# Time correlation of two $\gamma$ rays resulting from positronium annihilation

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(Received 14 February 2012; published 16 April 2012)

We have obtained the wave function and time correlation of two $\gamma$ rays resulting from the annihilation of a spin-singlet positronium. We have modified the calculations made by Gauthier and Hawton [Phys. Rev. A **81**, 062121 (2010)] in consideration of the real experimental conditions. It has been found that the time correlation is determined by the center-of-mass motion of the positronium, and that the exponential decay component shown by Gauthier and Hawton does not appear in the time-correlation function. We have also conducted an experiment focused on the exponential component in the time-correlation function. The experimental results are consistent with our calculation.

DOI: 10.1103/PhysRevA.85.042111 PACS number(s): 03.65.Ud, 36.10.Dr, 78.70.Bj

## I. INTRODUCTION

Two annihilation $\gamma$ rays resulting from a positron-electron pair have an energy of 511 keV each and therefore they behave more like particles than waves. For all practical purposes they can be considered as a pair of simultaneously generated particles. Still, the wave nature and the wave function of the $\gamma$-ray photons are at times essential, as in studies on time correlation or interference properties of $\gamma$ rays.

Recently, Gauthier and Hawton [1] developed an expression for the wave function of two $\gamma$ rays resulting from the annihilation of a spin-singlet positronium (para-positronium), which is the bound state of a positron and an electron. By means of the wave function and a few assumptions, they concluded that the time-correlation function is proportional to $\exp(-|t_1 - t_2|/\tau)$, where $\tau$ is 125 ps and $t_1, t_2$ are arrival times to the measuring points. Although their result is consistent with Irby's experiment [2,3], no other experimental support could be found. They detected $\gamma$ rays with multichannel plates. Since the signal-to-noise ratio and the detection efficiency of the multichannel plates are very low, the systematic and statistical accuracy is poor. It is known that a scintillation detector is best for $\gamma$-ray timing measurements [4,5]. Moreover, no care was taken to form positronium in the experiment. The $\gamma$ rays in the experiment mainly resulted from the annihilation of unbound positrons with electrons.

In the present paper, we obtain the time-correlation function of two annihilation $\gamma$ rays without the assumptions made by Gauthier and Hawton [1]. We also perform an experiment to accurately determine the exponential component in the time-correlation function.

## II. THEORY

### A. Wave function

The positronium wave function $\Psi_{\rm Ps}$ is the product of the center-of-mass wave function $\Psi_{\rm Ps,c}$ and the wave function of the relative motion $\Psi_{\rm Ps,r}$:

$$\Psi_{\rm Ps}(\boldsymbol{x}_{\rm p},\boldsymbol{x}_{\rm e}) = \Psi_{\rm Ps,c}(\boldsymbol{x}_{\rm c})\Psi_{\rm Ps,r}(\boldsymbol{x}_{\rm r}), \tag{1}$$

where $\boldsymbol{x}_{\rm p}$ and $\boldsymbol{x}_{\rm e}$ are the coordinates of the positron and the electron, respectively. The center-of-mass coordinate $\boldsymbol{x}_{\rm c}$ and the relative coordinate $\boldsymbol{x}_{\rm r}$ are defined as

$$\boldsymbol{x}_{\rm c} = \tfrac{1}{2}(\boldsymbol{x}_{\rm p} + \boldsymbol{x}_{\rm e}), \quad \boldsymbol{x}_{\rm r} = \boldsymbol{x}_{\rm p} - \boldsymbol{x}_{\rm e}. \tag{2}$$

In the momentum representation, Eq. (1) becomes

$$\psi_{\rm Ps}(\boldsymbol{p}_{\rm p},\boldsymbol{p}_{\rm e}) = \psi_{\rm Ps,c}(\boldsymbol{p}_{\rm c})\psi_{\rm Ps,r}(\boldsymbol{p}_{\rm r}), \tag{3}$$

where $\boldsymbol{p}_{\rm p}$ and $\boldsymbol{p}_{\rm e}$ are the momentum of the positron and the electron, respectively. The center-of-mass momentum $\boldsymbol{p}_{\rm c}$ and the relative momentum $\boldsymbol{p}_{\rm r}$ are defined as

$$\boldsymbol{p}_{\rm c} = \boldsymbol{p}_{\rm p} + \boldsymbol{p}_{\rm e}, \quad \boldsymbol{p}_{\rm r} = \tfrac{1}{2}(\boldsymbol{p}_{\rm p} - \boldsymbol{p}_{\rm e}).$$

The para-positronium annihilates into two $\gamma$ rays with a lifetime of 125 ps. The $\gamma$-ray wave function can be written similarly as

$$\phi_\gamma(\boldsymbol{k}_1,\boldsymbol{k}_2) = \phi_{\gamma,\rm c}(\boldsymbol{k}_c)\phi_{\gamma,\rm r}(\boldsymbol{k}_r), \tag{4}$$

where $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the wave vector of the first and the second $\gamma$ ray, respectively. $\boldsymbol{k}_c$ and $\boldsymbol{k}_r$ are defined as

$$\boldsymbol{k}_{\rm c} = \boldsymbol{k}_1 + \boldsymbol{k}_2, \quad \boldsymbol{k}_{\rm r} = \tfrac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2). \tag{5}$$

The center-of-mass momentum is unchanged before and after the annihilation. Therefore, we have

$$\boldsymbol{p}_{\rm c} = \hbar\boldsymbol{k}_{\rm c} \tag{6}$$

and

$$\psi_{\rm Ps,c}(\boldsymbol{p}_{\rm c}) = \phi_{\gamma,\rm c}(\boldsymbol{k}_{\rm c}). \tag{7}$$

These are the principles of the two-photon angular correlation method [6].

