Weak-interaction contributions to hyperfine splitting and Lamb shift in light muonic atoms

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Weak-interaction contributions to hyperfine splitting and Lamb shift in light electronic and muonic atoms are calculated. We notice that correction to hyperfine splitting turns into zero for deuterium. Weak correction to the Lamb shift in hydrogen is additionally suppressed in comparison with other cases by a small factor $(1-4\sin^2\theta_W)$.

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I. INTRODUCTION

Unexpected results of the Lamb shift measurement in muonic hydrogen at Paul Scherrer Institute (PSI) [1] gave rise to the proton radius puzzle. It consists of the five σ discrepancy in the value of the proton radius extracted from the muonic hydrogen experiment [1] on the one hand and the values extracted from the electronic hydrogen (see review in the Committee on Data for Science and Technology (CODATA) compilation [2]) and electron-proton scattering [3] on the other hand. A burst of theoretical activity followed the PSI experiment. Old results on the Lamb shift in muonic hydrogen were recalculated and confirmed (see, e.g., Ref. [4] and references therein), proton structure and polarizability corrections were critically reevaluated and improved (see Refs. [5,6] and references therein), and possible new physics explanations were explored (see Refs. [7,8] and references therein). Despite all these efforts no resolution of the proton radius puzzle was found. It seems now that the solution of this problem will require a lot of additional experimental and theoretical work, in particular, precise measurements of different transition frequencies in muonic hydrogen and other light muonic atoms. An experimental program on the measurement of transition frequencies in light muonic atoms is now in progress by the CREMA Collaboration at PSI. First experimental data on hyperfine splitting (HFS) in muonic hydrogen is coming soon, and the results for muonic deuterium and helium ion are to follow [9]. In anticipation of these experimental data we calculate below weak-interaction contributions to HFS and Lamb shift in light muonic atoms, generalizing old results for muonic hydrogen [10] (see also Ref. [11]).

An effective low-energy field-theoretic weak-interaction Hamiltonian due to neutral currents for the fundamental fermions has the form (see, e.g., Ref. [12])

$$H_Z=\frac{4G_F}{\sqrt{2}}\int d^3x\left(\sum_i\bar\psi_i\gamma^\mu(\widetilde{T}_3-\sin^2\theta_W Q)\psi_i\right)^2, \quad (1)$$

where θ_W is the Weinberg angle, Q is the charge operator in terms of the proton charge, $\widetilde{T}_3=T_3(1-\gamma^5)/2$, T_3 is the weak isospin, and summation goes over all species of fermions. Each current in this local four-fermion Hamiltonian contains a vector and an axial part. For nucleons, axial parts are renormalized by strong interactions and should be multiplied by $g_A=1.27$ (see, e.g., Ref. [12]). Specializing for the case of lepton-nucleon interaction the Hamiltonian in Eq. (1) reduces to [an extra factor of 2 arises because all fields enter each factor in Eq. (1)]

$$\begin{aligned}H_Z=\frac{G_F}{2\sqrt{2}}\int d^3x[\bar\psi_l\gamma^\mu\gamma^5\psi_l-\bar\psi_l\gamma^\mu(1-4\sin^2\theta_W)\psi_l]\\ \times[g_A\bar\psi_n\gamma_\mu\gamma^5\psi_n-\bar\psi_n\gamma_\mu\psi_n-g_A\bar\psi_p\gamma_\mu\gamma^5\psi_p\\ +\bar\psi_p\gamma_\mu(1-4\sin^2\theta_W)\psi_p],\end{aligned} \quad (2)$$

where ψ_l is the lepton (electron or muon) field, and ψ_p and ψ_n are the proton and neutron fields, respectively. This Hamiltonian generates all weak-interaction contributions considered below.

