. Applications of inequality (1) for LOCC discrimination of interesting multipartite ensembles having certain group symmetries can be found in Ref. [15].

The purpose of the present work is to investigate how local distinguishability of a given set of orthogonal mixed states depend on entanglement and mixedness of the states. We first show that local distinguishability of mixed states can be completely characterized by local distinguishability of their supports. In particular, we establish a simple equivalence between local discrimination of orthogonal states and subspaces in the sense that a given set of density matrices can be perfectly distinguished by LOCC if and only if their supports are also perfectly locally distinguishable, and moreover, if the states can be perfectly distinguished, then the separable measurement that distinguishes the states also distinguishes the supports and vice versa. We use this fact to obtain the following results:

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(a) state-specific properties such as inseparability and mixedness of the density matrices (whose local distinguishability is under consideration) do not have any special role in determining their local distinguishability, (b) local distinguishability of mixed states may be completely determined by maximal pure-state entanglement within their supports, and (c) the number of LOCC distinguishable orthogonal mixed states can be bounded by the quantities that are optimized over all orthogonal mixed-state ensembles having identical supports. We now briefly discuss results (a)–(c).

For result (a), we show that the state-specific properties such as inseparability and mixedness of the given mixed states do not have any fundamental role in determining their local distinguishability. To see this, suppose $\mathbb{S} = \{\sigma_1,\sigma_2,\ldots,\sigma_N\}$ is a set of orthogonal density matrices whose local distinguishability is under question. Let $\{s_1,s_2,\ldots,s_N\}$ be the set of orthogonal subspaces where s_i is the support of σ_i. It is clear that infinitely many sets like $\mathbb{S}$ exist, where each set contains N orthogonal density matrices with the property that s_i is the support of its ith element. Let $\mathbb{Q}$ be the collection of all such sets; that is, $\mathbb{Q} = \{\mathbb{S},\mathbb{S}',\ldots\}$. We then show that either every set in $\mathbb{Q}$ is perfectly distinguishable by LOCC or none of them are, regardless of how entangled or mixed the states in a given set are. That is, if $\mathbb{S}$ is perfectly distinguishable by LOCC, then so is any set, say $\mathbb{S}' \in \mathbb{Q}$ and vice versa, even though average entanglement or mixedness could be very different for the states in $\mathbb{S}$ and $\mathbb{S}'$ (for instance, the density matrices in $\mathbb{S}$ may be highly entangled, whereas the density matrices in $\mathbb{S}'$ may be very weakly entangled). We call this property subspace degeneracy. Thus the state-specific properties of the density matrices σ_i do not have any special role as far as their local distinguishability is concerned.

For results (b) and (c), we use the above observations to present upper bounds on the number of perfectly locally distinguishable orthogonal mixed states. In particular, we obtain two kinds of upper bounds. The first one shows that the number of orthogonal density matrices that can be perfectly distinguished locally is bounded above by the total dimension over the average of maximal pure state entanglement in the supports of the density matrices. This bound is not necessarily optimal but depends only on pure-state entanglement within the supports of the states and therefore may be easy to compute in many instances. This shows that local distinguishability of mixed states may be determined by pure-state entanglement alone. The second bound is optimal in the sense that it optimizes the bounding quantities over all orthogonal ensembles (satisfying certain conditions) admissible within the supports of the density matrices.

II. NECESSARY CONDITIONS FOR PERFECT LOCC STATE DISCRIMINATION

Let $\mathcal{H}$ be the Hilbert space of a composite quantum system and $D = \dim \mathcal{H}$. Throughout this paper we consider only finite-dimensional systems. We note that any measurement realized by LOCC is separable (the converse is not true [4]). A separable measurement $\Pi = \{\Pi_1,\Pi_2,\ldots,\Pi_n\}$ on $\mathcal{H}$ is a positive operator-valued measure (POVM) satisfying $\sum_{i=1}^{n} \Pi_i = \mathcal{I}_{\mathcal{H}}$, where Π_i is a separable, positive semidefinite operator for every i. Therefore, if a set of quantum states is perfectly distinguishable by LOCC, then there exists a separable measurement distinguishing the states. For a necessary and sufficient condition for perfect discrimination by separable measurements, see Ref. [16]. We now state two necessary conditions for perfect LOCC state discrimination. The first condition and its variants can be found in Refs. [12–14,16,19] and the second condition is due to Ref. [14].