The relative part of the $\gamma$-ray wave function represents the two $\gamma$ rays resulting from the annihilation of the para-positronium at rest ($\boldsymbol{k}_c = \vec{0}$). Therefore, the state vector $|\phi_{\gamma,r}\rangle$ which correspond to $\phi_{\gamma,\rm r}(\boldsymbol{k}_r)$ can be written as

$$|\phi_{\gamma,r}\rangle = C\sum_{\boldsymbol{k}} \phi_{\gamma,\rm r}(\boldsymbol{k})\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}}|0\rangle, \tag{8}$$

where $\hat{a}^\dagger_{\boldsymbol{k}}$ is the creation operator of a photon of the wave vector $\boldsymbol{k}$, $C$ is a constant parameter for normalization, and $|0\rangle$ is the vacuum state. In the present study, normalization is not important and the same notation $C$ is used for different functions.

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As shown later, $\phi_{\gamma,\mathrm{r}}(\boldsymbol{k})$ for large $t$ without a phase factor has the form

$$\phi_{\gamma,\mathrm{r}}(\boldsymbol{k}) = C\frac{1}{E(\boldsymbol{k}) - E_0 + i\Gamma/2}. \tag{9}$$

Here $\Gamma$ is the decay width of the para-positronium and is given by

$$\Gamma = \frac{\hbar}{\tau_{\mathrm{s}}} = \frac{\hbar}{125\ \mathrm{ps}}, \tag{10}$$

$E(\boldsymbol{k})$ is the total energy of the two $\gamma$ rays, which equals to $2\hbar c|\boldsymbol{k}|$ with $c =$ the speed of light in vacuum and $E_0$ is the rest mass of an electron and a positron, which is $2mc^2$. Equation (9) has the same form as Eq. (6) on page 181 of Ref. [7] by Heitler by setting $t = \infty$. It is also identical to equation (6.3.18) in Ref. [8]. The expression in Refs. [7,8] is obtained for the first-order process. As shown below, Eq. (9) also holds for the para-positronium annihilation into two $\gamma$-ray photons, which is the second-order process.

In this paragraph we perform the second-order calculation to obtain Eq. (9). According to Sec. 14 in Ref. [7], the Hamiltonian $H$ is

$$H = H_0 + H_{\mathrm{int}}. \tag{11}$$

$H_0$ is the Hamiltonian without the interaction between the radiation field and the electron (positron). $H_{\mathrm{int}}$ is the contribution from the interaction. Schrödinger's equation in the interaction representation is

$$\begin{aligned} i\hbar\frac{\partial\Phi'}{\partial t} &= H'_{\mathrm{int}}\Phi', \\ H'_{\mathrm{int}} &= \exp(iH_0t/\hbar)H_{\mathrm{int}}\exp(-iH_0t/\hbar). \end{aligned} \tag{12}$$

The solution of Eq. (12) can be expanded as

$$\Phi'(t) = \sum_n b_n(t)\Phi_n, \tag{13}$$

where $\Phi_n$ is the $n$th eigenstate of $H_0$ with the energy eigenvalue of $E_n$. $\Phi_0$ represents the initial state in which no photon is present. We have a positron and an electron. In the intermediate state $\Phi_{n'}$ we have one photon, a positron, and an electron. In the final state we have two photons and no positron-electron pair. The initial conditions for Schrödinger's equation are

$$b_0(t=0) = 1, \quad b_n(t=0) = 0 \quad (n \neq 0). \tag{14}$$

Since $H_0$ is diagonal for $\Phi_n$, we have

$$i\hbar\dot{b}_{n'} = H_{n'|0}b_0(t)\exp[i\,(E_{n'} - E_0)\,t/\hbar], \tag{15}$$

for the intermediate states and

$$i\hbar\dot{b}_n = \sum_{n'} H_{n|n'}b_{n'}(t)\exp[i(E_n - E_{n'})t/\hbar] \tag{16}$$

for the final states. $H_{i|j}$ is a matrix element of $H$. We know that the probability of finding the system in the initial state decays exponentially; thus, we have

$$b_0(t) = \exp\left(-\frac{\gamma}{2}t\right), \tag{17}$$

where $\gamma = 1/125$ ps $= \Gamma/\hbar$. From Eqs. (15) and (17), we obtain

$$b_{n'}(t) = H_{n'|0}\frac{\exp[i(E'_n - E_0)t/\hbar - \gamma t/2] - 1}{i(E'_n - E_0)/\hbar - \gamma/2}. \tag{18}$$

