One-dimensional multicomponent fermions with δ-function interaction in strong- and weak-coupling limits: Two-component Fermi gas

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The Fredholm equations for one-dimensional two-component fermions with repulsive and with attractive δ-function interactions are solved by an asymptotic expansion for strong repulsion, weak repulsion, weak attraction, and strong attraction. Consequently, we obtain the first few terms of the expansion of the ground-state energy for the Fermi gas with polarization for these regimes. We also prove that the two sets of Fredhom equations for weakly repulsive and attractive interactions are identical as long as the integration boundaries match each other between the two types. Thus the asymptotic expansions of the energies of repulsive and attractive fermions are identical to all orders in this region. The identity of the asymptotic expansions may not mean that the energy connects analytically.

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I. INTRODUCTION

One-dimensional (1D) Fermi gases with δ-function interaction are important exactly solvable quantum many-body systems and have had tremendous impact in quantum statistical mechanics. The two-component δ-function-interaction Fermi gas with arbitrary polarization was exactly solved by Yang [1] using the Bethe-ansatz hypothesis in 1967. Sutherland [2] generalized the ansatz to solve the 1D multicomponent Fermi gas with δ-function interaction in 1968. The study of multicomponent attractive Fermi gases was initiated by Yang [3] and by Takahashi [4] in 1970. Since then exactly solvable models have been extensively studied by a variety of methods developed in the context of mathematical physics; see [3,5,6]. In particular, recent breakthrough experiments on trapped fermionic atoms confined to one dimension [7] have provided a better understanding of significant quantum-statistical effects and the novel pairing nature in quantum many-body systems. The observed results are seen to be in good agreement with the results obtained using the analysis of exactly solved models [8–12].

Although the Bethe-ansatz equations for the 1D two-component δ-function-interaction Fermi gas with arbitrary polarization were found long ago [1], it was not until much later that this model began to receive more attention in the context of cold atoms [13]. The asymptotic ground-state energy of a Fermi gas with polarization was studied via the discrete Bethe-ansatz equations in strongly and weakly interacting regimes in [14]. But it turns out that the asymptotic expansion of the discrete Bethe-ansatz equations can be controlled only up to the leading-order correction to the interaction energy. However, the fundamental physics of integrable models is usually determined by the set of the generalized Fredholm integral equations in the thermodynamic limit; see the article by Yang [15]. The solutions of the Fredholm equations have not been thoroughly investigated analytically except in a few limiting cases [13,16]; and there are some numerical results [8,9,17,18]. It is usually a difficult task to solve those Fredeholm equations analytically. It is highly desirable to find a systematic way to treat the generalized Fredholm equations.

In the present paper, we develop a systematic method to solve asymptotically the Fredholm equations for a 1D two-component Fermi gas with δ-function interaction and with polarization in four regimes: the strongly repulsive, weakly repulsive, weakly attractive, and strongly attractive regimes. The first few terms of the expansion of the ground-state energy for the Fermi gas with polarization are obtained explicitly for these regimes. We also address the analytical behavior of the ground-state energy at vanishing interaction strength. This method which we develop can be directly applied to 1D multicomponent Fermi gases. The results for 1D κ-component fermions will be reported in the second paper of this study [19].

II. THE FREDHOLM EQUATIONS

The Hamiltonian for the 1D N-body problem [1,20],

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1D}\sum_{1\leqslant i<j\leqslant N}\delta(x_i - x_j), \qquad (1)$$

describes N fermions of the same mass m with two internal spin states confined to a 1D system of length L interacting via a δ-function potential. For an irreducible representation $[2^{N_\downarrow}, 1^{N_\uparrow - N_\downarrow}]$, the Young tableau has two columns. Here, $N_\uparrow$ and $N_\downarrow$ are the numbers of fermions in the two hyperfine levels $|\uparrow\rangle$ and $|\downarrow\rangle$ such that $N = N_\uparrow + N_\downarrow$. The coupling constant g_{1D} can be expressed in terms of the interaction strength $c = -2/a_{1D}$ as $g_{1D} = \hbar^2 c/m$, where a_{1D} is the effective 1D scattering length [21]. Let $2m = \hbar = 1$ for our convenience. We define a dimensionless interaction strength $\gamma = c/n$ for the physical analysis, with the linear density $n = N/L$. For repulsive fermions, $c > 0$ and for attractive fermions, $c < 0$.

