Optimal uncertainty relations for extremely coarse-grained measurements

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We derive two quantum uncertainty relations for position and momentum coarse-grained measurements. Building on previous results, we first improve the lower bound for uncertainty relations using the Rényi entropy, particularly in the case of coarse-grained measurements. We then sharpen a Heisenberg-like uncertainty relation derived previously in [Europhys. Lett. **97**, 38003 (2012)] that uses variances and reduces to the usual one in the case of infinite precision measurements. Our sharpened uncertainty relation is meaningful for any amount of coarse graining. That is, there is always a nontrivial uncertainty relation for coarse-grained measurement of the noncommuting observables, even in the limit of extremely large coarse graining.

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I. INTRODUCTION

Uncertainty relations play a central role in quantum physics. Generally, these are inequalities that limit the amount of information that can be obtained about noncommuting observables for identically prepared systems. From a fundamental point of view, the fact that measurements on quantum systems must obey uncertainty relations distinguishes them from their classical counterparts. Since all physical quantum states must obey them, uncertainty relations provide a method to check the validity of an inferred quantum state that was reconstructed from tomographic measurements. Uncertainty relations also play an important role in applications in quantum information science. In particular, the security of several quantum cryptography protocols are founded on uncertainty relations [1,2], as are several tests for detection of quantum entanglement [3–8] and Einstein-Podolsky-Rosen (EPR) steering correlations [9–12].

Historically, the most renowned uncertainty relation is

$$\sigma_x^2\sigma_p^2 \geqslant \frac{1}{4}|\langle[\hat{x},\hat{p}]\rangle|^2 = \frac{\hbar^2}{4}, \tag{1}$$

for position $\hat{x}$ and momentum $\hat{p}$ observables. Its existence was first suggested in Ref. [13] and since then has been called the Heisenberg uncertainty relation (HUR). It was proved by Kennard in Ref. [14], and, later, Robertson [15] extended its validity to arbitrary pairs of observables $\hat{A}$ and $\hat{B}$. The HUR of Eq. (1) applies to the variances $\sigma_x^2 \equiv \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2$ and $\sigma_p^2 \equiv \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2$, which quantify the uncertainty in position and momentum measurements, respectively. Here $\langle\ldots\rangle \equiv \mathrm{tr}[\hat{\varrho}\ldots]$ is the expectation value when the system is described by the quantum state $\hat{\varrho}$. This inequality follows from the fact that the complementary operators do not commute: $[\hat{x},\hat{p}] = i\hbar$. A number of additional uncertainty relations for position and momentum have been presented [16–24].

It is important to distinguish the uncertainty relation in Eq. (1) from the similar ones that appear (i) in the context of joint measurement of position and momentum [25,26] and (ii) in the error-disturbance process of position measurement [27,28]. In case (i) we have the *Heisenberg uncertainty relation for joint measurement* and in case (ii) the *Heisenberg noise-disturbance uncertainty relation*, which establishes a lower bound between the product of the estimate of a position measurement error and the estimate of the resulting disturbance in the momentum. Reference [27] provides an analysis of the connection between these two types of uncertainty relations and Ref. [29] describes an alternative approach to the trade-off relation between the measurement errors of noncommuting observables. For case (i), the joint measurement of position and momentum, it was shown in Ref. [25] that, independently of the strategy used to perform the joint measurement, the lower bound of the product of variances is incremented by a factor of 2. This is due to the fact that joint measurement of the original noncommuting variables requires that each one must interact with different "meter" variables that have to commute themselves in order to be jointly measurable (in principle with arbitrary high precision). It is this coupling of the original variables with the meter variables that introduce the additional noise.

Uncertainty relations are generally interpreted to express limits to the amount of information that one can obtain about complementary properties of a quantum system prepared in a given quantum state and, thus, naturally invoke the notion of measurement. However, most uncertainty relations, such as in Eq. (1), assume perfect knowledge of the quantities involved, which can only be obtained experimentally with infinite precision measurements. However, in any experiment, measurements are performed with a finite precision, and, thus, any consistent uncertainty relation applicable to "real-world" scenarios involving statistics of measurement results should include aspects of the measurement process. Several authors have considered finite precision, or "coarse-grained," measurements in the context of uncertainty relations involving the Shannon [18,19] and Rényi entropy functions [22]. These relations are meaningful up to a certain amount of coarse graining, after which they are trivially satisified [30].

The analysis of imprecision (coarse-grained measurement) in the context of joint measurement of position and momentum [case (i) above] was given in Ref. [26], and a HUR-type uncertainty relation for joint measurement was obtained for sufficiently small values of the resolution of the detectors. It is important to note that in this HUR inequality the joint

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measurement necessarily relates the widths of the resolutions of position and momentum measurements. The coarse-grained version of the HUR uncertainty relation in Eq. (1) that arises not from joint measurement but rather from the statistical results of the measurement of position and momentum in an identically prepared system was recently obtained in Ref. [30]. This inequality, in contrast to the one in Ref. [26], is indeed valid for arbitrary and independent values of the widths of the position and momentum detectors (although for large-enough values of coarse graining the inequality is trivially satisfied).

It is generally believed, and taught in many quantum-mechanics textbooks [31,32], that the quantum nature of a system is not observable when coarse-grained measurements are performed. This has been shown in the context of Bell's inequality violations [33] and for the precession of a single spin-j particle [34]. Thus, it seems natural to expect that uncertainty relations should at some point fail to be significant with an increasing amount of coarse graining. However, here we derive new quantum-mechanical uncertainty relations and show through optimization that coarse-grained measurements are *always* limited by some quantum uncertainty relation. We consider the HUR and a family of uncertainty relations using the Rényi entropy under coarse-grained sampling. First, we show that one can construct reliably estimated probability distributions for coarse-grained measurements in order to obtain reliable uncertainty relations. We provide an improved lower bound for the Bialynicki-Birula uncertainty relations [22] using discrete Rényi entropies and use this lower bound to derive a new Heisenberg-like uncertainty relation for coarse-grained measurements. By optimization, we provide a Heisenberg-like uncertainty relation that is meaningful for *any amount of coarse graining*. That is, the amount of information that can be obtained by coarse-grained measurements of noncommuting observables is always limited. We also prove that this optimized uncertainty relation reduces to the usual one in the limit of infinite precision measurements.

This paper is organized as follows. In Secs. II and III we introduce several definitions and previously established uncertainty relations. Section IV provides a method for constructing continuous probability density functions from coarse-grained measurements. In Sec. V we derive an improved lower bound for the Bialynicki-Birula uncertainty relation for discrete Rényi entropies. We show that this relation is optimal in the case $\alpha = 1/2$ and provides an improvement for the case of the Shannon entropy ($\alpha = 1$) for large coarse graining. In Sec. VI we derive the HUR for coarse graining. Applying these new results for entropic relations, we arrive at the HUR that restricts measurements for any value of coarse graining.