Proposition 1. If the orthogonal quantum states $\sigma_1,\sigma_2,\ldots,\sigma_N$ are perfectly distinguishable by LOCC, then it is necessary that there exists a separable POVM $\Pi = \{\Pi_1,\Pi_2,\ldots,\Pi_N\}$ such that

$$\mathrm{Tr}(\Pi_i \sigma_j) = \delta_{ij}. \tag{2}$$

Proposition 2. A necessary condition for perfect LOCC discrimination of the states $\sigma_1,\sigma_2,\ldots,\sigma_N$ by a separable POVM $\Pi = \{\Pi_1,\Pi_2,\ldots,\Pi_N\}$ is that the following inequality is satisfied:

$$\sum_{i=1}^{N} d(\sigma_i) \leqslant D, \tag{3}$$

where $d(\sigma_i) := \min \frac{\mathrm{Tr}(\Pi_i)}{\mathrm{Tr}(\sigma_i \Pi_i)}$ such that $0 \leqslant \frac{\Pi_i}{\mathrm{Tr}(\sigma_i \Pi_i)} \leqslant \mathcal{I}$.

Let us remark that the above necessary condition is particularly useful for bounding the number of states that can be perfectly discriminated by LOCC [14].

III. RESULTS

A. Local discrimination of orthogonal subspaces

We first explain what we mean by LOCC discrimination of orthogonal subspaces (for discrimination of nonorthogonal subspaces using global measurements, see Ref. [24]). In local discrimination of orthogonal subspaces, a pure quantum state shared between several observers is guaranteed to belong to a subspace chosen from a known collection of orthogonal subspaces. The goal is to determine by LOCC to which subspace the state belongs without making any error. We assume that within each subspace each state is equally likely and so are the subspaces. We will say that the subspaces $\{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_k\}$ are perfectly locally distinguishable if we can perfectly distinguish the set of density matrices $\{\rho_1,\rho_2,\ldots,\rho_k\}$ by LOCC, where ρ_i is the normalized projector onto the subspace $\mathcal{S}_i$. Clearly, the problem of local discrimination of orthogonal subspaces is a special case of the general problem. We begin with a simple but useful lemma.

Lemma 1. If the orthogonal subspaces $\{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_k\}$ are perfectly LOCC distinguishable, then so are the density matrices $\{\omega_1,\omega_2,\ldots,\omega_k\}$, where, $\omega_i \in \mathcal{S}_i$.

Proof. That the subspaces $\{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_k\}$ are perfectly LOCC distinguishable means that the set of orthogonal density matrices $\{\rho_1,\rho_2,\ldots,\rho_k\}$, the normalized projectors onto the subspaces, can be perfectly distinguished. Thus there exists a locally implementable separable POVM $\Pi = \{\Pi_1,\Pi_2,\ldots,\Pi_k\}$ such that $\mathrm{Tr}(\Pi_i \rho_j) = \delta_{ij}$. Denoting $\rho_j = \frac{1}{\dim \mathcal{S}_j}\Lambda_j$, where Λ_j is the projection operator onto $\mathcal{S}_j$, we get

$$\frac{1}{\dim \mathcal{S}_j}\mathrm{Tr}(\Pi_i \Lambda_j) = \delta_{ij}; \tag{4}$$

thus for $i \neq j$ the POVM elements $\{\Pi_i\}$ are all orthogonal to the subspace $\mathcal{S}_j$. Now for any density matrix Δ and a POVM $\mathcal{M} = \{\mathcal{M}_i : i = 1,\ldots,k\}$ the relation

$$\sum_i \mathrm{Tr}(\mathcal{M}_i \Delta) = 1 \tag{5}$$

is valid. The summation indicates that the sum of the probabilities must add up to 1 when the measurement $\mathcal{M}$ is performed on the state Δ, where $\mathrm{Tr}(\mathcal{M}_i \Delta)$ is the probability of obtaining outcome i. Suppose the POVM Π is implemented on the given state chosen from $\{\omega_1,\omega_2,\ldots,\omega_k\}$. We therefore have

$$\sum_i \mathrm{Tr}(\Pi_i \omega_j) = 1, \tag{6}$$

where $\mathrm{Tr}(\Pi_i \omega_j)$ is the probability of obtaining outcome i when the input state is ω_j. Because $\omega_j \in \mathcal{S}_j$, ω_j must be orthogonal to all POVM elements $\Pi_i; i \neq j$. Therefore,

$$\mathrm{Tr}(\Pi_i \omega_j) = \delta_{ij}. \tag{7}$$

Thus the POVM Π also perfectly distinguishes the states $\{\omega_1,\omega_2,\ldots,\omega_k\}$. ■