The energy eigenspectrum is given in terms of the quasi-momenta $\{k_i\}$ of the fermions via $E = \frac{\hbar^2}{2m}\sum_{j=1}^{N} k_j^2$, which in

terms of the function $e_b(x) = (x + ibc/2)/(x - ibc/2)$ satisfy the Bethe-ansatz (BA) equations

$$\exp(ik_iL) = \prod_{\alpha=1}^{N_\downarrow} e_1(k_i - \lambda_\alpha),$$
$$\prod_{j=1}^{N} e_1(\lambda_\alpha - k_j) = -\prod_{\beta=1}^{N_\downarrow} e_2(\lambda_\alpha - \lambda_\beta), \tag{2}$$

where $i = 1,2,\ldots,N$ and $\alpha = 1,2,\ldots,N_\downarrow$. The parameters $\{\lambda_\alpha\}$ are the rapidities for the internal hyperfine spin degrees of freedom. The fundamental physics of the model is determined by the set of transcendental equations which can be transformed to the generalized Fredholm types of equation in the thermodynamic limit. This transformation was found by Yang and Yang in a series of papers on the study of the spin XXZ model; see [15].

A. Repulsive regime

For repulsive interactions, it is shown from (2) that the Bethe-ansatz quasimomenta $\{k_i\}$ are real, but all $\{\lambda_\alpha\}$ are real only for the ground state; see Ref. [6]. There are complex roots of λ_α called spin strings for excited states. In the thermodynamic limit, i.e., $L,N \to \infty$, with N/L finite, the roots of the Bethe-ansatz equations (2) are dense enough in the parameter space. Therefore we can define the particle quasimomentum distribution function $r_1(k_i) = 1/[L(k_i - k_{i+1})]$ in the quasimomentum space. Here k_i and k_{i+1} are two conjunction quaisimomenta. Similarly, the distribution function of the spin rapidity is defined as $r_2(\lambda_i) = 1/[L(\lambda_i - \lambda_{i+1})]$ in spin parameter space. In order to unify the notations in the Fredholm equations, we replace the parameter λ by k for the distribution function of the spin rapidity. Thus the above Bethe-ansatz equations (2) can be written as the generalized Fredholm equations

$$r_1(k) = \frac{1}{2\pi} + \int_{-B_2}^{B_2} K_1(k-k')r_2(k')dk', \tag{3}$$
$$r_2(k) = \int_{-B_1}^{B_1} K_1(k-k')r_1(k')dk - \int_{-B_2}^{B_2} K_2(k-k')r_2(k')dk'. \tag{4}$$

The associated integration boundaries B_1 and B_2 are determined by the relations

$$n :\equiv N/L = \int_{-B_1}^{B_1} r_1(k)dk,$$
$$n_\downarrow :\equiv N_\downarrow/L = \int_{-B_2}^{B_2} r_2(k)dk, \tag{5}$$

where n denotes the linear density while $n_\downarrow$ is the density of spin-down fermions. The boundary B_1 characterizes the Fermi point in the quasimomentum space, whereas the boundary B_2 characterizes the spin rapidity distribution interval with respect to the polarization. They can be obtained by solving Eqs. (5). In the above equations, we denote the kernel function as

$$K_\ell(x) = \frac{1}{2\pi}\frac{\ell c}{(\ell c/2)^2 + x^2}. \tag{6}$$

The ground-state energy per unit length is given by

$$E = \int_{-B_1}^{B_1} k^2 r_1(k)dk. \tag{7}$$

The magnetization per length is defined by $s_z = (n - 2n_\downarrow)/2$. Through the boundary conditions (5), the ground-state energy (7) can be expressed as a function of total particle density n and magnetization s_z. In the grand-canonical ensemble, we can also get the magnetic field h and chemical potential μ via

$$h = 2\frac{\partial E(n,s_z)}{\partial s_z}, \qquad \mu = \frac{\partial E(n,s_z)}{\partial n}. \tag{8}$$