II. UNCERTAINTY RELATIONS ASSOCIATED WITH CONTINUOUS PROBABILITY DISTRIBUTIONS OF POSITION AND MOMENTUM VARIABLES

For a quantum state described by a density operator $\hat{\varrho}$, the probability densities describing measurements of $\hat{x}$ and $\hat{p}$ are given by

$$\rho(x) = \langle x|\hat{\varrho}|x\rangle \quad \text{and} \quad \tilde{\rho}(p) = \langle p|\hat{\varrho}|p\rangle. \tag{2}$$

Since $\rho(x)$ and $\tilde{\rho}(p)$ are probability distributions corresponding to a quantum state, they obey uncertainty relations such as the HUR of Eq. (1), which involves variances that can be calculated from continuous probability distributions $\rho(x)$ and $\tilde{\rho}(p)$ as

$$\sigma_z^2[f] \equiv \int_{\mathbb{R}} dz\, z^2 f(z) - \left(\int_{\mathbb{R}} dz\, z\, f(z)\right)^2, \tag{3}$$

where $f = \rho, \tilde{\rho}$ and $z = x, p$. Probability densities $\rho(x)$ and $\tilde{\rho}(p)$ will also obey the uncertainty relation for the continuous Rényi entropies ($1/\alpha + 1/\beta = 2$, $\beta \geqslant 1$) [22,35],

$$h_\alpha[\rho] + h_\beta[\tilde{\rho}] \geqslant -\frac{1}{2(1-\alpha)} \ln\left(\frac{\alpha}{\pi\hbar}\right) - \frac{1}{2(1-\beta)} \ln\left(\frac{\beta}{\pi\hbar}\right), \tag{4}$$

where the continuous Rényi entropy is defined as [36]

$$h_\lambda[f] \equiv \frac{1}{1-\lambda} \ln\left(\int_{\mathbb{R}} dz\, [f(z)]^\lambda\right). \tag{5}$$

The limit $\lambda \to 1$ corresponds to the continuous Shannon entropy

$$\lim_{\lambda\to 1} h_\lambda[f] = h[f] \equiv -\int_{\mathbb{R}} dz\, f(z)\, \ln[f(z)], \tag{6}$$

and so the entropic uncertainty relation for Shannon entropies is [16,19]

$$h[\rho] + h[\tilde{\rho}] \geqslant \ln \pi e\hbar. \tag{7}$$

Note that the uncertainty relations (1), (4), and (7) involve the perfect knowledge of the continuous probability distributions $\rho(x)$ and $\tilde{\rho}(p)$ that can be obtained only from measurements with infinite precision.

III. UNCERTAINTY RELATIONS AND COARSE-GRAINED MEASUREMENTS OF POSITION AND MOMENTUM

In general, measurements are performed with finite precision, so any uncertainty relation involving infinite precision quantities has to be adapted for experimentally obtained quantities that depend on the coarse-grained nature of the measurement. The usual uncertainty relations should be recovered from the coarse-grained uncertainty relations in the limit of infinite precision. This leads to an interesting question: What is the minimum precision (or maximum coarse graining) that still allows for a legitimate uncertainty relation? or, equivalently, What is the minimum number of measurements, whose results belong to a fixed range of eigenvalues of the complementary observables x and p that allows one to verify an uncertainty relation? The coarse-grained measurement process is equivalent to considering the probability distributions $\rho(x)$ and $\tilde{\rho}(p)$ sampled in bins, with finite width Δ and δ for position and momentum, respectively. These parameters shall coincide with the finite widths of the detectors used. Due to these finite widths, the operators that are in fact measured are the coarse-grained position and momentum operators, defined as [30],

$$\hat{x}_\Delta = \sum_k x_k \int_{(k-1/2)\Delta}^{(k+1/2)\Delta} dx\, |x\rangle\langle x| \tag{8}$$

and

$$\hat{p}_\delta = \sum_l p_l \int_{(l-1/2)\delta}^{(l+1/2)\delta} dp\, |p\rangle\langle p|, \tag{9}$$

where $x_k = k\Delta$ and $p_l = l\delta$ are the coordinates at the center of the sampling windows. From repeated measurements over identically prepared systems, we can construct the probabilities r_k^Δ and s_l^δ to obtain the results x_k and p_l, respectively. If the position and momentum measurements are repeated with sufficient statistics, these probabilities are expected to be very close to the actual values. Since $\langle \hat{x}_\Delta\rangle = \mathrm{Tr}(\hat{\varrho}\cdot\hat{x}_\Delta)$ and $\langle \hat{p}_\delta\rangle = \mathrm{Tr}(\hat{\varrho}\cdot\hat{p}_\delta)$, we obtain

$$\langle \hat{x}_\Delta\rangle = \sum_k x_k r_k^\Delta, \quad \langle \hat{p}_\delta\rangle = \sum_l p_l s_l^\delta, \tag{10}$$

where

$$r_k^\Delta = \int_{(k-1/2)\Delta}^{(k+1/2)\Delta} dx\, \rho(x), \quad s_l^\delta = \int_{(l-1/2)\delta}^{(l+1/2)\delta} dp\, \tilde{\rho}(p). \tag{11}$$

The discrete variances that correspond to the measurements of the coarse-grained position and momentum operators are

$$\sigma_{x_\Delta}^2 \equiv \langle \hat{x}_\Delta^2\rangle - \langle \hat{x}_\Delta\rangle^2 = \sum_k x_k^2 r_k^\Delta - \left(\sum_k x_k r_k^\Delta\right)^2 \tag{12}$$

and

$$\sigma_{p_\delta}^2 \equiv \langle \hat{p}_\delta^2\rangle - \langle \hat{p}_\delta\rangle^2 = \sum_l p_l^2 s_l^\delta - \left(\sum_l p_l s_l^\delta\right)^2. \tag{13}$$

In the case where the widths Δ and δ are sufficiently small, these discrete variances are approximations of the continuous variances $\sigma_x^2 \equiv \sigma_x^2[\rho]$ and $\sigma_p^2 \equiv \sigma_p^2[\tilde{\rho}]$ in Eq. (1):

$$\lim_{\Delta\to 0} \sigma_{x_\Delta}^2 = \sigma_x^2, \quad \lim_{\delta\to 0} \sigma_{p_\delta}^2 = \sigma_p^2. \tag{14}$$

However, as the sampling widths increase, the inferred variances $\sigma_{x_\Delta}^2$ and $\sigma_{p_\delta}^2$ begin to underestimate the true variances σ_x^2 and σ_p^2. In fact, we have the limit

$$\lim_{\Delta\to\infty} \sigma_{x_\Delta}^2 = 0 = \lim_{\delta\to\infty} \sigma_{p_\delta}^2. \tag{15}$$

One possible adaptation of the HUR of Eq. (1) to finite coarse-grained measurements could be the usual Heisenberg-type uncertainty relation associated with any noncommuting observables,

$$\sigma_{x_\Delta}^2 \sigma_{p_\delta}^2 \geqslant \tfrac{1}{4}|\langle[\hat{x}_\Delta,\hat{p}_\delta]\rangle|^2. \tag{16}$$

Unfortunately, this lower bound depends on the state of the quantum system and is not useful from an experimental point of view. Furthermore, one can show that, due to the coarse graining ($\Delta \neq 0$ or $\delta \neq 0$), there are always families of localized states for which the right-hand side of Eq. (16) becomes equal to 0.

Another way to obtain uncertainty relations associated with coarse-grained measurements is to investigate the properties of the probability distributions $\{r_k^\Delta\}$ and $\{s_l^\delta\}$ using the discrete Rényi entropies [36],

$$H_\alpha[r_k^\Delta] = \frac{1}{1-\alpha}\ln\sum_{k=-\infty}^{\infty}(r_k^\Delta)^\alpha, \tag{17}$$

$$H_\beta[s_l^\delta] = \frac{1}{1-\beta}\ln\sum_{l=-\infty}^{\infty}(s_l^\delta)^\beta. \tag{18}$$

In the limit $\alpha\to 1$, $\beta\to 1$ these definitions recover the usual Shannon entropies $H[r_k^\Delta] = \lim_{\alpha\to 1} H_\alpha[r_k^\Delta] = -\sum_k r_k^\Delta \ln r_k^\Delta$, $H[s_l^\delta] = \lim_{\beta\to 1} H_\beta[s_l^\delta] = -\sum_l s_l^\delta \ln s_l^\delta$. As in the case of the variances above, the discrete Rényi entropy starts to underestimate the continuous entropy when the widths Δ and δ are large. In fact, when the sampling widths are extremely large, we have one r_k^Δ and one s_l^δ with near unit probabilities, which are responsible for the zero uncertainty in the discrete variables, i.e.,

$$\lim_{\Delta\to\infty} H_\alpha[r_k^\Delta] = 0 = \lim_{\delta\to\infty} H_\beta[s_l^\delta]. \tag{19}$$

But in this case we have also the opposite situation, i.e., when the coarse-grained measurement is fine, $H[r_k^\Delta]$ and $H[s_l^\delta]$ start to superestimate the Shannon entropies $h[\rho]$ and $h[\tilde{\rho}]$ respectively. In fact, we have the limit situation $\lim_{\Delta\to 0}\lim_{\delta\to 0}(H[r_k^\Delta]+H[s_l^\delta]) = \infty$ [37]. A first attempt to establish an uncertainty relation for coarse-grained measurement involving the discrete Rényi entropies, $H_\alpha[r_k^\Delta]$ and $H_\beta[s_l^\delta]$, was done by Bialynicki-Birula [22]. In Sec. V, we will derive an improved lower bound for this uncertainty relation.