II. WEAK-INTERACTION CONTRIBUTIONS TO HYPERFINE SPLITTING

The leading weak-interaction contribution to HFS arises from interaction of axial currents in Eq. (2). In the leading nonrelativistic approximation only spatial components of axial neutral currents give nonzero contributions [10], and the Hamiltonian simplifies to

$$\begin{aligned}H_Z&\to\frac{g_AG_F}{2\sqrt{2}}\int d^3x(\bar\psi_l\gamma^\mu\gamma^5\psi_l)(\bar\psi_n\gamma_\mu\gamma^5\psi_n-\bar\psi_p\gamma_\mu\gamma^5\psi_p)\\ &\to-\frac{g_AG_F}{2\sqrt{2}}\int d^3x(\bar\psi_l\gamma^i\gamma^5\psi_l)(\bar\psi_n\gamma^i\gamma^5\psi_n-\bar\psi_p\gamma^i\gamma^5\psi_p).\end{aligned} \quad (3)$$

For a nucleus with Z protons and A-Z neutrons this field-theoretic Hamiltonian in the nonrelativistic limit reduces to the quantum mechanical Hamiltonian

$$H_Z=\frac{g_AG_F}{2\sqrt{2}}\boldsymbol{\sigma}_l\cdot\left(\sum_p\boldsymbol{\sigma}_p-\sum_n\boldsymbol{\sigma}_n\right)\delta^{(3)}(\boldsymbol{r}). \quad (4)$$

Matrix elements of this operator give the leading weak-interaction contributions to HFS that is nonzero only in S states. The only remaining task is to calculate the expectation value of the scalar product taking into account the nuclear wave function. We consider below in parallel light electronic and muonic atoms and ions, but numerical results are provided only for muonic systems.

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A. Hydrogen

In the case of muonic (electronic) hydrogen there is only one term in the nuclear factor in Eq. (4), and we immediately obtain the leading weak-interaction contribution to HFS splitting in the nS state [10] in the form

$$\Delta E_Z(nS) = \frac{g_A G_F}{2\sqrt{2}} |\psi_n(0)|^3 (\boldsymbol{\sigma}_e \cdot \boldsymbol{\sigma}_p)\Big|_{F=0}^{F=1}, \quad (5)$$

where $\psi_n(0) = \sqrt{(Z\alpha m_r)^3/(\pi n^3)}$ is the Coulomb-Schrödinger wave function at the origin ($Z = 1$ for hydrogen), $m_r = m_l m_p/(m_l + m_p)$ is the reduced mass, $\boldsymbol{J} = \boldsymbol{\sigma}_l/2$ is the lepton spin operator, $\boldsymbol{I} = \boldsymbol{\sigma}_p/2$ is the proton (nucleus) spin operator, and $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{J}$ is the total angular momentum. Obviously, $(\boldsymbol{\sigma}_e \cdot \boldsymbol{\sigma}_p)|_{F=0}^{F=1} = 4$ and

$$\Delta E_Z(nS) = \frac{2 g_A G_F}{\sqrt{2}} \frac{(Z\alpha m_r)^3}{\pi n^3}. \quad (6)$$

Numerically for $n = 2$ in muonic hydrogen the weak contribution is

$$\Delta E_Z(2S) = 2.8 \times 10^{-4} \text{ meV}, \quad (7)$$

which is at least an order of magnitude smaller than the uncertainty in HFS due to proton structure contributions [13].

To elucidate the magnitude of the weak-interaction contribution let us compare it with the dominant Fermi contribution to HFS in muonic hydrogen (see, e.g., [14,15]),

$$E_F = \frac{4}{3} g_p \frac{\alpha (Z\alpha)^3 m_r^3}{m_l m_p} \approx 182.44 \text{ meV}, \quad (8)$$

where $g_p \approx 5.58\ldots$ [2] is the proton g factor in nuclear magnetons.