In the final state we have two $\gamma$ rays with wave vectors $\boldsymbol{k}_{n'}$ and $-\boldsymbol{k}_{n'}$ due to the momentum conservation law. This implies that the summation over $n'$ in Eq. (16) is not necessary. The intermediate and final states are in one-to-one correspondence. Then, for the final state, we have

$$\begin{aligned} i\hbar\dot{b}_n &= H_{n|n'}b_{n'}(t)\exp[i(E_n - E_{n'})t/\hbar], \\ E_n &= 2\hbar c|\boldsymbol{k}_n|, \\ E_{n'} &= 2mc^2 + \hbar c|\boldsymbol{k}_n|. \end{aligned} \tag{19}$$

$E_{n'}$ is approximately $3mc^2$. Using Eqs. (18) and (19), we have

$$b_n(t) \propto \frac{\exp[i(E_n - E_0)t/\hbar - \gamma t/2] - 1}{i(E_n - E_0)/\hbar - \gamma/2}\,\frac{1}{i(E'_n - E_0)/\hbar - \gamma/2} + \text{the second term}. \tag{20}$$

The second term is negligible. Because $\gamma \ll mc^2/\hbar$, the energy dependence in $H_{n|n'}$ and $H_{n'|0}$ is neglected. In addition, $E_{n'} - E_0$ is approximately equal to $mc^2$, and the denominator $i(E'_n - E_0)/\hbar - \gamma/2$ is almost constant. Letting $t \to \infty$ in Eq. (20) we obtain Eq. (9).

From Eq. (9) we obtain

$$\phi_{\gamma,\mathrm{r}}(\boldsymbol{k}) = C\frac{1}{2}\frac{1}{\hbar c|\boldsymbol{k}| - mc^2 + i\Gamma/4} \tag{21}$$

and

$$|\phi_{\gamma,\mathrm{r}}\rangle = C\sum_{\boldsymbol{k}}\frac{1}{\hbar c|\boldsymbol{k}| - mc^2 + i\Gamma/4}\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}}|0\rangle. \tag{22}$$

Equation (22) has the same form as Eq. (10) in Ref. [1]. However, the coefficient of $\Gamma$ in Ref. [1] is twice that of $\Gamma$ in Eq. (22), and is not explained in Ref. [1]. This difference will affect the remaining results.

### B. Time-correlation function

The time-correlation function can be obtained from the second-order Glauber correlation function [9]

$$\begin{aligned} G_{12}(\boldsymbol{r}_1,t_1,\boldsymbol{r}_2,t_2) = \langle\Phi|\hat{E}^{(-)}_1(\boldsymbol{r}_1,t_1)\hat{E}^{(-)}_2(\boldsymbol{r}_2,t_2) \\ \times \hat{E}^{(+)}_2(\boldsymbol{r}_2,t_2)\hat{E}^{(+)}_1(\boldsymbol{r}_1,t_1)|\Phi\rangle, \end{aligned} \tag{23}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the position vectors of detectors 1 and 2, respectively. The electric field operators $\hat{E}^{(\pm)}_i(\boldsymbol{r}_i,t_i)$, $i = 1,2$ are given by

$$\hat{E}^{(+)}_i(\boldsymbol{r}_i,t_i) = \frac{1}{L^{3/2}}\sum_k\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\hat{a}_k\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}_i - \omega t_i)] \tag{24}$$

and

$$\hat{E}^{(-)}_i(\boldsymbol{r}_i,t_i) = \frac{1}{L^{3/2}}\sum_k\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\hat{a}^\dagger_k\exp[-i(\boldsymbol{k}\cdot\boldsymbol{r}_i - \omega t_i)], \tag{25}$$

where $\omega = c|\boldsymbol{k}|$, $\epsilon_0$ is the vacuum dielectric constant, and $L$ is the period of the boundary condition.

The time-correlation function is given by

$$R_{12}(\tau) = \int dt_1 G_{12}(\boldsymbol{r}_1,t_1,\boldsymbol{r}_2,t_1+\tau). \tag{26}$$

We can limit ourselves to one-dimensional problems by putting the $\gamma$-ray detectors on the $x$ axis at $x_1$ ($\gg$0) and $x_2$ ($\ll$0). The detectors detect the $\gamma$ rays having $\boldsymbol{k}$ vectors parallel to the $x$ axis.

First let us try to put Eq. (22) into Eq. (23). We calculate the right part $|\mathrm{R}\rangle$ of Eq. (23) as

$$\begin{aligned}|\mathrm{R}\rangle &= E_2^{(+)}(x_2,t_2)E_1^{(+)}(x_1,t_1)|\phi_{\gamma,\mathrm{r}}\rangle\\ &= \frac{1}{L^3}\left\{\sum_k \sqrt{\frac{\hbar\omega}{2\epsilon_0}}\hat{a}_k \exp\left[i(kx_2-\omega t_2)\right]\right\}\\ &\quad\times\left\{\sum_k \sqrt{\frac{\hbar\omega}{2\epsilon_0}}\hat{a}_k \exp\left[i(kx_1-\omega t_1)\right]\right\}\\ &\quad\times\left\{\sum_{k_\mathrm{r}} \frac{C}{\hbar c|k_\mathrm{r}|-mc^2+i\Gamma/4}\hat{a}^\dagger_{k_r}\hat{a}^\dagger_{-k_r}|0\rangle\right\}.\end{aligned} \tag{27}$$