In this paper, we will be concerned with sampling widths that could be extremely large. In the next section we will present the basic ingredients in order to obtain reliable uncertainty relations for the coarse-grained measurements.

IV. CONTINUOUS COARSE-GRAINED PROBABILITY DISTRIBUTION FUNCTIONS FOR POSITION AND MOMENTUM

In calculations based on experimental data, we will show that it is advantageous to adopt the following approximated probability density functions (PDFs):

$$w_\Delta(x) = \sum_{k=-\infty}^{\infty} r_k^\Delta D_\Delta(x,x_k) \tag{20}$$

and

$$\tilde{w}_\delta(p) = \sum_{l=-\infty}^{\infty} s_l^\delta D_\delta(p,p_l). \tag{21}$$

The $D_\Delta(x,x_k)$ and $D_\delta(p,p_l)$, which we shall call generalized histogram functions (GHFs), are two independent *approximation to identity functions* [38] with width parameters Δ and δ, respectively. These are normalized functions,

$$\int dz\, D_\eta(z,z_j) = 1, \tag{22}$$

that converge to the Dirac δ function: $\lim_{\eta\to 0} = D_\eta(z,z_j) = \delta(z-z_j)$. In this limit, we have $\lim_{\Delta\to 0} w_\Delta(x) = \rho(x)$ and $\lim_{\delta\to 0}\tilde{w}_\delta(p) = \tilde{\rho}(p)$. Additionally, we require that $D_\Delta(x,x_k)$ and $D_\delta(p,p_l)$ have finite support on the intervals $[(k-1/2)\Delta, (k+1/2)\Delta]$ and $[(l-1/2)\delta,(l+1/2)\delta]$, respectively. We also impose the restriction that the bins are centered at the

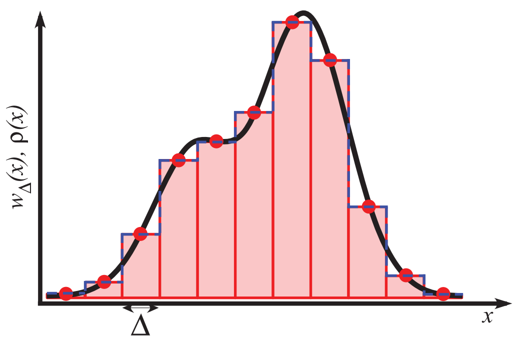

FIG. 1. (Color online) Example of a continuous $w_\Delta(x)$ distribution function (blue dashed line) approximating the original distribution function $\rho(x)$ (black line) constructed from rectangle functions.

same points z_j as the functions $D_\eta(z,z_j)$, so we can recover these values through

$$z_j = \int_{\mathbb{R}} dz\, z\, D_\eta(z,z_j) = \int_{(j-1/2)\eta}^{(j+1/2)\eta} dz\, z\, D_\eta(z,z_j). \tag{23}$$

Moreover, we assume that these functions are translationally invariant,

$$D_\eta(z+z_j-z_m,z_j) = D_\eta(z,z_m), \tag{24}$$

and we place no restriction on the variance of the function $D_\eta(z,z_j)$.

Perhaps the simplest example of a GHF with these properties is the normalized rectangle function,

$$\mathrm{Rect}_\eta(z,z_j) = \begin{cases} 1/\eta & \text{for } z \in \left[\left(j-\frac{1}{2}\right)\eta,\left(j+\frac{1}{2}\right)\eta\right] \\ 0 & \text{elsewhere} \end{cases}. \tag{25}$$

An example of a $w_\Delta(x)$ distribution function constructed with rectangle functions is illustrated in Fig. 1. The PDFs $w_\Delta(x)$ and $\tilde{w}_\delta(p)$ represent approximations to the actual PDFs $\rho(x)$ and $\tilde{\rho}(p)$ that are based on the results of the discretely sampled measurements.

The advantage given by these new PDFs and the introduction of the histogram functions is that they will allow us to derive reliable and optimal uncertainty relations for the coarse-grained variances of Eqs. (12) and (13). In the first step we shall show that there exists a simple relation between the uncertainty associated with the approximate distribution, say, $w_\Delta(x)$, the uncertainty of the associated discrete distribution r_k^Δ, and the uncertainty of the GHF $D_\Delta(x,x_k)$. Let us show this connection only for the position variable since the calculations for the momentum variable are analogous.

The variance of $w_\Delta(x)$ is

$$\sigma_x^2[w_\Delta] \equiv \int_{\mathbb{R}} dx\, x^2 w_\Delta(x) - \left(\int_{\mathbb{R}} dx\, x\, w_\Delta(x)\right)^2$$
$$= \sum_k r_k^\Delta \langle x^2\rangle_{k,D} - \left(\sum_k r_k^\Delta \langle x\rangle_{k,D}\right)^2, \tag{26}$$

where

$$\langle x^2\rangle_{k,D} \equiv \int_{(k-1/2)\Delta}^{(k+1/2)\Delta} dx\, x^2 D(x,x_k,\Delta) \tag{27}$$

and

$$\langle x\rangle_{k,D} \equiv \int_{(k-1/2)\Delta}^{(k+1/2)\Delta} dx\, x\, D(x,x_k,\Delta). \tag{28}$$

The variance of the function $D_\Delta(x,x_k)$ is, then,

$$\sigma_\Delta^2 \equiv \langle x^2\rangle_{k,D} - \langle x\rangle_{k,D}^2. \tag{29}$$

Note that, due to the translational invariance of the GHF function of Eq. (24), σ_Δ^2, in fact, does not depend on k and we can write,

$$\sigma_\Delta^2 = \sum_k r_k^\Delta \langle x^2\rangle_{k,D} - \sum_k r_k^\Delta \langle x\rangle_{k,D}^2. \tag{30}$$

Using Eq. (30) to work out the total variance $\sigma_x^2[w_\Delta]$ we find that

$$\sigma_x^2[w_\Delta] = \sigma_\Delta^2 + \sum_k r_k^\Delta \langle x\rangle_{k,D}^2 - \left(\sum_k r_k^\Delta \langle x\rangle_{k,D}\right)^2. \tag{31}$$

Taking into account the property Eq. (23) of the GHF, we substitute $\langle x\rangle_{k,D} = x_k$ and, finally, obtain

$$\sigma_x^2[w_\Delta] = \sigma_{x_\Delta}^2 + \sigma_\Delta^2, \tag{32}$$

where the variance $\sigma_{x_\Delta}^2$ was defined in Eq. (12). From Eq. (32) it is easy to understand the limits in Eq. (15) if we interpret the two contributions to the variance in Eq. (32) (and in an analogous expression for the momentum) in the following way. First, we set the phase space origin at the center of the bins that contain $\langle\hat{x}_\Delta\rangle$ and $\langle\hat{p}_\delta\rangle$ (i.e., the central bins). Thus, the first contribution in Eq. (32) is given by the discrete variances $\sigma_{x_\Delta}^2$ corresponding to the coarse-grained measurements $x_k = k\Delta$ outside the central bin, since the central bin has no contribution to the discrete variance in this case. The other contribution, σ_Δ^2, can be interpreted as the variance of the GHF of the central bin. For increasing values of coarse graining, the contribution to $\sigma_x^2[w_\Delta]$ from the central bin grows and the contribution from discrete measurements outside the central bin decreases.