For excited states the dominant contribution to HFS scales as $1/n^3$ and for the state with an arbitrary principal quantum number n the ratio of the weak and the dominant contributions to HFS in muonic hydrogen is

$$\frac{n^3 \Delta E_Z(nS)}{E_F} = \frac{3}{2\sqrt{2}\pi} \frac{g_A G_F m_\mu m_p}{g_p \alpha} \approx 1.2\ldots \times 10^{-5}. \quad (9)$$

B. Deuterium

Deuteron is a spin-1 loosely bound system of two nonrelativistic nucleons that are predominantly described by the S-state wave function. Respective spin wave function is symmetric and the matrix element of the spin-symmetric deuteron nuclear factor of the effective Hamiltonian in Eq. (4) in this approximation is equal to zero:

$$\langle \boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n \rangle = 0. \quad (10)$$

This conclusion remains valid even after account of the admixture of the D wave in the deuteron wave function, since the D-wave spin function is also symmetric with respect to spin variables (see, e.g., Ref. [16]). Hence, the weak-interaction contribution to HFS in electronic and muonic deuterium in the leading nonrelativistic approximation is zero.

C. Tritium

Triton is a spin-$\frac{1}{2}$ ($I = 1/2$) system of one proton and two neutrons ($Z = 1$, $A = 3$). The third component of isospin for the triton is minus one half ($T_3 = 1/2 - 1/2 - 1/2 = -1/2$). It is predominantly described by a product of the S-wave coordinate wave function and a completely antisymmetric spin-isospin wave function. Obviously, in this approximation $\langle \boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{n_1} - \boldsymbol{\sigma}_{n_2} \rangle = 2\boldsymbol{I}$. A more accurate analysis taking into account the other components of the triton wave function produces [17]

$$\langle \boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{n_1} - \boldsymbol{\sigma}_{n_2} \rangle = 2\boldsymbol{I}\left(1 - \tfrac{4}{3} P_{S'} - \tfrac{2}{3} P_D\right) = 2c\boldsymbol{I}, \quad (11)$$

where $c \approx 0.92$.

Further calculations go exactly like in the hydrogen case above and we obtain

$$\Delta E_Z(nS) = \frac{2c g_A G_F}{\sqrt{2}} \frac{(Z\alpha m_r)^3}{\pi n^3}. \quad (12)$$

Numerically for $n = 2$ in muonic tritium the weak contribution is

$$\Delta E_Z(n = 2) = 3.1 \times 10^{-4} \text{ meV}. \quad (13)$$

Like in the hydrogen case we compare the weak contribution with the dominant Fermi energy in muonic tritium:

$$E_F = \frac{4}{3} g_t \frac{\alpha (Z\alpha)^3 m_r^3}{m_l m_p} = 239.919\ldots \text{ meV}, \quad (14)$$

where $g_t = 5.957\,924\,896(76)$ [2] is the triton g factor in nuclear magnetons.

For excited states the dominant contribution to HFS scales as $1/n^3$, and for the state with an arbitrary principal quantum number n the ratio of the weak and the dominant contributions to HFS in muonic tritium is

$$\frac{n^3 \Delta E_Z(nS)}{E_F} = \frac{3}{2\sqrt{2}\pi} \frac{c g_A G_F m_\mu m_p}{g_t \alpha} \approx 1.0\ldots \times 10^{-5}. \quad (15)$$

D. Helium ion

Helion is a spin-$\frac{1}{2}$ ($I = 1/2$) system of two protons and a neutron ($Z = 2$, $A = 3$). The third component of isospin for the helion is one half ($T_3 = 1/2 + 1/2 - 1/2 = 1/2$). Like the triton the helion is predominantly described by a product of the S-wave coordinate wave function and a completely antisymmetric spin-isospin wave function. Obviously, in this approximation $\langle \boldsymbol{\sigma}_{p_1} + \boldsymbol{\sigma}_{p_2} - \boldsymbol{\sigma}_n \rangle = -2\boldsymbol{I}$. A more accurate analysis taking into account the other components of the helion wave function produces [17]

$$\langle \boldsymbol{\sigma}_{p_1} - \boldsymbol{\sigma}_{p_2} - \boldsymbol{\sigma}_n \rangle = -2\boldsymbol{I}\left(1 - \tfrac{4}{3} P_{S'} - \tfrac{2}{3} P_D\right) = -2c\boldsymbol{I}. \quad (16)$$