Here, $k_\mathrm{r}$ is the $x$ component of $\boldsymbol{k}_\mathrm{r}$. Due to the symmetry, the summations over $k_\mathrm{r} > 0$ and $k_\mathrm{r} < 0$ are equal. We calculate for $k_\mathrm{r} > 0$. $|\mathrm{R}\rangle$ is divided into two parts as

$$|\mathrm{R}\rangle = |\mathrm{R}_+\rangle + |\mathrm{R}_-\rangle \tag{28}$$

and

$$\begin{aligned}|\mathrm{R}_+\rangle &\propto \sum_{k_\mathrm{r}>0} \exp\left[i(-k_\mathrm{r}x_2-\omega t_2)\right]\exp\left[i(k_\mathrm{r}x_1-\omega t_1)\right]\\ &\quad\times\frac{1}{\hbar c k_\mathrm{r}-mc^2+i\Gamma/4}|0\rangle\\ &= \sum_{k_\mathrm{r}>0}\frac{\exp\left[ik_\mathrm{r}(x_1-x_2-ct_1-ct_2)\right]}{\hbar c k_\mathrm{r}-mc^2+i\Gamma/4}.\end{aligned} \tag{29}$$

Similarly,

$$|\mathrm{R}_-\rangle \propto \sum_{k_\mathrm{r}>0}\frac{\exp[ik_\mathrm{r}(x_2-x_1-ct_1-ct_2)]}{\hbar c k_\mathrm{r}-mc^2+i\Gamma/4}. \tag{30}$$

We used $\hat{a}_k\hat{a}^\dagger_l - \hat{a}^\dagger_l\hat{a}_k = \delta_{kl}$ and $\hat{a}|0\rangle = 0$. Since $\Gamma \ll mc^2$, the energy dependence in $\sqrt{\omega}$ is negligible. The right-hand sides of Eqs. (29) and (30) can be determined via the residue theorem. We have a pole at

$$\hbar c k_\mathrm{r} = mc^2 - i\frac{\Gamma}{4}. \tag{31}$$

If $x_1 - x_2 - ct_1 - ct_2 > 0$, $|\mathrm{R}_+\rangle$ equals zero. Similarly, $|\mathrm{R}_-\rangle$ is zero when $x_2 - x_1 - ct_1 - ct_2 > 0$. After integration we obtain

$$\begin{aligned}|\mathrm{R}\rangle \propto |\mathrm{R}_+\rangle + |\mathrm{R}_-\rangle &= \Theta\left(-(x_1-x_2-ct_1-ct_2)\right)\\ &\quad\times\exp\left[\frac{\gamma}{4c}(x_1-x_2-ct_1-ct_2)\right]\\ &\quad+\Theta\left(-(x_2-x_1-ct_1-ct_2)\right)\\ &\quad\times\exp\left[\frac{\gamma}{4c}(x_2-x_1-ct_1-ct_2)\right]\end{aligned} \tag{32}$$

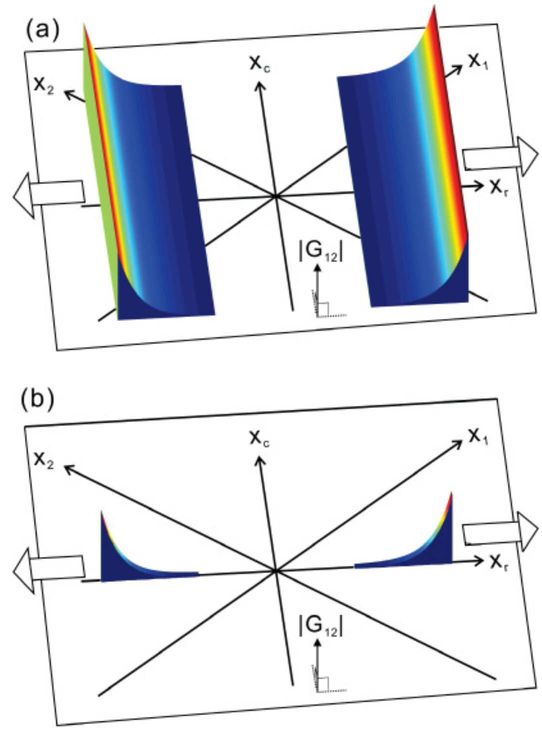


FIG. 1. (Color online) Second-order Glauber correlation function plotted against $x_1$ and $x_2$ for $t = t_1 = t_2$. Panel (a) is for Eq. (33) and panel (b) is for Eq. (41).

and

$$\begin{aligned}G_{12}(x_1,x_2,t_1,t_2) &\propto \Theta\left((-x_1+x_2)/c+t_1+t_2\right)\\ &\quad\times\exp\left\{-\frac{\gamma}{2}\left[(-x_1+x_2)/c+t_1+t_2\right]\right\}\\ &\quad+\Theta\left((-x_2+x_1)/c+t_1+t_2\right)\\ &\quad\times\exp\left\{-\frac{\gamma}{2}\left[(-x_2+x_1)/c+t_1+t_2\right]\right\}.\end{aligned} \tag{33}$$

When we put detectors at $x_1$ ($>$0) and $x_2$ ($<$0), time correlation is determined by the first term of (33) as

$$R_{12}(\tau) = \int dt G_{12}(x_1,t,x_2,t+\tau) = \text{constant}. \tag{34}$$

This equation gives no information on the time correlation. The reason is clear from Fig. 1(a) in which $G_{12}(x_1,x_2,t_1,t_2)$ from Eq. (33) at $t = t_1 = t_2$ is shown as a function of $x_1$ and $x_2$. In this calculation we use only $\phi_{\gamma,\mathrm{r}}$ in Eq. (27) and assume $k_c = 0$. This implies that the spatial distribution of the initial positronium is infinite and wider than the detector distance. Therefore, this calculation does not correspond to the real experimental conditions.