For the uncertainty quantified by the continuous Shannon entropy, we have

$$h[\omega_\Delta] \equiv -\int_{\mathbb{R}} dx\, w_\Delta(x)\, \ln[w_\Delta(x)]$$
$$= -\sum_k \int_k dx\, r_k^\Delta D_\Delta(x,x_k) \ln\left[r_k^\Delta D_\Delta(x,x_k)\right]$$
$$= H\left[r_k^\Delta\right] + \sum_k r_k^\Delta\, h[D_\Delta(x,x_k)]$$
$$= H\left[r_k^\Delta\right] + h_\Delta, \tag{33}$$

where the second line follows from the fact that the GHF has a compact support on the interval, and only one term inside the logarithm survives. Here $h_\Delta \equiv h[D_\Delta(x,x_k)]$ is the Shannon entropy of the continuous probability distribution $D(x,x_k,\Delta)$, which, according to Eq. (24), also does not depend on the index k. As mentioned above, similar results are found for the momentum distribution:

$$\sigma_p^2[\tilde{w}_\delta] = \sigma_{p_\delta}^2 + \sigma_\delta^2, \tag{34}$$

$$h[\tilde{w}_\delta] = H\left[s_l^\delta\right] + h_\delta. \tag{35}$$

It is important to realize that both σ_η^2 and h_η do not depend on the specific value z_j of the center of each bin, so the uncertainty measured by these quantities is associated with *a generic bin of the experimental sampling*. Thus, the variance (Shannon entropy) of the approximated PDFs are given by the sum of the discrete variance (Shannon entropy) of the experimental points and the GHFs used. This important property will allow us to construct consistent uncertainty relations in the next sections.

V. ENTROPIC UNCERTAINTY RELATIONS FOR COARSE-GRAINED OBSERVABLES

We will, first, derive new uncertainty relations for coarse-grained measurements based on the Rényi entropy. It was shown by Bialynicki-Birula that the discrete Rényi entropies of Eqs. (17) and (18) satisfy the following entropic uncertainty relation [22] ($1/\alpha + 1/\beta = 2$, $\beta \geqslant 1$):

$$H_\alpha[r_k^\Delta] + H_\beta[s_l^\delta] \geqslant \mathcal{B}_\alpha, \tag{36}$$

where

$$\mathcal{B}_\alpha = -\frac{1}{2}\left(\frac{\ln\alpha}{1-\alpha} + \frac{\ln\beta}{1-\beta}\right) - \ln\left(\frac{\Delta\delta}{\pi\hbar}\right). \tag{37}$$

We note a full symmetry between the parameters α and β, and, since $\beta = \alpha/(2\alpha - 1)$, we shall treat the lower bound $\mathcal{B}_\alpha$ and further results as α-dependent increasing functions ($1/2 \leqslant \alpha \leqslant 1$),

$$-\ln\left(\frac{\Delta\delta}{2\pi\hbar}\right) = \mathcal{B}_{1/2} \leqslant \mathcal{B}_\alpha \leqslant \mathcal{B}_1 = -\ln\left(\frac{\Delta\delta}{\pi e\hbar}\right). \tag{38}$$

Here we prove a new uncertainty relation hypothesized in Ref. [38],

$$H_\alpha[r_k^\Delta] + H_\beta[s_l^\delta] \geqslant -\ln\left[\frac{\Delta\delta}{2\pi\hbar}\left[R_{00}\left(\frac{\Delta\delta}{4\hbar},1\right)\right]^2\right] \equiv \mathcal{R}, \tag{39}$$

where $R_{00}(\xi,\eta)$ [40] denotes one of the radial prolate spheroidal wave functions of the first kind [41]. The full proof of this new relation is given in Appendix A. Since this relation is valid independently of Eq. (36), an improved lower bound for the sum of the Rényi entropies Eqs. (17) and (18) reads,

$$H_\alpha[r_k^\Delta] + H_\beta[s_l^\delta] \geqslant L_\alpha \geqslant 0, \tag{40}$$

where $L_\alpha \equiv \max\{\mathcal{B}_\alpha, \mathcal{R}\}$. When $\Delta\delta \ll \hbar$ we have

$$\mathcal{R} \approx -\ln\left(\frac{\Delta\delta}{2\pi\hbar}\right) = \mathcal{B}_{1/2}, \tag{41}$$

so the final lower bound $L_{1/2}$ is a smooth function of $\Delta\delta/\hbar$. For other values of α, especially $\alpha = 1$, the L_α versus $\Delta\delta/\hbar$ curve is not smooth. Figure 2 shows a plot of $\mathcal{R}$, $\mathcal{B}_{1/2}$, and $\mathcal{B}_1$ as functions of $\Delta\delta/\hbar$. Note that for $\alpha = 1$ and $\Delta\delta/\hbar \gtrsim 7$, we improve the lower bound $\mathcal{B}_1$ in Eq. (36) by $L_1 = \mathcal{R}$. We also note that for all values of α we have a nontrivial uncertainty relation since, unlike $\mathcal{B}_1$ and $\mathcal{B}_{1/2}$, $L_\alpha > 0$ for $\Delta\delta < \infty$ (see Fig. 2).

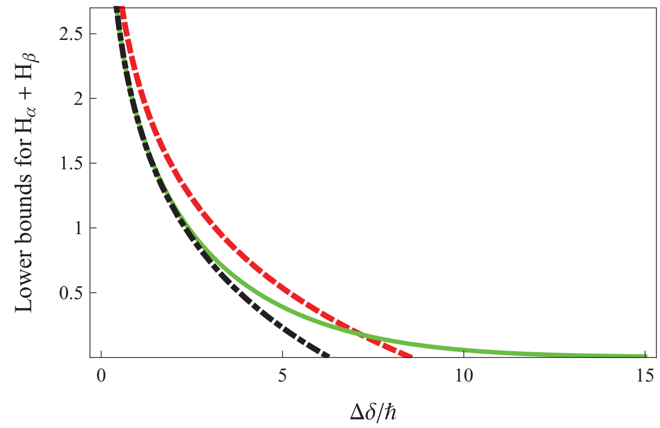

FIG. 2. (Color online) We plot the lower bounds $\mathcal{R}$ (green solid line), $\mathcal{B}_1$ (red dashed line), and $\mathcal{B}_{1/2}$ (black dashed-dotted line). Note that for $\alpha = 1/2$ we have $L_{1/2} = \mathcal{R}$.