B. Equivalence of LOCC discrimination of states and subspaces

For a given set of orthogonal density matrices $\{\rho_i : i = 1,\ldots,N\}$, consider the set of subspaces $\{\mathcal{S}_1,\ldots,\mathcal{S}_N\}$, where $\mathcal{S}_i$ is the support of ρ_i. The orthogonality of the density matrices implies that the supports are orthogonal. We now show that the problems of local discrimination of orthogonal states and subspaces are equivalent in the following sense.

Theorem 1. The density matrices $\rho_1,\ldots,\rho_N$ are perfectly distinguishable by LOCC if and only if their supports are. Moreover, if the states are perfectly distinguishable, then the measurement that distinguishes the states also distinguishes their supports and vice versa.

Proof. Let $\Pi = \{\Pi_i : i = 1,\ldots,N\}$ be the POVM that perfectly distinguishes the set of density matrices $\{\rho_i : i = 1,\ldots,N\}$ by LOCC. Therefore, $\mathrm{Tr}(\Pi_i \rho_j) = \delta_{ij}$. Let s_j be the support of ρ_j and $\varrho_j = \frac{1}{|\mathcal{P}_j|}\mathcal{P}_j$, where $\mathcal{P}_j$ is the projector onto the subspace s_j and $|\mathcal{P}_j| = \dim s_j$. To prove that the POVM Π also perfectly distinguishes the subspaces $\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_N$, we only need to show that $\mathrm{Tr}(\Pi_i \varrho_j) = \delta_{ij}$. That the POVM is locally implementable holds by the assumption that it perfectly distinguishes $\{\rho_i\}$. Consider first the diagonal decomposition of ρ_j:

$$\rho_j = \sum_{l=1}^{|\mathcal{P}_j|} p_l^j |\phi_l^j\rangle\langle\phi_l^j|. \tag{8}$$

From Eq. (5) for every l we have

$$\sum_i \mathrm{Tr}\big(\Pi_i |\phi_l^j\rangle\langle\phi_l^j|\big) = 1. \tag{9}$$

Also $\mathrm{Tr}(\Pi_i \rho_j) = \delta_{ij}$ implies that all POVM elements Π_i ($i \neq j$) are orthogonal to the states $\{|\phi_l^j\rangle : l = 1,\ldots,d_j\}$. Using this fact, the above equation reduces to

$$\mathrm{Tr}\big(\Pi_i |\phi_l^j\rangle\langle\phi_l^j|\big) = \delta_{ij} : \forall l. \tag{10}$$

Noting that ϱ_j, the normalized projector onto the subspace s_j, can be written as

$$\varrho_j = \frac{1}{|\mathcal{P}_j|} \sum_{l=1}^{|\mathcal{P}_j|} |\phi_l^j\rangle\langle\phi_l^j|, \tag{11}$$

we immediately obtain $\mathrm{Tr}(\Pi_i \varrho_j) = \delta_{ij}$ using Eqs. (10) and (11). Thus the subspaces can be perfectly distinguished, and the POVM Π distinguishes them. The rest of the proof, namely, the POVM that perfectly distinguishes the orthogonal subspaces $\{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_N\}$ also distinguishes the orthogonal density matrices $\{\rho_1,\ldots,\rho_N\}$, follows from Lemma 1. ■

The condition in Theorem 1, though remarkably simple and intuitive, is able to capture the essence of local state discrimination and, in particular, the role of entanglement therein. In particular, Theorem 1 leads to what we call "subspace degeneracy," which is discussed in the next section.