Instead we should use

$$\Phi = \phi_{\gamma,\mathrm{r}}\phi_{\gamma,\mathrm{c}} \tag{35}$$

and

$$|\Phi\rangle = \sum_{k_r}\sum_{k_c}\phi_{\gamma,\mathrm{c}}(k_c)\phi_{\gamma,\mathrm{r}}(k_r)\hat{a}^\dagger_{k_\mathrm{r}+k_c/2}\hat{a}^\dagger_{-k_\mathrm{r}+k_c/2}|0\rangle \tag{36}$$

in Eq. (23). We obtain

$$|\mathrm{R}'\rangle = E_2^{(+)}(x_2,t_2)E_1^{(+)}(x_1,t_1)|\Phi\rangle = |\mathrm{R}'_+\rangle + |\mathrm{R}'_-\rangle \propto \left\{\sum_k \hat{a}_k \exp[i(kx_2-\omega t_2)]\right\}\left\{\sum_k \hat{a}_k \exp[i(kx_1-\omega t_1)]\right\} \times \left\{\sum_{k_r}\sum_{k_c}\phi_{\gamma,c}(k_c)\frac{1}{\hbar c|k_r| - mc^2 + i\Gamma/4}\hat{a}^\dagger_{k_r+k_c/2}\hat{a}^\dagger_{-k_r+k_c/2}|0\rangle\right\}. \quad (37)$$

Using $\hat{a}_k\hat{a}_l^\dagger - \hat{a}_l^\dagger\hat{a}_k = \delta_{kl}$ and $\hat{a}|0\rangle = 0$, we obtain

$$|\mathrm{R}'_+\rangle \propto \sum_{\hbar k_r\approx mc}\sum_{k_c}\exp\left[i\left(k_r+\frac{k_c}{2}\right)(x_1-ct_1) - \left(k_r-\frac{k_c}{2}\right)(x_2+ct_2)\right]\phi_{\gamma,c}(k_c)\frac{1}{\hbar ck_r - mc^2 + i\Gamma/4}|0\rangle$$
$$= \sum_{\hbar k_r\approx mc}\sum_{k_c}\exp\left[ik_r(x_1-ct_1-x_2-ct_2) + i\frac{k_c}{2}(x_1-ct_1+x_2+ct_2)\right]\phi_{\gamma,c}(k_c)\frac{1}{\hbar ck_r - mc^2 + i\Gamma/4}|0\rangle. \quad (38)$$

This can be separated into two parts as

$$|\mathrm{R}'_+\rangle \propto \left\{\sum_{\hbar k_r\approx mc}\frac{\exp[ik_r(x_1-ct_1-x_2-ct_2)]}{\hbar ck_r - mc^2 + i\Gamma/4}\right\}\left\{\sum_{k_c}\exp\left[i\frac{k_c}{2}(x_1-ct_1+x_2+ct_2)\right]\phi_{\gamma,c}(k_c)|0\rangle\right\}. \quad (39)$$

From Eq. (7) $\phi_{\gamma,c} = \psi_{\mathrm{Ps,c}}$. Putting this into Eq. (39), we obtain

$$|\mathrm{R}'_+\rangle \propto \Theta(-(x_1-x_2-ct_1-ct_2)) \times \exp\left[\frac{\gamma}{4c}(x_1-x_2-ct_1-ct_2)\right] \times \Psi_{\mathrm{Ps,c}}\left[\frac{1}{2}(x_1-ct_1+x_2+ct_2)\right]. \quad (40)$$

$|\mathrm{R}'_-\rangle$ is obtained by interchanging $(x_1,t_1)$ and $(x_2,t_2)$ in the above.

Thus we have

$$G_{12}(x_1,x_2,t_1,t_2) = \langle \mathrm{R}'_+|\mathrm{R}'_+\rangle + \langle \mathrm{R}'_-|\mathrm{R}'_-\rangle \propto \Theta(-(x_1-x_2-ct_1-ct_2)) \times \exp\left[\frac{\gamma}{2c}(x_1-x_2-ct_1-ct_2)\right] \times \left|\Psi_{\mathrm{Ps,c}}\left(\frac{x_1+x_2-ct_1+ct_2}{2}\right)\right|^2 + \Theta(-(x_2-x_1-ct_1-ct_2)) \times \exp\left[\frac{\gamma}{2c}(x_2-x_1-ct_1-ct_2)\right] \times \left|\Psi_{\mathrm{Ps,c}}\left(\frac{x_1+x_2+ct_1-ct_2}{2}\right)\right|^2. \quad (41)$$

$G_{12}(x_1,x_2,t_1,t_2)$ in Eq. (41) is shown in Fig. 1(b). In the figure we assume that Ps is confined between $-L_{\mathrm{Ps}}$ and $+L_{\mathrm{Ps}}$ ($L_{\mathrm{Ps}} \ll c\tau_s = 3.75$ cm) and $|\Psi_{\mathrm{Ps,c}}|^2$ is constant for $-L_{\mathrm{Ps}} < x_c < L_{\mathrm{Ps}}$.