VI. HEISENBERG UNCERTAINTY RELATIONS FOR COARSE-GRAINED OBSERVABLES

The improved uncertainty relation Eq. (40) for the case of the Shannon entropy ($\alpha = 1$) will allow us to derive a Heisenberg-like uncertainty relation for variances that is valid for any amount of coarse graining. We, first, apply the reversed logarithmic Sobolev inequality [40] to the approximated PDFs [Eqs. (20) and (21)] as follows:

$$\tfrac{1}{2}\ln\left(2\pi e\,\sigma_x^2[w_\Delta]\right) \geqslant h[w_\Delta], \tag{42}$$

$$\tfrac{1}{2}\ln\left(2\pi e\,\sigma_p^2[\tilde{w}_\delta]\right) \geqslant h[\tilde{w}_\delta]. \tag{43}$$

We now shall add these two inequalities, make use of Eqs. (33) and (35) for the continuous entropies and apply the entropic uncertainty relation of Eq. (40) for $\alpha = \beta = 1$. We find that

$$\sigma_x^2[w_\Delta]\sigma_p^2[\tilde{w}_\delta] \geqslant \frac{\exp(2L_1)}{(2\pi e)^2}e^{2h_\Delta+2h_\delta} = \frac{\hbar^2}{4}\frac{e^{2h_\Delta+2h_\delta}}{\Delta^2\delta^2}g\left(\frac{\Delta\delta}{\hbar}\right), \tag{44}$$

where

$$g\left(\frac{\Delta\delta}{\hbar}\right) \equiv \max\left\{1, \left(\frac{2}{e}\right)^2\left[R_{00}\left(\frac{\Delta\delta}{4\hbar},1\right)\right]^{-4}\right\}. \tag{45}$$

In the case where the GHFs are the normalized rectangle functions in Eq. (25) ($\eta = \Delta,\delta$), we have the following Heisenberg-like uncertainty relation:

$$\sigma_x^2[w_\Delta]\sigma_p^2[\tilde{w}_\delta] \geqslant \frac{\hbar^2}{4}g\left(\frac{\Delta\delta}{\hbar}\right). \tag{46}$$

When $\Delta\delta/\hbar < 6$ we have $g(\Delta\delta/\hbar) = 1$; thus, this uncertainty relation coincides with the result presented recently in Ref. [30] as follows:

$$\left(\sigma_{x_\Delta}^2 + \frac{\Delta^2}{12}\right)\left(\sigma_{p_\delta}^2 + \frac{\delta^2}{12}\right) \geqslant \frac{\hbar^2}{4}. \tag{47}$$

Note that the uncertainty relation of Eq. (47) is satisfied trivially when $\Delta\delta/\hbar \geqslant 6$, so the uncertainty relation of Eq. (46) seems to be an improvement of Eq. (47) when $\Delta\delta/\hbar \geqslant 6$. However, we have to realize that both sides of Eq. (46) grow with coarse graining and the lower bound in Eq. (46) grows slower than the left-hand side. As a result, the uncertainty relation of Eq. (46) is also trivially satisfied for $\Delta\delta/\hbar \geqslant 6$, and there is no improvement with respect to the previous result.

Nevertheless, the right-hand side of the inequality in Eq. (44) contains information about the GHFs that can be optimized, since the variances $\sigma_x^2[w_\Delta]$ and $\sigma_p^2[\tilde{w}_\delta]$ are inferred directly from measurements. We can now ask the following question: What choice of GHFs gives us the optimal uncertainty relation? To answer this, we perform an optimization procedure over the possible functional forms of the functions D_η and also on the values of their variances. All details are presented in Appendix B. The solution of this optimization procedure is obtained for a GHF given by a Gaussian function whose support is in the interval $[-\eta/2,\eta/2]$,

$$D_\eta^{\text{opt}}(z,0) = \sqrt{\frac{a_\eta}{\pi}}\,\frac{e^{-a_\eta z^2}}{\text{Erf}(\eta\sqrt{a_\eta}/2)}, \quad a_\eta \in \mathbb{R}, \tag{48}$$

where Erf is the usual error function and a_η is an optimization parameter related to the variance σ_η^2. With this optimal GHF we arrive at the optimal coarse-grained version of the Heisenberg uncertainty relation (see Appendix B),

$$K\left(\frac{\sigma_{x_\Delta}^2}{\Delta^2}\right)K\left(\frac{\sigma_{p_\delta}^2}{\delta^2}\right) \geqslant \exp(2L_1) = \left(\frac{\pi e\hbar}{\Delta\delta}\right)^2 g\left(\frac{\Delta\delta}{\hbar}\right), \tag{49}$$

where

$$K(u) = \frac{\exp[2u\mathcal{M}^{-1}(u)]}{\text{Erf}^2[\sqrt{\mathcal{M}^{-1}(u)}/2]}, \tag{50}$$

and $\mathcal{M}^{-1}(\cdot)$ denotes the inverse of the following invertible function $\mathcal{M}(\cdot)$:

$$\mathcal{M}(t) = \frac{\exp(-t/4)}{2\sqrt{\pi t}\,\text{Erf}(\sqrt{t}/2)}. \tag{51}$$

A plot of $\mathcal{M}$ and $\mathcal{M}^{-1}$ is shown in Fig. 3. In Fig. 4 we plot the function $K(u)$ in comparison to the linear function $1 + 2\pi e\, u$. In the next section, we will analyze the new uncertainty relation (49).

A. Analysis of the uncertainty relation Eq. (49)

A first observation about the uncertainty relation of Eq. (49) is that it is valid for any finite value of coarse graining such that $\Delta \neq 0$ and $\delta \neq 0$, because in its derivation no restrictions were made on their possible values. In the case of the limiting situation where $\Delta,\delta \to 0$ we recover the infinite precision Heisenberg uncertainty relation of Eq. (1). In order to prove this we avoid the divergence on the right-hand side of Eq. (49) by multiplying both sides by the factor $(\Delta\delta)^2$ and calculate the limits $\Delta,\delta \to 0$ in the following way ($g(0) = \max\{1,(2/e)^2\} = 1$):

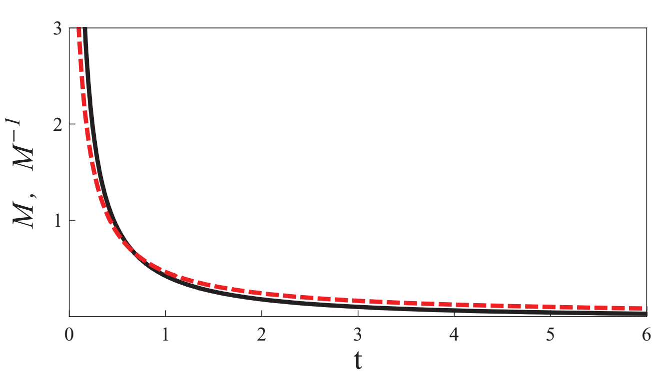

FIG. 3. (Color online) Plots of the $\mathcal{M}(t)$ function (black solid line) and the $\mathcal{M}^{-1}(t)$ inverse function (red dashed line).

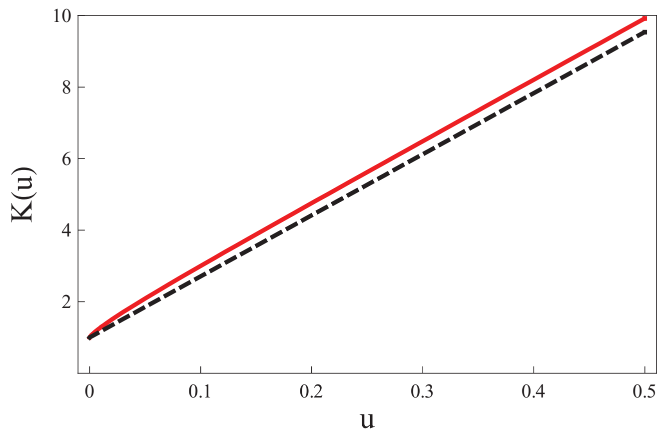

FIG. 4. (Color online) Comparison between the function $K(u)$ (red solid line) and the linear function $1 + 2\pi e\, u$ (black dashed line).

$$\lim_{\Delta\to 0}\left[\Delta^2 K\left(\sigma_{x_\Delta}^2/\Delta^2\right)\right]\lim_{\delta\to 0}\left[\delta^2 K\left(\sigma_{p_\delta}^2/\delta^2\right)\right] \geqslant (\pi e\hbar)^2. \tag{52}$$

We can perform both limits in Eq. (52) separately. Let us now introduce a new variable $v = \mathcal{M}^{-1}(\sigma_{x_\Delta}^2/\Delta^2)$. Taking into account the limit of Eq. (14), we obtain

$$\begin{aligned}\lim_{\Delta\to 0}\left[\Delta^2 K\left(\sigma_{x_\Delta}^2/\Delta^2\right)\right] &= \sigma_x^2 \lim_{v\to 0}\left[\frac{1}{\mathcal{M}(v)}\frac{\exp[2v\mathcal{M}(v)]}{\text{Erf}^2(\sqrt{v}/2)}\right] \\ &= 2\pi e\sigma_x^2.\end{aligned} \tag{53}$$

The same result can be obtained for the limit $\delta \to 0$, thus Eq. (52) reads $(2\pi e\sigma_x\sigma_p)^2 \geqslant (\pi e\hbar)^2$, which is equivalent to the usual HUR in Eq. (1).