C. Subspace degeneracy

As noted in the Introduction, intuitively, subspace degeneracy means that any given set $\mathbb{S}$ of orthogonal density matrices belongs to a collection of infinitely many sets having identical distinguishability properties no matter how different the average entanglement of the individual sets are. For a given set $\mathbb{S} = \{\sigma_i : i = 1,\ldots,N\}$ whose local distinguishability is under consideration, consider another orthogonal set $\mathbb{S}' = \{\sigma_i' : i = 1,\ldots,N\}$ with the property that for every i, σ_i and σ_i' have identical support. Let $\mathbb{Q} = \{\mathbb{S}'\}$ be the collection of all such orthogonal sets $\mathbb{S}'$. Clearly, $\mathbb{S}$ is also a member of $\mathbb{Q}$. In other words, given a set of orthogonal subspaces $S = \{s_i : i = 1,\ldots,N\}$, $\mathbb{Q}$ is simply the collection of only those sets $\mathbb{S}' = \{\sigma_i' : i = 1,\ldots,N\}$ with the properties that the for every i, $\sigma_i' \in s_i$ and $\mathrm{rank}(\sigma_i') = \dim s_i$. By simple application of Theorem 1 we obtain the next result.

Proposition 3. All orthogonal sets in $\mathbb{Q}$ are either perfectly LOCC distinguishable or none of them are, regardless of the average entanglement of the individual sets. Furthermore, if the sets can be perfectly distinguished by LOCC, then there is a separable measurement $\Pi_{\mathbb{Q}}$ that distinguishes every set in $\mathbb{Q}$.

Simply put, no matter how different the average entanglement of the sets might be, as far as perfect local distinguishability is concerned, they are either equally hard or equally easy to distinguish. This is in sharp contrast to pure states, where the result in Ref. [14] implies different upper bounds for pure ensembles of the same cardinality but having different average entanglement. Thus, unlike pure states, there cannot be any direct correlation between entanglement (under any reasonable measure) of the states and their local distinguishability. Furthermore, entanglement or mixedness of the states in $\mathbb{S}$ is not crucial in determining whether $\mathbb{S}$ can be perfectly distinguished or not by LOCC. We use this fact (Proposition 3) to obtain two kinds of upper bounds on the number of perfectly LOCC distinguishable orthogonal states.

D. Upper bounds on the number of perfectly LOCC distinguishable orthogonal states

In this section we give two kinds of upper bounds on the number of perfectly LOCC distinguishable density matrices. In the first one, the bounding quantities depend only on the maximal pure-state entanglement in the supports of the density matrices, whereas in the second we use Proposition 3 to optimize the bounding quantities over all sets in $\mathbb{Q}$.

We first show how local distinguishability of a set of orthogonal density matrices can be related to the local distinguishability of a set of orthogonal pure states satisfying certain conditions. For the orthogonal density matrices $\sigma_1,\ldots,\sigma_N$, let $S=\{|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_N\rangle\}$ be a collection of orthogonal pure states such that for every i, $|\psi_i\rangle\in s_i$, where s_i is the support of σ_i.

Proposition 4. If the states $|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_N\rangle$ are not perfectly distinguishable by LOCC, then the density matrices $\sigma_1,\ldots,\sigma_N$ cannot be perfectly distinguished by LOCC.

Proof. The proof of the second statement is by contradiction. Suppose states $|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_N\rangle$ cannot be perfectly distinguished by LOCC but there is a LOCC protocol that perfectly distinguishes the density matrices $\sigma_1,\ldots,\sigma_N$. From Theorem 1 we know that if the density matrices $\sigma_1,\ldots,\sigma_N$ are perfectly LOCC distinguishable, then one can also perfectly distinguish the orthogonal subspaces $s_1,s_2,\ldots,s_N$, where s_i is the support of σ_i. This implies, by Lemma 1, that the states $|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_N\rangle$ can also be perfectly distinguished locally because for every i, $|\psi_i\rangle\in s_i$, which contradicts our assumption. ■

It is important to note that if the states $|\psi_{1}\rangle,|\psi_2\rangle,\ldots,|\psi_N\rangle$ can be perfectly distinguished locally, then it does *not* mean that the density matrices $\sigma_1,\ldots,\sigma_N$ can also be reliably distinguished. For example, consider the following density matrices in $2\otimes 2$: $\sigma_1=\alpha|\Phi^+\rangle\langle\Phi^+|+(1-\alpha)|01\rangle\langle 01|$ and $\sigma_2=\beta|\Phi^-\rangle\langle\Phi^-|+(1-\beta)|10\rangle\langle 10|$, where $\alpha\neq 0$ and $\beta\neq 0$. Clearly, $s_1=\text{span}\{|\Phi^+\rangle,|01\rangle\}$ and $s_2=\text{span}\{|\Phi^-\rangle,|10\rangle\}$. While any two orthogonal vectors $|\psi_1\rangle,|\psi_2\rangle$, where $|\psi_i\rangle\in s_i$ for $i=1,2$, are perfectly LOCC distinguishable, the density matrices σ_1 and σ_2 are not. The reason is that neither of the subspaces can be spanned only by product states, thereby violating a necessary condition for perfect local discrimination by separable measurements [16].