B. Attractive regime

For the attractive regime, i.e., $c < 0$, it is found from (2) that complex string solutions of k_i also satisfy the Bethe-ansatz equations. Thus the quasimomenta $\{k_i\}$ of the fermions with different spins form two-body bound states [4,22], i.e., $k_i = k_i' \pm i\frac{1}{2}c$, accompanied by the real spin parameter k_i'. Here $i = 1,\ldots,N_\downarrow$. The excess fermions have real quasimomenta $\{k_j\}$ with $j = 1,\ldots,N - 2N_\downarrow$. Thus the Bethe-ansatz equations are transformed into the Fredholm equations for the density of the pairs $\rho_2(k)$ and the density of single Fermi atoms $\rho_1(k)$. They satisfy the following Fredholm equations [3,4]:

$$\rho_1(k) = \frac{1}{2\pi} + \int_{-Q_2}^{Q_2} K_1(k-k')\rho_2(k')dk', \tag{9}$$
$$\rho_2(k) = \frac{2}{2\pi} + \int_{-Q_1}^{Q_1} K_1(k-k')\rho_1(k')dk' + \int_{-Q_2}^{Q_2} K_2(k-k')\rho_2(k')dk'. \tag{10}$$

Here $c < 0$ in the kernel functions $K_\ell(x)$. The integration boundaries Q_1 and Q_2 are the Fermi points of the single particles and pairs, respectively. They are determined by

$$n \equiv: \frac{N}{L} = 2\int_{-Q_2}^{Q_2} \rho_2(k)dk + \int_{-Q_1}^{Q_1} \rho_1(k)dk,$$
$$n_\downarrow \equiv: \frac{N_\downarrow}{L} = \int_{-Q_2}^{Q_2} \rho_2(k)dk. \tag{11}$$

The ground-state energy per length is given by

$$E = \int_{-Q_2}^{Q_2} (2k^2 - c^2/2)\rho_2(k)dk + \int_{-Q_1}^{Q_1} k^2\rho_1(k)dk. \tag{12}$$

In a similar way, the magnetic field and chemical potential can be determined from the relations (8). In the next section, we discuss the solutions and analytical behavior of the Fredholm equations.

III. ASYMPTOTIC SOLUTIONS OF THE FREDHOLM EQUATIONS

A. Strong repulsion

The strong-coupling condition $cL/N \gg 1$ naturally gives the condition $c \gg B_1$, where the Fermi boundary $B_1 \propto n\pi$ according to the Fermi statistics. For the balanced case, the numbers of spin-up and -down fermions are equal. Thus there are no finite "Fermi" points in spin parameter space, i.e., the

boundary $B_2 \to \infty$. Taking a Taylor expansion with the kernel function $K_1(k-k')$ in (4) at $k'=0$, we obtained with an accuracy up to the order of $1/c^4$

$$r_2(k) \approx nK_1(k) + \frac{E}{2\pi}\left[\frac{3c}{\left(\frac{c^2}{4}+k^2\right)^2} - \frac{c^3}{\left(\frac{c^2}{4}+k^2\right)^3}\right] - \int_{-\infty}^{\infty} K_2(k-k')r_2(k')dk'. \quad (13)$$