In the opposite limit of infinite coarse graining, the discrete variances go to 0, as we mentioned in Eq. (15) and discussed after Eq. (32). Even for finite coarse graining, it is possible that $\sigma_{x_\Delta}^2 = 0$ *or* $\sigma_{p_\delta}^2 = 0$, if the quantum state is localized in position or momentum (position or momentum probability distribution has a compact support). However, the quantum state cannot be simultaneously localized in both position and momentum spaces. Therefore, for finite coarse graining it is forbidden that $\sigma_{x_\Delta}^2 = 0 = \sigma_{p_\delta}^2$. Let us show that this fact is present in our uncertainty relation Eq. (49). To this end, we shall calculate the limit $\sigma_{x_\Delta},\sigma_{p_\delta} \to 0$. Using the same variable v as above, we have

$$\lim_{\sigma_{x_\Delta}\to 0} K\left(\sigma_{x_\Delta}^2/\Delta^2\right) = \lim_{v\to\infty}\left[\frac{\exp[2v\mathcal{M}(v)]}{\text{Erf}^2(\sqrt{v}/2)}\right] = 1. \tag{54}$$

The same result can be obtained for the limit $\sigma_{p_\delta} \to 0$. Next, we note that for $\Delta < \infty$ and $\delta < \infty$ we have $L_1 > 0$. Finally, when we put $\sigma_{x_\Delta}^2 = 0 = \sigma_{p_\delta}^2$, we obtain from Eq. (49) the hierarchy of contradictory inequalities,

$$1 \geqslant \exp(2L_1) > 1. \tag{55}$$

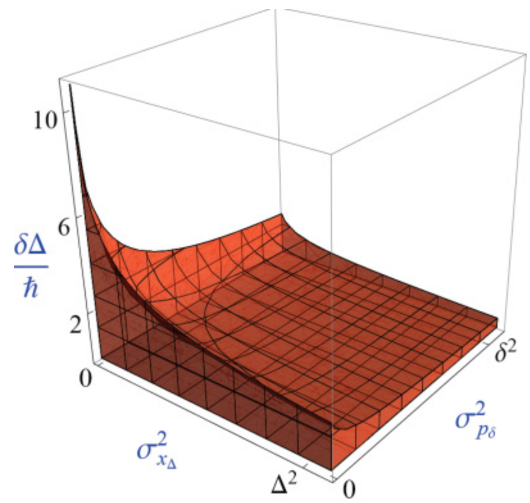

FIG. 5. (Color online) Plot of the uncertainty relation of Eq. (49) as a function of the discrete variances $\sigma^2_{x_\Delta}$ and $\sigma^2_{p_\delta}$. The red forbidden region shows that there is always an uncertainty relation for any size coarse graining.

Thus, zero variance in both discrete variables, $\sigma^2_{x_\Delta} = 0 = \sigma^2_{p_\delta}$, is prohibited.

The coarse-grained Heisenberg uncertainty relation is shown graphically in Fig. 5. The red area represents forbidden values of $\sigma^2_{x_\Delta}$ and $\sigma^2_{p_\delta}$. The narrow peak of forbidden values for small $\sigma^2_{x_\Delta}$ and $\sigma^2_{p_\delta}$ illustrates the result Eq. (55). Though this forbidden region gets smaller as $\Delta\delta/\hbar$ grows, there is always some forbidden region which limits the information that can be obtained about the noncommuting observables. Thus, there always exists an uncertainty relation, regardless of the size of the coarse graining.

This is surprising since it is commonly argued that in the large quantum number limit (semiclassical regime) we should recover classical mechanics for coarse-grained averaging due to finite-precision detectors [32,34]. We can follow the argumentation in a simple example of a particle with mass M in a one-dimensional infinite square-well potential [32]. The energy eigenstates are $\Psi_n(x) = \sqrt{2/L}\,\sin(n\pi x/L)$ $(0 \leqslant x \leqslant L)$. The eigenenergies are $E_n = p_n^2/2M$ and the two values of the momentum $p_n = \pm\hbar n\pi/L$ are equally probable since $\tilde{\rho}_n(p) = |\tilde{\Psi}_n(p)|^2$ consists of a symmetric probability distribution function peaked at p_n, with oscillatory tails that go to $\pm\infty$. It is easy to see that, for some finite coarse graining, we recover the classical probability distributions inside each bin, both in position and momentum, in the limit of large quantum numbers, i.e., $n \to \infty$. In the position representation we recover a constant value inside each bin. In the momentum representation we obtain a zero value for each bin, except for the bins that contain the values p_n, corresponding to Dirac δ functions that move away to infinity. However, the limit $n \to \infty$ has only a formal meaning and, in fact, does not appear in real systems. For large but finite quantum number n, the position and momentum probability distributions are close but not equal to the classical distributions. Thus, even if we choose coarse graining in the position representation that is equal to the size L of the potential well, so $\sigma^2_{x_\Delta} = 0$, and extremely large coarse graining in momentum, the value of $\sigma^2_{p_\delta}$ will contain contributions from the bins at the tails of the distribution $\tilde{\rho}_n(p)$ that differ from zero for $-\infty < p < \infty$. In other words, $\sigma^2_{p_\delta} \neq 0$ and, in fact, must be limited by the uncertainty relation in Eq. (49).

VII. CONCLUSIONS

We have derived several new uncertainty relations for continuous-variable quantum systems when coarse-grained measurements are performed. First, we show a new bound for the uncertainty relation involving Rényi entropy. This bound is an improvement for all entropy orders α for larger coarse graining and is optimal for order $\alpha = 1/2$. Using this result, we derive a new Heisenberg-like uncertainty relation. Surprisingly, there is always a meaningful uncertainty relation for any amount of finite coarse graining. Thus, even coarse-grained measurements never commute, and information obtained about one observable increases uncertainty in the other. These results are interesting from a fundamental point of view and also may find application in a quantum information scenario. In particular, the security of several quantum key distribution schemes with continuous variables rely on uncertainty relations.