The time correlation $R_{12}(\tau)$ is determined by the first term of Eq. (41) because we put detectors at $x_1$ (>0) and $x_2$ (<0). We have

$$R_{12}(\tau) = \int dt G_{12}(x_1,t,x_2,t+\tau) \propto \left|\Psi_{\mathrm{Ps,c}}\left(\frac{x_1+x_2+c\tau}{2}\right)\right|^2. \quad (42)$$

The time-correlation function $R_{12}(\tau)$ is determined by the initial Ps center-of-mass distribution $|\Psi_{\mathrm{Ps,c}}|^2$. An example is shown in Fig. 2.

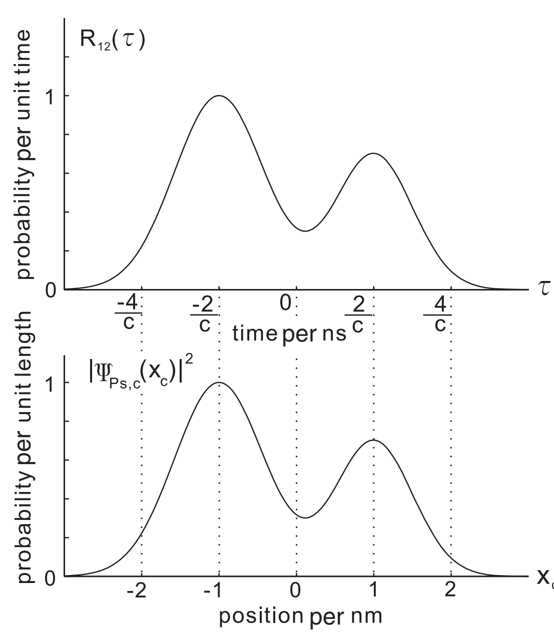


FIG. 2. Relation between the center-of-mass wave function of positronium and the time-correlation function.

### C. Positron annihilation lifetime spectrum

The positron annihilation lifetime spectrum can be obtained from $G_{12}$. At $t = 0$ we have a para-positronium confined between $-L_{\mathrm{Ps}}$ and $L_{\mathrm{Ps}}$. Since $L_{\mathrm{Ps}}$ is small (typically $L_{\mathrm{Ps}} \approx 1$ Å $\ll c\tau_s$), we have

$$G_{12}(x_1,x_2,t_1,t_2) \propto \Theta\left(-(x_1 - x_2 - ct_1 - ct_2)\right) \times \exp\left[\frac{\gamma}{2c}(x_1 - x_2 - ct_1 - ct_2)\right] \times \delta\left(\frac{1}{2}(x_1 + x_2 - ct_1 + ct_2)\right). \quad (43)$$

The positron annihilation lifetime spectrum $P(t_1,x_1)$ is proportional to the probability for detecting the $\gamma$ ray at $t_1$ and $x_1$. This probability is obtained by integrating $G_{12}$ over $t_2$. Thus we have

$$P(t_1,x_1) \propto \int dt_2 G_{12} = \Theta\left(-2(x_1 - ct_1)\right) \exp\left[\frac{\gamma}{c}(x_1 - ct_1)\right]. \quad (44)$$

Thus we should obtain the mean lifetime $\tau$ from the positron annihilation lifetime spectrum as

$$\tau = \frac{1}{\gamma} = 125 \text{ ps}. \quad (45)$$

The same value is obtained in Ref. [1]. This is because two effects due to error or artifact in Ref. [1] cancel each other out. The linewidth in relative motion in Ref. [1] is as twice that of ours and no explanation is given in Ref. [1]. This factor of 2 is canceled during the integration of the wave function in Eq. (22) in Ref. [1]. Their wave function, which is essentially the same as Eq. (33) in the present paper, seems not to be consistent with the real experiment.

## III. EXPERIMENT

Positronium atoms were formed using a $^{22}$Na positron source (700 kBq) and MgO ultrafine powder or silica aerogel. To enhance the intensity of the para-positronium, the sample and source assembly was placed in air. Spin-triplet positronium was converted into spin-singlet positronium via positronium-$O_2$ collisions. The time-correlation function was measured with a digital positron lifetime spectrometer using $BaF_2$ scintillators and fast photomultipliers [4,5]. The full width at half maximum of the time resolution was 180 to 190 ps. Two detectors were set at a distance of 20 cm from the positronium. We performed two measurements. In the first run, positronium was formed using MgO ultrafine powder. The intensity of the para-positronium was estimated to be 30% taking into account the effect of the spin conversion. In the second run, positronium was formed using silica aerogel under a magnetic field of 0.16 tesla. The intensity of the para-positronium was estimated to be 35%. The magnetic field was applied to induce Zeeman mixing of the positronium. Each run lasted about 10 days.