Further calculations go exactly like in the hydrogen and tritium cases above and we obtain

$$\Delta E_Z(nS) = -\frac{2c g_A G_F}{\sqrt{2}} \frac{(Z\alpha m_r)^3}{\pi n^3}. \quad (17)$$

Numerically for $n = 2$ in muonic helium the weak contribution is

$$\Delta E_Z(n = 2) = -2.5\ldots \times 10^{-3} \text{ meV}. \quad (18)$$

Like in the hydrogen and tritium cases we compare the weak contribution with the dominant Fermi energy in muonic helium:

$$E_F = \frac{4}{3} g_h \frac{\alpha (Z\alpha)^3 m_r^3}{m_\mu m_p} = -1370.8\ldots \text{ meV}, \tag{19}$$

where $g_h = -4.255\,250\,613$ [2] is the helion g factor in nuclear magnetons.

For excited states the dominant contribution to HFS as $1/n^3$, and for the state with an arbitrary principal quantum number n the ratio of the weak and the dominant contributions to HFS in muonic helium is

$$\frac{n^3 \Delta E^Z}{E_F} = -\frac{3}{2\sqrt{2}\pi} \frac{c g_A G_F m_\mu m_p}{g_h Z\alpha} \approx 1.5\ldots \times 10^{-5}. \tag{20}$$

E. Helium ions e^4He^+ and μ^4He^+

The spin of the α particle is zero and there is no hyperfine structure in e^4He^+ and μ^4He^+ helium ions and no weak-interaction contribution to hyperfine structure.

III. LEADING WEAK-INTERACTION CONTRIBUTION TO LAMB SHIFT

The leading weak-interaction contribution to the Lamb shift arises from interaction of vector currents in Eq. (2). In the leading nonrelativistic approximation only time components give nonzero contributions [10], and the interaction Hamiltonian simplifies to

$$\begin{aligned} H_Z \to{}& \frac{G_F}{2\sqrt{2}} \int d^3x [\bar{\psi}_l \gamma^\mu (1 - 4\sin^2\theta_W)\psi_l] \\ &\times [\bar{\psi}_n \gamma_\mu \psi_n - \bar{\psi}_p \gamma_\mu (1 - 4\sin^2\theta_W)\psi_p] \\ \to{}& \frac{G_F}{2\sqrt{2}} \int d^3x [\bar{\psi}_l \gamma^0 (1 - 4\sin^2\theta_W)\psi_l] \\ &\times [\bar{\psi}_n \gamma_0 \psi_n - \bar{\psi}_p \gamma_0 (1 - 4\sin^2\theta_W)\psi_p]. \end{aligned} \tag{21}$$

For a nucleus with Z protons and A-Z neutrons this field-theoretic Hamiltonian in the nonrelativistic limit reduces to the quantum mechanical Hamiltonian

$$H_Z = \frac{G_F}{2\sqrt{2}} (1 - 4\sin^2\theta_W)[A - Z - Z(1 - 4\sin^2\theta_W)]\delta^{(3)}(\boldsymbol{r}). \tag{22}$$

The matrix element of this operator gives the leading weak-interaction contributions to the Lamb shift that is nonzero only in S states. We obtain an explicit expression for the leading weak correction in all light atoms in the form

$$\begin{aligned} \Delta E_Z(nS) ={}& \frac{G_F}{2\sqrt{2}} (1 - 4\sin^2\theta_W)[(A - Z) - Z(1 - 4\sin^2\theta_W)] \\ &\times \frac{(m_r Z\alpha)^3}{\pi n^3}. \end{aligned} \tag{23}$$

For $A = Z = 1$ this result was obtained in Ref. [10]. It is interesting to notice that in muonic hydrogen due to $A = Z = 1$ the weak contribution to the Lamb shift is additionally suppressed by a small factor, $1 - 4\sin^2\theta_W \approx 0.08$. This suppression disappears for all other light muonic systems.