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APPENDIX A: DERIVATION OF THE ENTROPIC UNCERTAINTY RELATION (39)

To begin, let us consider a normalized, pure quantum state described in position space by a one-dimensional wave function $\psi(x)$. The same state is described in momentum space by $\tilde{\psi}(p)$, the Fourier transform of $\psi(x)$. These wave functions provide two probability distributions,

$$g(x) = |\psi(x)|^2, \quad \tilde{g}(p) = |\tilde{\psi}(p)|^2, \tag{A1}$$

in position and momentum space, respectively. For Eq. (A1) we shall define, analogously to Eq. (11), the following discrete probability distributions:

$$q_k = \int_{(k-1/2)\Delta}^{(k+1/2)\Delta} dx\, g(x), \quad p_l = \int_{(l-1/2)\delta}^{(l+1/2)\delta} dp\, \tilde{g}(p). \tag{A2}$$

The starting point of our derivation shall be the definition of two new probability distributions $\{|a_{km}|^2\}$ and $\{|b_{ln}|^2\}$ that are distinct from Eq. (A2), where [39]

$$a_{km} = \int_{(k-1/2)\Delta}^{(k+1/2)\Delta} dx\, \psi(x)\varphi^*_{km}(x), \tag{A3}$$

$$b_{ln} = \int_{(l-1/2)\delta}^{(l+1/2)\delta} dp\, \tilde{\psi}(p)\theta^*_{ln}(p). \tag{A4}$$

Functions $\varphi_{km}(x)$ have been arbitrarily chosen to form an orthonormal basis in the kth bin as follows:

$$\int_{(k-1/2)\Delta}^{(k+1/2)\Delta} dx\, \varphi_{km}(x)\varphi_{km'}^*(x) = \delta_{mm'}. \tag{A5}$$

Similarly, in momentum space, we introduce the functions $\theta_{ln}(p)$, which are orthonormal in the lth bin,

$$\int_{(l-1/2)\delta}^{(l+1/2)\delta} dp\, \theta_{ln}(p)\theta_{ln'}^*(p) = \delta_{nn'}. \tag{A6}$$

For the probability distributions $\{|a_{km}|^2\}$ and $\{|b_{ln}|^2\}$, the Riesz theorem [43] reads $(1/\alpha + 1/\beta = 2, \beta \geqslant 1)$,

$$\left[\mathcal{C}\sum_n |b_{ln}|^{2\beta}\right]^{1/\beta} \leqslant \left[\mathcal{C}\sum_m |a_{km}|^{2\alpha}\right]^{1/\alpha}, \tag{A7}$$

where the constant $\mathcal{C}$ is

$$\mathcal{C} = \sup_{(k,m,l,n)} \left| \int_{(k-1/2)\Delta}^{(k+1/2)\Delta} dx \int_{(l-1/2)\delta}^{(l+1/2)\delta} dp\, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \varphi_{km}^*(x)\theta_{ln}(p) \right|. \tag{A8}$$

Using the well-known Maassen-Uffink result [21] it was shown that [39]

$$H_\alpha[|a_{km}|^2] + H_\beta[|b_{ln}|^2] \geqslant -2\ln\mathcal{C}, \tag{A9}$$

where the Rényi entropies $H_\alpha[|a_{km}|^2]$ and $H_\beta[|b_{ln}|^2]$ are related to the probability distributions $\{|a_{km}|^2\}$ and $\{|b_{ln}|^2\}$. To obtain this uncertainty relation one needs to take the logarithm of both sides of Eq. (A7) and recognize the definitions of the Rényi entropies. In [39,44] it was also shown that

$$\mathcal{C} < \exp(-\mathcal{R}/2) \equiv \sqrt{\frac{\Delta\delta}{2\pi\hbar}} R_{00}\left(\frac{\Delta\delta}{4\hbar}, 1\right). \tag{A10}$$

The integral equation leading to this result appears in signal processing theory and also in entropic uncertainty relations [18]. Since the inequality (A10) is independent of the choice of the functions $\varphi_{km}(x)$ and $\theta_{ln}(p)$, we can take

$$\varphi_{km}(x) = \begin{cases} \psi(x)/\sqrt{q_k} & m = 0 \\ \text{orthogonal functions} & m \neq 0 \end{cases} \tag{A11}$$

and

$$\theta_{ln}(p) = \begin{cases} \tilde{\psi}(p)/\sqrt{p_l} & n = 0 \\ \text{orthogonal functions} & n \neq 0 \end{cases}. \tag{A12}$$

In this particular choice, we have

$$a_{km} = \delta_{0m}\sqrt{q_k}, \quad b_{ln} = \delta_{0n}\sqrt{p_l} \tag{A13}$$

and

$$H_\alpha[|a_{km}|^2] = H_\alpha[q_k], \quad H_\beta[|b_{ln}|^2] = H_\beta[p_l]. \tag{A14}$$

This observation, together with Eqs. (A9) and (A10), leads to the result $H_\alpha[q_k] + H_\beta[p_l] \geqslant \mathcal{R}$.

In order to extend this uncertainty relation to the case of the probability distributions Eq. (2) related to the mixed state density operator $\hat{\varrho}$, we note that Eq. (2) can always be represented in the following way ($\sum_i \lambda_i = 1$):

$$\rho(x) = \sum_i \lambda_i g_i(x) \quad \text{and} \quad \tilde{\rho}(p) = \sum_i \lambda_i \tilde{g}_i(p), \tag{A15}$$

where $g_i(x)$ and $\tilde{g}_i(p)$ are probability distributions of the form of Eq. (A1). Equation (A15) immediately implies the same decomposition of the probability distributions $\{r_k^\Delta\}$ and $\{s_l^\delta\}$,

$$r_k^\Delta = \sum_i \lambda_i q_k^i \quad \text{and} \quad s_l^\delta = \sum_i \lambda_i p_l^i, \tag{A16}$$

where q_k^i and p_l^i are calculated for the probability distributions $g_i(x)$ and $\tilde{g}_i(p)$, respectively. Using the arguments provided in Refs. [22,45], it follows from the Minkowski inequality that

$$H_\alpha[r_k^\Delta] + H_\beta[s_l^\delta] \geqslant \sum_i \lambda_i \left(H_\alpha[q_k^i] + H_\beta[p_l^i]\right), \tag{A17}$$

which finishes the derivation.

APPENDIX B: OPTIMIZATION LEADING TO EQS. (49)–(51)

In the uncertainty relation of Eq. (44) there are two pairs of parameters that are independent of the state $\hat{\varrho}$, i.e., $\{h_\Delta, \sigma_\Delta^2\}$ and $\{h_\delta, \sigma_\delta^2\}$. We will perform an optimization over these parameters, but since they are not independent among themselves we shall do this procedure in two steps. First, we maximize the Shannon entropies h_Δ and h_δ, keeping the variances σ_Δ^2, σ_δ^2 constant. We can consider the position and momentum variables separately; thus, we will perform all calculations for the general case of the $D_\eta(z,0)$ function. To this end, we shall solve the variational equation as follows:

$$\frac{\delta}{\delta D_\eta(z,0)}\left(-\int_{-\eta/2}^{\eta/2} dz\, D_\eta(z,0)\ln(D_\eta(z,0)) - \lambda_\eta \int_{-\eta/2}^{\eta/2} dz\, D_\eta(z,0) - a_\eta \int_{-\eta/2}^{\eta/2} dz\, z^2 D_\eta(z,0)\right) = 0. \tag{B1}$$

Here λ_η and a_η are η-dependent Lagrange multipliers associated with the normalization and constant variance constraints. The solution is a Gaussian function with the support only in the interval $[-\eta/2, \eta/2]$,

$$D_\eta(z,0) = \sqrt{\frac{a_\eta}{\pi}} \frac{e^{-a_\eta z^2}}{\mathrm{Erf}(\eta\sqrt{a_\eta}/2)}, \quad a_\eta \in \mathbb{R}, \tag{B2}$$

where $\mathrm{Erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y dt\, \exp(-t^2)$ denotes the usual error function. The normalization constraint gives the value of λ_η as a function of the a_η parameter

$$\lambda_\eta(a_\eta) = \ln\left[\sqrt{\frac{\pi}{a_\eta}}\mathrm{Erf}\left(\frac{\eta\sqrt{a_\eta}}{2}\right)\right] - 1. \tag{B3}$$

The variance constraint imposes a relation between a_η and the variance σ_η^2 of the following form:

$$\sigma_\eta^2(a_\eta) = \frac{1}{2a_\eta}\left[1 - \sqrt{\frac{a_\eta}{\pi}} \frac{\eta\exp(-a_\eta\eta^2/4)}{\mathrm{Erf}(\eta\sqrt{a_\eta}/2)}\right]. \tag{B4}$$

The right-hand side is a monotonically decreasing invertible function of the a_η parameter. Moreover, this relation restricts the values of the variance to $0 \leqslant \sigma_\eta^2 < \eta^2/4$, which is a natural consequence of the fact that the maximal value of z^2 on the interval $[-\eta/2,\eta/2]$ is equal to $\eta^2/4$. It is worth noting that the case $\sigma_\eta^2 = \eta^2/12$, which is equivalent to $a_\eta = 0$, describes the case of the rectangle function of Eq. (25). Therefore, according to the relation of Eq. (B4), we now shall use the a_η parameter instead of the variance σ_η^2. Due to the results of Eqs. (B2) and (B4) we have

$$h_\eta(a_\eta) = 1 + \lambda_\eta(a_\eta) + a_\eta \sigma_\eta^2(a_\eta). \tag{B5}$$

This entropy attains its maximal value equal to $\ln \eta$ for $a_\eta = 0$ [the rectangle function (25)].