To arrive at our upper bound we will use the previous proposition and inequality (1). For a set of orthogonal pure states $\{|\phi_1\rangle,|\phi_2\rangle,\ldots,|\phi_N\rangle\}$ inequality (1) becomes

$$N\leqslant\frac{D}{\overline{1+R(|\phi_i\rangle)}}\leqslant\frac{D}{\overline{2^{E_R(|\phi_i\rangle)}}}\leqslant\frac{D}{\overline{2^{E_g(|\phi_i\rangle)}}},\tag{12}$$

where the corresponding bounding quantities have been defined before.

Now, if the states $|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_N\rangle$ as defined in Proposition 4 violate the above inequality, then we can certainly conclude that the density matrices $\sigma_1,\ldots,\sigma_N$ are not perfectly distinguishable by LOCC. Therefore, if the density matrices $\sigma_1,\sigma_2,\ldots,\sigma_N$ are perfectly LOCC distinguishable, then the following inequality holds:

$$N\leqslant\frac{D}{\overline{1+R(|\psi_i\rangle)}}\leqslant\frac{D}{\overline{2^{E_R(|\psi_i\rangle)}}}\leqslant\frac{D}{\overline{2^{E_g(|\psi_i\rangle)}}}.\tag{13}$$

Note that the inequality may still be satisfied even if the density matrices are locally indistinguishable. The example given after Proposition 4 conforms to this fact. The crucial point is that if the density matrices are locally distinguishable, then the inequality will not be violated.

Naturally, we would like to maximize the bounding quantities over all orthogonal pure state ensembles like S. Let $S_{\max}=\{|\Psi_1\rangle,|\Psi_2\rangle,\ldots,|\Psi_N\rangle\}$ be the set of orthogonal pure states with the properties that, for every i, (a) $|\Psi_i\rangle\in s_i$ and (b) $R(|\Psi_i\rangle)=\max_{\psi\in s_i}R(|\psi\rangle)$. The first condition ensures that the states belong to the supports of the density matrices, so that Proposition 4 is applicable. The second condition reflects the fact that for every i, $|\Psi_i\rangle$ is the state with maximum pure state entanglement in the support of σ_i. Thus by replacing the pure state ensemble S by $S_{\max}$ in Eq. (13) we have the following result.

Theorem 2. If the density matrices $\sigma_1,\sigma_2,\ldots,\sigma_N$ are perfectly LOCC distinguishable, then

$$N\leqslant\frac{D}{\overline{1+R(|\Psi_i\rangle)}}\leqslant\frac{D}{\overline{2^{E_R(|\Psi_i\rangle)}}}\leqslant\frac{D}{\overline{2^{E_g(|\Psi_i\rangle)}}},\tag{14}$$

where, for every i, $|\Psi_i\rangle\in s_i$, $R(\Psi_i)=\max_{\psi\in s_i}R(|\psi\rangle)$.

Let us note that an exact analytical formula for robustness R is known for pure bipartite states [21]. Therefore, for any given set of bipartite orthogonal density matrices, the upper bound can be explicitly calculated (one needs to optimize to get the best possible bound). Inequality (14) shows that mixed states also admit pure-state-like correlation between entanglement and the number of locally distinguishable states. This allows us to make a general statement on the connection between entanglement and local distinguishability: The number of perfectly LOCC distinguishable quantum states, pure or mixed, is bounded above by the total dimension over the average of maximal pure-state entanglement in the supports of the states. It is, however, important to note that the quantity, second from left, in inequality (1) is always stronger than the leftmost quantity, in Eq. (14) [25].

Our second bound can be considered to be the optimized version of the general mixed-state bound given by (1). From Proposition 3 we know that if set $\mathbb{S}$ is perfectly LOCC distinguishable, then so is any set $\mathbb{S}'\in\mathbb{Q}$, and the measurement that distinguishes $\mathbb{S}$ also distinguishes any $\mathbb{S}'$ and vice versa. Noting that the sets are of same cardinality, it simply follows that an upper bound on the number of perfectly LOCC distinguishable states for any $\mathbb{S}'$ is also an upper bound for $\mathbb{S}$. The optimal bound is thus obtained by maximizing the bounding quantities over all $\mathbb{S}'$.