Here E is the ground-state energy per length. By taking a Fourier transformation of (13), we may obtain the distribution function

$$\tilde{r}_2(\omega) = \left(n - \frac{E\omega^2}{2}\right) \Big/ \left(2\cosh\frac{\omega c}{2}\right). \quad (14)$$

Substituting (14) into the Fredholm equation (3), we have

$$r_1(k) = \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(1/2)c|\omega|}\tilde{r}_2(\omega)e^{i\omega k}d\omega = \frac{1}{2\pi} + \frac{n}{2\pi}\left(Y_0(k) - \frac{E}{2n}Y_2(k)\right), \quad (15)$$

where

$$Y_\alpha(k) \approx \int_{-\infty}^{\infty} \frac{e^{i\omega k}\omega^\alpha d\omega}{1+e^{|\omega|c}}.$$

After some algebra, we obtain

$$Y_0(k) = \frac{2\ln 2}{c} - \frac{3k^2}{2c^3}\zeta(3), \qquad Y_2(k) \approx \frac{3}{c^3}\zeta(3).$$

Here $\zeta(z)$ is the Riemann zeta function. Then we obtain

$$r_1(k) = \frac{1}{2\pi} + \frac{n\ln 2}{\pi c} - \frac{3n\zeta(3)}{4\pi c^3}\left(k^2 + \frac{E}{n}\right) + O(c^{-4}). \quad (16)$$

We see clearly that for strong repulsion the distribution of $r_1(k)$ is very flat and it is a constant up to a correction of the order of $1/c^3$. This naturally suggests that 1D fermions with strong repulsion can be treated as ideal particles with fractional statistics. Substituting (16) into the linear density (5) and energy (7), we obtain

$$n = \frac{B_1}{\pi}\left(1 + \frac{2n\ln 2}{c} - \frac{3E\zeta(3)}{2c^3} - \frac{n^3\pi^2\zeta(3)}{2c^3}\right),$$
$$E = \frac{B_1^3}{3\pi}\left(1 + \frac{2n\ln 2}{c} - \frac{3E\zeta(3)}{2c^3} - \frac{9n^3\pi^2\zeta(3)}{10c^3}\right),$$

which give

$$B_1 = n\pi\left[1 - \frac{2\ln 2}{\gamma}\left(1 - \frac{2\ln 2}{\gamma}\right) - \frac{8(\ln 2)^3}{\gamma^3} + \frac{\pi^2\zeta(3)}{\gamma^3}\right] + O(c^{-4}), \quad (17)$$

$$E = \frac{n^3\pi^2}{3}\left[1 - \frac{4\ln 2}{\gamma} + \frac{12(\ln 2)^2}{\gamma^2} - \frac{32(\ln 2)^3}{\gamma^3} + \frac{8\pi^2\zeta(3)}{5\gamma^3}\right] + O(c^{-4}). \quad (18)$$

We see that the energy is given in terms of the dimensionless strength $\gamma = c/n$. The correction to leading order in $1/\gamma$ was found in [13,14]. Actually, the two sets of the Fredholm equations can be converted into dimensionless units. Therefore the ground-state energy can be written as an analytical function of γ except at $\gamma = 0$. This ground-state energy is a good approximation for the balanced Fermi gas with a strongly repulsive interaction (good agreement is seen for $cL/N > 8$); see Figs. 1 and 2. In these figures, the solid lines are obtained from the ground-state energy (7) and (12) with the numerical solutions to the two sets of Fredholm equations (3) and (4) for the repulsive regime and (9) and (10) for the attractive regime. The dashed lines are plotted from the asymptotic ground-state energy for the four regimes.