Thus, we can rewrite the uncertainty relation of Eq. (44) optimized with respect to both entropies in the following way:

$$\frac{\sigma_{x_\Delta}^2 + \sigma_\Delta^2(a_\Delta)}{\exp[2h_\Delta(a_\Delta)]} \frac{\sigma_{p_\delta}^2 + \sigma_\delta^2(a_\delta)}{\exp[2h_\delta(a_\delta)]} \geqslant \frac{\exp(2L_1)}{(2\pi e)^2}. \tag{B6}$$

Substituting the results of Eqs. (B4) and (B5), this relation reads,

$$F\left(\frac{\sigma_{x_\Delta}^2}{\Delta^2}, \Delta^2 a_\Delta\right) F\left(\frac{\sigma_{p_\delta}^2}{\delta^2}, \delta^2 a_\delta\right) \geqslant \exp(2L_1), \tag{B7}$$

where

$$F(u,t) = \frac{2t(u - \mathcal{M}(t)) + 1}{\mathrm{Erf}^2(\sqrt{t}/2)} \exp(2t\mathcal{M}(t)), \tag{B8}$$

and the $\mathcal{M}(\cdot)$ function has been defined in Eq. (51). The second task in the optimization procedure is to find the minimal value of $F(u,t)$ with respect to $t \in \mathbb{R}$, where $u \geqslant 0$ plays the role of an independent parameter. Differentiation of Eq. (B8) leads to the minimum (the second derivative with respect to t is a positive function for $u \geqslant 0$) at point $t_{\min} = \mathcal{M}^{-1}(u)$. Unfortunately, there is no analytical expression for the $\mathcal{M}^{-1}(\cdot)$ function. In order to finish the derivation of the optimal uncertainty relation of Eq. (49), we shall take the function of Eq. (50) to be $K(u) = F(u, \mathcal{M}^{-1}(u))$.

[1] M. D. Reid, Phys. Rev. A **62**, 062308 (2000).
[2] F. Grosshans and N. J. Cerf, Phys. Rev. Lett. **92**, 047905 (2004).
[3] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[4] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[5] S. Mancini, V. Giovannetti, D. Vitali, and P. Tombesi, Phys. Rev. Lett. **88**, 120401 (2002).
[6] H. Nha and M. S. Zubairy, Phys. Rev. Lett. **101**, 130402 (2008).
[7] S. P. Walborn, B. G. Taketani, A. Salles, F. Toscano, and R. L. de Matos Filho, Phys. Rev. Lett. **103**, 160505 (2009).
[8] A. Saboia, F. Toscano, and S. P. Walborn, Phys. Rev. A **83**, 032307 (2011).
[9] M. D. Reid, Phys. Rev. A **40**, 913 (1989).
[10] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).
[11] M. D. Reid, P. D. Drummond, W. P. Bowen, E. G. Cavalcanti, P. K. Lam, H. A. Bachor, U. L. Anderson, and G. Leuchs, Rev. Mod. Phys. **81**, 1727 (2010).
[12] S. P. Walborn, A. Salles, R. M. Gomes, F. Toscano, and P. H. Souto Ribeiro, Phys. Rev. Lett. **106**, 130402 (2011).
[13] W. Heisenberg, Z. Phys. **43**, 122 (1927).
[14] E. H. Kennard, Z. Phys. **44**, 326 (1927).
[15] H. P. Robertson, Phys. Rev. **34**, 163 (1929).
[16] I. Bialynicki-Birula and J. Mycielski, Commun. Math. Phys. **44**, 129 (1975).
[17] D. Deutsch, Phys. Rev. Lett. **50**, 631 (1983).
[18] M. H. Partovi, Phys. Rev. Lett. **50**, 1883 (1983).
[19] I. Bialynicki-Birula, Phys. Lett. A **103**, 253 (1984).
[20] K. Kraus, Phys. Rev. D **35**, 3070 (1987).
[21] H. Maassen and J. B. M. Uffink, Phys. Rev. Lett. **60**, 1103 (1988).
[22] I. Bialynicki-Birula, Phys. Rev. A **74**, 052101 (2006).
[23] J. I. de Vicente and J. Sánchez-Ruiz, Phys. Rev. A **77**, 042110 (2008).
[24] S. Zozor, M. Portesi, P. Sánchez-Moreno, and J. S. Dehesa, Phys. Rev. A **83**, 052107 (2011).
[25] E. Arthurs and J. L. Kelly, Bell Syst. Tech. **44**, 725 (1965).
[26] M. Raymer, Am. J. Phys. **62**, 986 (1994).
[27] M. Ozawa, Phys. Rev. A **67**, 042105 (2003).
[28] J. Erhart, S. Sponar, G. Sulyok, G. Badurek, M. Ozawa, and Y. Hasegawa, Nature Phys. **8**, 185 (2012).
[29] Y. Watanabe, T. Sagawa, and M. Ueda, Phys. Rev. A **84**, 042121 (2011).
[30] Ł. Rudnicki, S. P. Walborn, and F. Toscano, Europhys. Lett. **97**, 38003 (2012).
[31] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1995).
[32] L. Ballentine, *Quantum Mechanics: A Modern Development* (World Scientific, Singapore, 1998).
[33] A. Peres, Found. Phys. **22**, 819 (1992).
[34] J. Kofler and Č. Brukner, Phys. Rev. Lett. **99**, 180403 (2007).
[35] S. Zozor and C. Vignat, Physica A **375**, 499 (2007).
[36] A. Rényi, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1 (University of California Press, Berkeley, 1961), pp. 547–561.
[37] Note that for a discrete entropy $H[f_j^\eta]$, where $f_j^\eta \equiv \int_{(j-1/2)\eta}^{(j+1/2)\eta} dz\, f(z)$, we have in the limit $\lim_{\eta\to 0}(H[f_j^\eta] + \ln(\eta)) \to h[f]$ [46].
[38] W. Rudin, *Functional Analysis,* 2nd ed. (McGraw-Hill, New York, 1991).
[39] Ł. Rudnicki, *Uncertainty Related to Position and Momentum Localization of a Quantum State*, e-print arXiv:1010.3269v1 (2010).
[40] In the Wolfram MATHEMATICA notation it reads `SpheroidalS1[0,0,ξ,η]`.
[41] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).
[42] D. Chafaï, Séminaire de Probabilité, XXXVI, Lecture Notes in Math. **1801** (Springer, Berlin, 2003), pp. 194–200.
[43] M. Riesz, Acta Math. **49**, 465 (1927).
[44] T. Schürmann, Act. Phys. Pol. B **39**, 587 (2008).
[45] I. Bialynicki-Birula and Ł. Rudnicki, in *Statistical Complexity*, edited by K. D. Sen (Springer, Berlin, 2011), pp. 1–34.
[46] Cover and Thomas, *Elements of Information Theory* (John Wiley & Sons, New York, 2006).