For the given set $\mathbb{S}=\{\sigma_i;i=1,\ldots,N\}$, let Q_i be the set of all density matrices in s_i having rank equal to $\dim s_i$, where s_i is the support of σ_i. Define the following quantities:

$$\mathcal{R}_i=\max_{\sigma''\in Q_i}\mathcal{R}(\sigma''),$$
$$\mathcal{E}_i=\max_{\sigma''\in Q_i}[E_R(\sigma'')+S(\sigma'')],$$
$$\mathcal{G}_i=\max_{\sigma''\in Q_i}G(\sigma''),$$

where $\mathcal{R}(\sigma'') := \alpha^{-1}[1 + R_g(\sigma'')]$, with α being the maximum eigenvalue of σ'', $R_g(\sigma'')$ is the *global* robustness of entanglement [21], $E_R(\sigma'')$ is the relative entropy [22], $S(\sigma'')$ is the von Neumann entropy, and $G(\sigma'')$ is the geometric measure [14].

Theorem 3. If the set of states $\mathbb{S} = \{\sigma_i; i = 1,\ldots,N\}$ is perfectly distinguishable by LOCC, then the number of states is bounded by

$$N \leqslant D/\overline{d(\sigma_i)} \leqslant D/\overline{\mathcal{R}_i} \leqslant D/\overline{2^{\mathcal{E}_i}} \leqslant D/\overline{2^{\mathcal{G}_i}}, \tag{15}$$

where $\overline{x_i} = \frac{1}{N}\sum_{i=1}^{N} x_i$ denotes the average.

Proof. It is sufficient to show that

$$d(\sigma_i) \geqslant \mathcal{R}_i \geqslant \mathcal{E}_i \geqslant \mathcal{G}_i. \tag{16}$$

Inequality (15) is then obtained by combining Proposition 2 and the above inequality and dividing by N. The proof follows along lines very similar to that in Ref. [14].

It was shown in Ref. [14] that we can write $d(\sigma_i) = \min(\frac{1}{\lambda})$ such that $\exists\, \varrho'$, satisfying

$$\tilde{\Pi}_i = \lambda_i \mathcal{P}_i + (1 - \lambda_i|\mathcal{P}_i|)\,\varrho' \in \mathfrak{S} \tag{17}$$

along with the conditions

$$\mathrm{Tr}(\mathcal{P}_i \varrho') = 0, \tag{18}$$

$$\langle\phi|\tilde{\Pi}_i|\phi\rangle \geqslant \lambda \ \forall|\phi\rangle, \tag{19}$$

where $\tilde{\Pi}_i = \Pi_i/\mathrm{Tr}(\Pi_i)$, $\mathcal{P}_i$ is the projector onto s_i (the support of σ_i), $|\mathcal{P}_i| = \dim s_i$, and $\mathfrak{S}$ is the set of separable density matrices. Recall that Q_i is the set of all density matrices in s_i having rank equal to $|\mathcal{P}_i|$. Now observe that for any density matrix $\sigma'' \in Q_i$, we have $d(\sigma'') = d(\sigma_i)$ [26]. Now σ'' can be expressed as

$$\sigma'' = \alpha\mathcal{P}_i - \beta\sigma''', \tag{20}$$

where, α is the maximum eigenvalue of σ'', $\beta = |\mathcal{P}_i|\alpha - 1$, and $\sigma''' \in s_i$. Equation (17) can therefore be rewritten in the form,

$$\tilde{\Pi}_i = \frac{\lambda_i}{\alpha}(\sigma'' + \gamma\varrho'') \in \mathfrak{S} \tag{21}$$

where $\gamma = (\alpha - \lambda_i)/\lambda_i$. Noting that the generalized (or global) robustness of entanglement [21] of $R_g(\sigma)$ of any state σ is defined by $R_g(\sigma) = \min t$ such that there exists a state ϱ satisfying

$$\frac{1}{1+t}(\sigma + t\varrho) \in \varsigma, \tag{22}$$

where ς is a separable state, we immediately obtain $R_g(\sigma'') \leqslant \gamma$. Thus, for any $\sigma'' \in Q_i$, we have