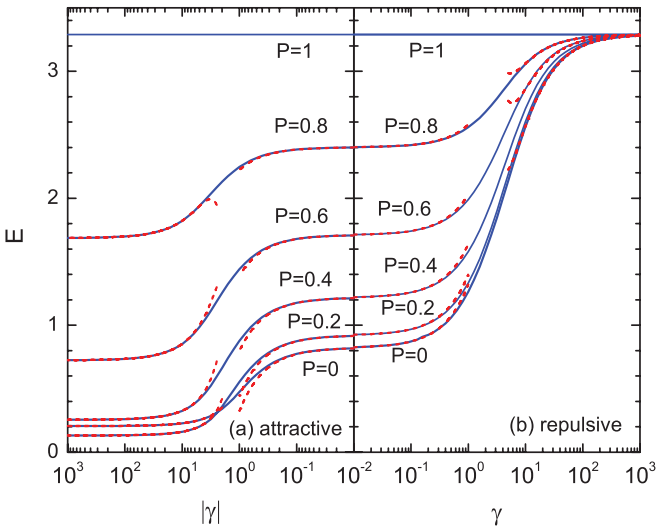

FIG. 1. (Color online) The ground-state energy per length vs logarithmic $\gamma = cL/N$ in units of $\hbar^2N^3/2mL^2$: comparison between the asymptotic and numerical solutions of the Fredholm equations for polarization $P = 0,0.2,0.4,0.6,0.8,1.0$. In the attractive regime, the binding energy $\varepsilon_b = -c^2/2$ was subtracted. The crossing of the two lowest curves in the attractive regime indicates a relation between the critical polarization and interaction, where the chemical potential of single fermions exceeds the chemical potential of the pairs. An excellent agreement between our asymptotic results and numerical plots is seen for strong repulsion, weak repulsion, weak attraction, and strong attraction

However, it is extremely hard to obtain a closed form of the ground-state energy of the gas with an arbitrary spin-population imbalance in the repulsive regime. This is mainly because the distribution function $r_2(k)$ spans the region $-B_2 < k < B_2$, where the integration boundary B_2 can vary from zero to infinity as the polarization changes. The integration boundary B_2 decreases as the polarization increases. An intuitive way of understanding this point is that zero polarization corresponds to $B_2 = \infty$ while the fully polarized case corresponds to $B_2 = 0$. From dressed-energy formalism [10], we can easily see this monotonic relation between the Fermi boundary and polarization by analyzing the band filling under an external field. For high polarization and strong repulsion (i.e., $N_\downarrow \ll N$), we have the conditions $c \gg B_1, B_2$, which allows us to make the following Taylor

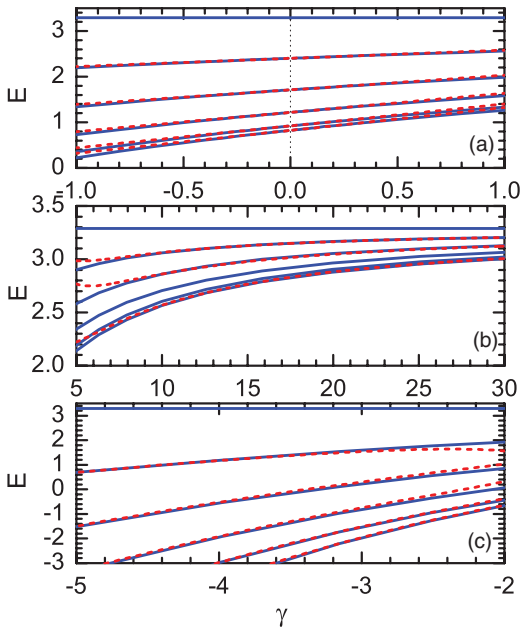

FIG. 2. (Color online) The ground-state energy per length vs $\gamma = cL/N$ in units of $\hbar^2N^3/2mL^2$: comparison between the asymptotic and numerical solutions of the Fredholm equations for polarization $P = 0,0.2,0.4,0.6,0.8,1.0$. An excellent agreement between our asymptotic expansion results and numerical plots is seen in (a) for the weakly repulsive and attractive regimes, (b) for the strongly repulsive regime, and (c) the strongly attractive regime.