The measured time-correlation function is plotted in Figs. 3(a) and 3(b) for the first and second runs, respectively.

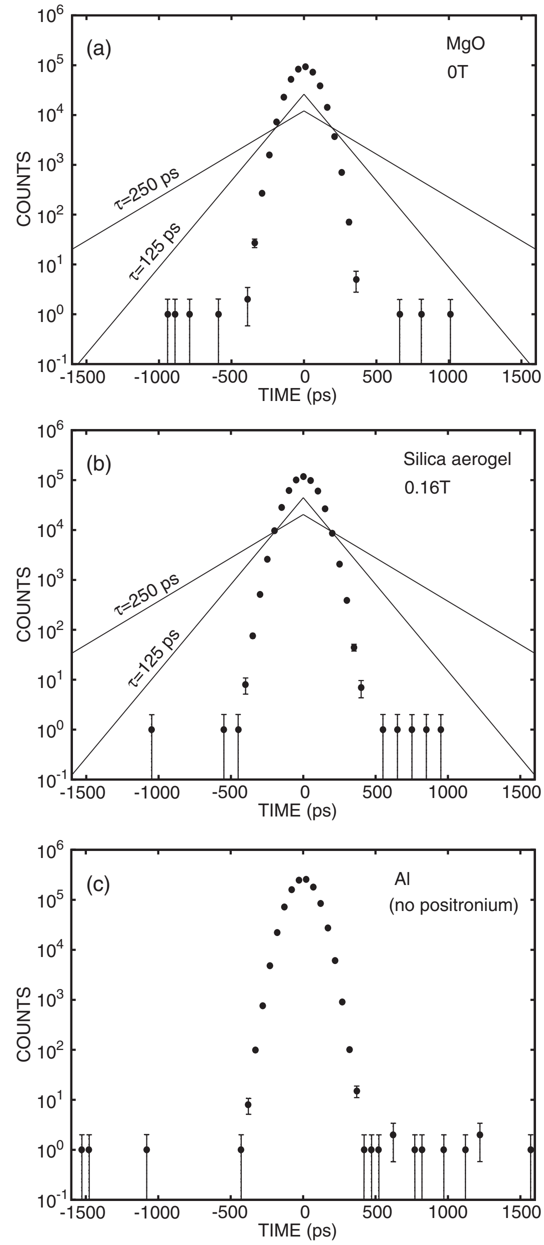


FIG. 3. Time correlation of the two $\gamma$ rays detected by two $BaF_2$ scintillators. A $^{22}$Na positron source was placed at the center of the two detectors, and was (a) surrounded by MgO ultrafine powder, (b) sandwiched by silica aerogel, and (c) sandwiched by Al plates. Time per channel equals 50 ps. Data with zero count are not plotted. The exponential components obtained in Ref. [1] are shown for $\tau = 125$ and 250 ps by solid lines with the expected intensity of spin-singlet positronium in each condition.

The random coincidence background, estimated to be less than 0.5 counts per channel, is not subtracted in Fig. 3. Data with zero count are not plotted. This measurement is very sensitive for the long exponential component shown in Ref. [1].

We also measured the time correlation for $\gamma$ rays due to unbound positrons. The $^{22}$Na positron source was sandwiched by Al. The result is shown in Fig. 3(c).

As can be seen, Figs. 3(a), 3(b), and 3(c) are almost identical. Lines with slopes of 125 and 250 ps are guides for the eye. No long exponential component with a decay of 125 or 250 ps was found. One hundred and twenty-five ps were proposed in Ref. [1], and it should be modified to 250 ps due to Eq. (22). Thus, this experiment supports our calculations.

When looking in more detail, all spectra have short single-exponential components in the tail part. The decay rates are $16 \pm 3$ ps, $23 \pm 3$ ps, and $19 \pm 3$ ps for Figs. 3(a), 3(b), and 3(c), respectively. From these values, the upper limit of the decay of the exponential component is less than 10 ps, which is not consistent with [1].

## IV. DISCUSSION

We have seen that the center of mass of the positronium is of great importance for two-photon time correlation. This permits us to consider the time correlation from a classical mechanical viewpoint with Newton's laws of motion and special relativity. Suppose that we have a positronium atom at rest at $x = 0$. This is possible in classical mechanics. We have two detectors on the $x$ axis. The first detector $\mathrm{M}_1$ having a mass of $M_1$ is put at $L_1$ ($>0$). The second detector $\mathrm{M}_2$ of $M_2$ is put at $L_2 = -L_1(M_1/M_2)$. In the initial state the center of mass of the total system is at $x = 0$. The positronium annihilates into two $\gamma$ rays. Each $\gamma$ ray has an energy of $mc^2$. The momentum of the first $\gamma$ ray is $mc\boldsymbol{u}_x$ and that of the second $\gamma$ ray is $-mc\boldsymbol{u}_x$, where $\boldsymbol{u}_x$ is the unit vector in the $x$ direction. At $t = t_1$, $\mathrm{M}_1$ absorbs the first $\gamma$ ray. The mass of $\mathrm{M}_1$ increases from $M_1$ to $M_1 + m$. The velocity of $\mathrm{M}_1$ becomes $mc/(M_1 + m)$. Similarly, the second $\gamma$ ray arrives on $\mathrm{M}_2$ at $t_2$. The center of mass at $t$ ($>t_1$, $t_2$) is unchanged. Therefore,

$$0 = (M_1 + m)\left[L_1 + \frac{mc}{M_1 + m}(t - t_1)\right] + (M_2 + m)\left[L_2 - \frac{mc}{M_2 + m}(t - t_2)\right]. \tag{46}$$