$$d(\sigma'') \geqslant \alpha^{-1}[1 + R_g(\sigma'')] \tag{23}$$

Thus,

$$d(\sigma_i) \geqslant \mathcal{R}_i = \max_{\sigma'' \in Q_i} \mathcal{R}(\sigma''), \tag{24}$$

where $\mathcal{R}(\sigma'') := \alpha^{-1}[1 + R_g(\sigma'')]$. The rest of the proof is straightforward. It is easy to show that for any density matrix $\sigma'' \in Q_i$, the following inequality holds:

$$r(\sigma_i) \geqslant \mathcal{E}(\sigma'') \geqslant \mathcal{G}(\sigma''). \tag{25}$$

Thus we obtain

$$\mathcal{R}_i \geqslant r(\sigma_i) \geqslant \mathcal{E}_i = \max_{\sigma'' \in Q_i} \mathcal{E}(\sigma'') \geqslant \mathcal{G}_i = \max_{\sigma'' \in Q_i} \mathcal{G}(\sigma''). \tag{26}$$

Combining Eqs. (24) and (26), we get Eq. (16). This concludes the proof. ■

A few remarks are in order.

(i) By construction, for every bounding quantity, $\mathcal{R}$, $\mathcal{E}$, and $\mathcal{G}$, there always exists a set of orthogonal quantum states $\mathbb{S}' = \{\sigma_i'; i = 1,...,N\} \in \mathbb{Q}$ maximizing it, which is the essence of the entire optimality argument. For example, one can construct an orthogonal set $\mathbb{S}'_{\mathcal{R}}(\sigma') = \{\sigma_i'; i = 1,\ldots,N\} \in \mathbb{Q}$, such that for every i, $\mathcal{R}_i = \mathcal{R}(\sigma_i')$, and similarly for the quantities $\mathcal{E}$ and $\mathcal{G}$.

(ii) A nice feature of the above inequality is that the hierarchical form holds even when the bounding quantities are independently maximized (this is clear from the proof), and different sets may maximize different quantities.

(iii) For any set $\mathbb{S}'' = \{\sigma_i''; i = 1,\ldots,N\} \in \mathbb{Q}$, the following inequality holds [inequality (15) is simply the optimized version of the following one]:

$$N \leqslant D/\overline{d(\sigma_i)} \leqslant D/\overline{\mathcal{R}(\sigma'')} \leqslant D/\overline{2^{\mathcal{E}(\sigma'')}} \leqslant D/\overline{2^{\mathcal{G}(\sigma'')}}. \tag{27}$$

IV. CONCLUSIONS

We have considered the problem of local distinguishability of orthogonal mixed states. In particular, we have investigated how entanglement and mixedness of the states influence their local distinguishability. We have shown a general equivalence between local discrimination of orthogonal states and subspaces which in turn implies that local distinguishability of mixed states is completely determined by whether or not their supports are also locally distinguishable. This led to the following results: (a) state-specific properties such as inseparability and mixedness of the density matrices do not have any special role in determining their local distinguishability, (b) local distinguishability of mixed states may be completely determined by maximal pure-state entanglement within their supports, and (c) an upper bound on the number of perfectly locally distinguishable orthogonal mixed states is given where the bounding quantities are optimized over all orthogonal mixed-state ensembles having identical supports.

Although the results obtained in this paper and in Ref. [14] show that entanglement is a significant factor in local distinguishability, many questions still remain open. For example, there are orthogonal product states known to be locally indistinguishable [4,5], despite being completely unentangled. Whether there is a deeper reason behind this phenomena or it is just a consequence of the fact that not all separable measurements are locally implementable is not known yet. Obviously, entanglement of the states is a nonissue here, but entanglement could still be important because, to implement such separable measurements by LOCC, one is expected to consume auxiliary entanglement. Thus, it is necessary to quantify the entanglement cost of such separable measurements. Another interesting class of states requiring further investigation are those which, despite being entangled and locally indistinguishable, do not violate the inequalities presented in this paper or in Ref. [14]. All these examples show that there is more to local distinguishability of quantum states than what can be captured through entanglement only.

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[26] Note that the definition of $d(\sigma)$ involves minimization over all separable POVMs satisfying Eq. (2). From Theorem 2 we know that the POVM that distinguishes $\mathbb{S}$ also distinguishes any $\mathbb{S}' \in \mathbb{Q}$.