expansion:

$$r_2(k) = \frac{1}{2\pi}\int_{-B_1}^{B_1}\frac{cr_1(k')}{\frac{c^2}{4}+k^2}\left[1-\frac{-2kk'+k'^2}{\frac{c^2}{4}+k^2}+\cdots\right]dk' - \int_{-B_2}^{B_2}K_2(k-k')r_2(k')dk' = n\left(1-\frac{4E}{c^2n}\right)K_1(k) - n_\downarrow K_2(k) + O(c^{-4}). \quad (19)$$

Here we denote $n_\downarrow = N_\downarrow/L$. We notice that the leading order of the distribution function $r_2(k)$ is proportional to $1/c$. Furthermore, taking a Taylor expansion in (3), we obtain

$$r_1(k) \approx \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-B_2}^{B_2}\frac{cr_2(k')}{\frac{c^2}{4}+k^2}\left[1-\frac{-2kk'+k'^2}{\frac{c^2}{4}+k^2}\right]dk' = \frac{1}{2\pi}\left[1+\frac{cn_\downarrow}{\frac{c^2}{4}+k^2}\right] + O(c^{-4}). \quad (20)$$

From the asymptotic distribution functions (19) and (20), we calculate the density

$$n = \int_{-B_1}^{B_1} r_1(k)dk \approx \frac{B_1}{\pi}\left(1+\frac{4n_\downarrow}{c}-\frac{16B_1^2n_\downarrow}{3c^3}\right)$$

which gives

$$B_1 \approx n\pi\left(1-\frac{4n_\downarrow}{c}+\frac{16n_\downarrow^2}{c^2}+\frac{16n^2n_\downarrow\pi^2}{3c^3}-\frac{64n_\downarrow^3}{c^3}\right). \quad (21)$$

From the energy (7) and the distribution function $r_1(k)$ (20), we may obtain an asymptotic ground-state energy of a highly polarized Fermi gas with strong repulsion ($c \gg B_1, B_2$),

$$E \approx \frac{1}{3}n^3\pi^2\left[1-\frac{8n_\downarrow}{c}+\frac{48n_\downarrow^2}{c^2}-\frac{1}{c^3}\left(256n_\downarrow^3-\frac{32}{5}\pi^2n^2n_\downarrow\right)\right]. \quad (22)$$

In fact, for strong repulsion, the interaction energy in the ground state of the highly polarized Fermi gas solely depends on the BA quantum number $N_\downarrow$. A similar structure can be found for 1D κ-component fermions [19]. By numerical checking, we see that for $\gamma > 8$ and polarization $P = (N_\uparrow - N_\downarrow)/(N_\uparrow + N_\downarrow) \geqslant 0.5$, the energy (22) is very accurate; see Fig. 2.

B. Weak repulsion

For the weakly repulsive regime, it is convenient to rewrite the Fredholm equations (3) and (4) as

$$r_1(k) = \frac{1}{2\pi} + \int_{-B_2}^{B_2}K_1(k-k')r_2(k')dk', \quad (23)$$

$$r_2(k) = \frac{1}{2\pi} - \int_{|k'|>B_1}K_1(k-k')r_1(k')dk'. \quad (24)$$

The derivation of (24) is straightforward via the Fourier transform of the Fredholm equations (3) and (4), where for our convenience in the study, we actually used

$$r_{m\,\mathrm{in}}(k) = \begin{cases} r_m(k), & |k| \leqslant B_m, \\ 0, & |k| > B_m, \end{cases}$$
$$r_{m\,\mathrm{out}}(k) = \begin{cases} r_m(k), & |k| > B_m, \\ 0, & |k| \leqslant B_m, \end{cases} \quad (25)$$

with $m = 1,2$ in the Fourier transformation. These Fredholm equations are valid for arbitrary polarization including the balanced case. In the following unification of the ground-state energy, we assume $B_1 > B_2$ as an ansatz. In the light of Takahashi's unification of the ground-state energy [23], we give the ground-state energy per length as