We obtain

$$t_2 - t_1 = -\frac{1}{c}(L_1 + L_2) = \frac{1}{c}(|L_2| - |L_1|). \tag{47}$$

This shows clearly that the time difference is determined by the detector distance. Any inconsistent result with Eq. (47) contradicts Newton's law of motion. Equation (47) is evidently consistent with our calculation (42).

Irby [2] states that "collapse of the spatial part of the photon's wave function, due to detection of the other photon, does not occur." Now we can study this statement using our wave function. We put a detector at $x_1$ ($>0$). When we detect the first $\gamma$ ray at $x_1$ and $t_1$, the $\gamma$-ray state reduces to $\delta(x - x_1)$. The state vector $|\phi_1\rangle$ is thus given by

$$|\phi_1\rangle = \hat{\phi}_1|0\rangle = \sum_k \exp(-ikx_1)\hat{a}_k^{\dagger}|0\rangle. \tag{48}$$

After the detection of the first $\gamma$ ray, the state vector of the second $\gamma$ ray $|\phi_2\rangle$ becomes

$$|\phi_2\rangle = \hat{\phi}_1^{+}|\Phi\rangle, \tag{49}$$

where $|\Phi\rangle$ is the two-photon wave function which is given by

$$|\Phi\rangle = \sum_{k_r}\sum_{k_c}\phi_{\gamma,\mathrm{c}}(k_c)\phi_{\gamma,\mathrm{r}}(k_r)\hat{a}^{\dagger}_{k_r+k_c/2}\hat{a}^{\dagger}_{-k_r+k_c/2}|0\rangle. \tag{50}$$

The probability of finding the second $\gamma$ ray at $x_2$ and $t_2$ is

$$P'(x_2,t_2) = \langle\phi_2|\hat{E}_2^{(-)}(x_2,t_2')\hat{E}_2^{(+)}(x_2,t_2')|\phi_2\rangle, \tag{51}$$

where $t_2' = t_2 - t_1$. Using Eqs. (22), (24), (48)–(51) and the calculations similar to (35)–(42), we can obtain

$$P'(x_2,t_2) \propto \left|\Psi_{\mathrm{Ps,c}}\left(\frac{x_2 + x_1 + c(t_2 - t_1)}{2}\right)\right|^2. \tag{52}$$

As expected, this is identical to the time-correlation function in Eq. (42). This is the way to collapse the wave function of one photon due to the detection of the position of another photon. This shows clearly that the statement by Irby [2] is not appropriate. In order to measure the arrival time of the $\gamma$ ray, confinement of positronium is essential. The arrival time has no meaning without the confinement, and the confinement cannot coexist with the fixed total momentum due to the uncertainty principle. Therefore, speculations with the fixed total momentum, as in Ref. [2], do not correspond with the real experimental condition.

The wave function of two $\gamma$ rays resulting from the positronium annihilation is completely different from that of another entangled two-photon system, such as emitted with parametric down conversion [10] and from a $2s$-$1s$ transition of a hydrogen atom. The reason for the "simultaneous detection of two photons" is also completely different. The main difference lies in the fact that the total momentum is not conserved in these processes. In parametric down conversion, the two-photon wave function is written as

$$|\Phi\rangle = \sum g(\omega)|E - \omega\rangle|\omega\rangle. \tag{53}$$

The time correlation is determined by $g(\omega)$. When the linewidth is broad, $\Phi$ can be expanded by two nearly localized photons, and we have a narrow time correlation. Compared to the positronium case, the contrast is very striking.

The wave function is indispensable to consider the Bragg reflection of $\gamma$ rays. Para-positronium formed in some single crystals is known to have a spatially broad wave function. In some cases the diffusion length reaches 1 $\mu$m [11]. It is possible that the annihilation $\gamma$ rays from these positronium atoms have high coherence and that the Bragg reflection of these $\gamma$ rays can be prominent. This has been ignored so far because the $\gamma$-ray Bragg peaks are hidden by the positronium Bragg peak [12].

## V. CONCLUSION

We have obtained the wave function of two $\gamma$-ray photons resulting from the annihilation of spin-singlet positronium. The theory proposed by Gauthier and Hawton is modified to fit the real experimental conditions. It is found that the width in the two-photon time-correlation function equals $2/c$ times the spatial distribution of the positronium center of mass. The spatial distribution of the positronium center of mass also determines the coherent length of the annihilation $\gamma$ ray. The experimental results are consistent with the theory.


## ACKNOWLEDGMENTS

H.S. is indebted to Professor T. Kuga and Dr. Y. Torii for valuable discussions. This work was partially supported by KAKENHI (23654114) from Japan Society of for the Promotion of Science (JSPS).



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