Let us compare the weak contribution to the Lamb shift with the dominant contribution. The principal contribution to the Lamb shift in light muonic atoms is generated by the diagram with the electron vacuum polarization insertion in the Coulomb photon and was calculated a long time ago [18] (see also reviews in Refs. [14,15,19]):

$$\Delta E_{nl} = -\frac{8\alpha (Z\alpha)^2 m_r}{3\pi n^3} Q_{nl}^{(1)}(\beta), \tag{24}$$

where

$$\begin{aligned} Q_{nl}^{(1)}(\beta) \equiv{}& \int_0^\infty \rho d\rho \int_1^\infty d\zeta f_{nl}^2\left(\frac{\rho}{n}\right) e^{-2\rho\zeta\beta} \\ &\times \left(1 + \frac{1}{2\zeta^2}\right) \frac{\sqrt{\zeta^2 - 1}}{\zeta^2}, \end{aligned} \tag{25}$$

$$f_{nl}\left(\frac{\rho}{n}\right) \equiv \sqrt{\frac{(n-l-1)!}{n[(n+l)!]^3}} \left(\frac{2\rho}{n}\right)^l e^{-\frac{\rho}{n}} L_{n-l-1}^{2l+1}\left(\frac{2\rho}{n}\right), \tag{26}$$

$L_{n-l-1}^{2l+1}(x)$ is the associated Laguerre polynomial [20] and $\beta = m_e/(m_r Z\alpha)$.

For the experimentally relevant interval $2P - 2S$ we obtain

$$\begin{aligned} \Delta E(2P - 2S) &= \Delta E_{21} - \Delta E_{20} \\ &= \frac{\alpha (Z\alpha)^2 m_r}{3\pi} \left[Q_{20}^{(1)}(\beta) - Q_{21}^{(1)}(\beta)\right] \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\frac{\Delta E_Z(L,2S)}{\Delta E(2P - 2S)} \\ &= \frac{3 G_F m_r^2 Z(1 - 4\sin^2\theta_W)[(A - Z) - Z(1 - 4\sin^2\theta_W)]}{16\sqrt{2}\left[Q_{20}^{(1)}(\beta) - Q_{21}^{(1)}(\beta)\right]}. \end{aligned} \tag{28}$$

This expression demonstrates once again that due to the condition $A = Z = 1$ the weak interaction contribution to the Lamb shift is additionally suppressed by an extra factor, $1 - 4\sin^2\theta_W \approx 0.08$, in comparison with the weak-interaction contribution in other light muonic systems. For muonic hydrogen, $\beta \approx 0.7$, $Q_{20}^{(1)}(\beta) = 0.056$, $Q_{21}^{(1)}(\beta) = 0.0037$, and we obtain

$$\frac{\Delta E_Z(L,n = 2)}{\Delta E(2P - 2S)} \approx -1.7 \times 10^{-9}. \tag{29}$$

We see that the weak correction to the Lamb shift in muonic hydrogen is orders of magnitude smaller than the relative error of the Lamb shift measurement [1]. It is also much smaller than the uncertainties of the proton structure corrections [5].

IV. CONCLUSIONS

We calculated the leading weak contributions to HFS and Lamb shift in light muonic atoms and ions. The leading correction to HFS in deuterium is zero because the deuteron weak-interaction Hamiltonian is antisymmetric with respect to nucleon spin variables while the deuteron spin wave function is symmetric. Corrections to Lamb shift in hydrogen are additionally suppressed by the small factor $(1 - 4\sin^2\theta_W)$. This happens because all other nuclei contain neutrons that weakly interact with leptons without this suppression factor. In all cases weak corrections are much smaller than current experimental and theoretical errors.

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