$$E = \frac{B_1^2}{3\pi} + \frac{1}{2\pi}\int_{-B_2}^{B_2}H(k,B_1)dk - \int_{-B_2}^{B_2}\left[\int_{|k'|>B_1}K_1(k-k')r_1(k')dk'\right]H(k,B_1)dk, \quad (26)$$

where

$$H(x,y) = \frac{1}{\pi}\left[\left(x^2-\frac{c^2}{4}\right)\pi g_y(x) + yc + \frac{1}{2}xc\ln\frac{4(x-y)^2+c^2}{4(x+y)^2+c^2}\right],$$
$$g_y(x) = 1 - G_+(y,x),$$
$$G_\pm(x,y) = \tan^{-1}\frac{c}{2(x-y)} \pm \tan^{-1}\frac{c}{2(x+y)}.$$

From (5), we find that the integration boundaries B_1 and B_2 satisfy the following conditions:

$$\frac{N_\uparrow}{L} = \frac{B_1}{\pi} - \frac{1}{\pi}\int_{-B_2}^{B_2}r_2(k)G_+(B_1,k)dk, \quad (27)$$

$$\frac{N_\downarrow}{L} = \frac{B_2}{\pi} - \frac{1}{\pi}\int_{|k|>B_1} r_1(k)G_-(k,B_2)dk, \tag{28}$$

in this weakly repulsive regime. Using the condition (28), we may obtain the integration boundary B_2 (up to an order-of-c contribution),

$$\begin{aligned} B_2 \approx n_\downarrow\pi &+ \frac{c}{4\pi}\ln\frac{4(B_1+B_2)^2+c^2}{4(B_1-B_2)^2+c^2} \\ &- \frac{(B_1-B_2)}{\pi}\tan^{-1}\frac{c}{2(B_1-B_2)} \\ &+ \frac{(B_1+B_2)}{\pi}\tan^{-1}\frac{c}{2(B_1+B_2)}, \end{aligned} \tag{29}$$

or

$$B_2 = n_\downarrow\pi + \frac{c}{2\pi}\ln\frac{|B_1+B_2|}{|B_1-B_2|} + O(c^2). \tag{30}$$

The logarithmic term in (29) converges as $B_1 = B_2$. However, the logarithmic term in (30) becomes divergent as $B_1 = B_2$. This divergent term in the ground-state energy can be canceled out. Here we see a subtlety of this asymptotic expansion. Similarly, we calculate the Fermi momentum B_1 by definition (5) and the distribution (23),

$$\begin{aligned} B_1 = n_\uparrow\pi &+ \frac{c}{4\pi}\ln\frac{4(B_1+B_2)^2+c^2}{4(B_1-B_2)^2+c^2} \\ &- \frac{(B_1-B_2)}{\pi}\tan^{-1}\frac{c}{2(B_1-B_2)} \\ &+ \frac{(B_1+B_2)}{\pi}\tan^{-1}\frac{c}{2(B_1+B_2)}, \end{aligned} \tag{31}$$

or

$$B_1 \approx n_\uparrow\pi + \frac{c}{2\pi}\ln\frac{|B_1+B_2|}{|B_1-B_2|} + O(c^2). \tag{32}$$

The ground-state energy per length in the weakly repulsive coupling limit can be expressed in terms of the Fermi boundaries,

$$\begin{aligned} E \approx &\frac{B_1^3}{3\pi} + \frac{B_2^3}{3\pi} + \frac{2c}{\pi^2}B_1B_2 \\ &- \frac{c}{2\pi^2}\left(B_1^2+B_2^2\right)\ln\frac{|B_1+B_2|}{|B_1-B_2|}. \end{aligned} \tag{33}$$

Substituting (30) and (32) into (33), we obtain the ground-state energy of the Fermi gas with a weakly repulsive interaction and with polarization

$$E = \tfrac{1}{3}n_\uparrow^3\pi^2 + \tfrac{1}{3}n_\downarrow^3\pi^2 + 2cn_\uparrow n_\downarrow + O(c^2). \tag{34}$$

This leading-order correction to the interaction energy indicates a mean-field effect. By numerical checking, we see that the energy (34) agrees well with the numerical results in this weak-coupling regime; see Fig. 2.

For the balanced case, the integration boundary B_2 tends to infinity. It is different from the above setting where we consider $B_1 > B_2$. The Fredholm equations (3) and (4) [or (23) and (24)] can be simplified with the help of a Fourier transformation,

$$r_1(k) = \frac{1}{\pi} - \int_{|k'|>B_1} K_2(k-k')r_{1\,\mathrm{out}}(k')dk'. \tag{35}$$

However, for $|k| > B_1$, we find

$$r_{1\,\mathrm{out}}(k) = \frac{1}{2\pi} - \int_{-\infty}^{\infty} R(k-k')r_{1\,\mathrm{in}}(k')dk', \tag{36}$$

where the function $R(k)$ is given by

$$\begin{aligned} R(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+e^{-|\omega|c}}e^{-i\omega k}d\omega \\ &= -\frac{1}{\pi}\left(\frac{c}{4k^2} + \frac{c^3}{8k^4} + \cdots\right). \end{aligned}$$

We see clearly that the second term in (36) gives a contribution $O(c)$. Substituting the leading term $r_1(k) = 1/2\pi$ for the region $|k| > B_1$ into the distribution function $r_1(k)$ (35), we obtain the distribution function $r_1(k)$ for $|k| < B_1$,

$$r_1(k) \approx \frac{1}{\pi} - \frac{1}{2\pi^2}\left[\tan^{-1}\frac{c}{k+B_1} + \tan^{-1}\frac{c}{B_1-k}\right].$$

From the relation $n = \int_{-B_1}^{B_1} r_1(k)dk$, we find

$$B_1 \approx \frac{n\pi}{2}\left[1 + \frac{c}{4\pi B_1}\ln\frac{4B_1^2+c^2}{c^2} + \frac{1}{\pi}\tan^{-1}\frac{c}{2B_1}\right]. \tag{37}$$

Then we obtain the balanced ground-state energy for a weakly repulsive interaction,

$$\begin{aligned} E = \frac{2}{3\pi}B_1^3\Bigg[1 - \frac{3}{4\pi}\Bigg(&\frac{c}{B_1}\ln\frac{4B_1^2+c^2}{c^2} \\ &+ \frac{4}{3}\tan^{-1}\frac{c}{2B_1} - \frac{8}{3}\frac{c}{B_1}\Bigg)\Bigg] + O(c^2). \end{aligned} \tag{38}$$

It is clearly seen that up to the order $O(c^2)$ the ground-state energy of a balanced gas with weak repulsion converges as $c \to 0$ (or say $cL/N \to 0$ by a rescaling in the above equations). By substituting B_1 into (38), we find that the logarithmic term is canceled. The ground-state energy for the balanced case is given by

$$E = \tfrac{1}{12}n^3\pi^2 + \tfrac{1}{2}n^2c + O(c^2). \tag{39}$$

We further discuss the continuity of the energy at vanishing interaction strength in the next section.

C. Weak attraction

For the weakly attractive regime, the Fredholm equations (9) and (10) are rewritten as

$$\rho_1(k) = \frac{1}{2\pi} + \int_{-Q_2}^{Q_2} K_1(k-k')\rho_2(k')dk', \tag{40}$$

$$\rho_2(k) = \frac{1}{2\pi} - \int_{|k'|>Q_1} K_1(k-k')\rho_1(k')dk'. \tag{41}$$

These Fredholm equations are valid for arbitrary polarization. It is seen clearly that the Fredholm equations (23) and (24) for the repulsive regime and (40) and (41) for the attractive regime are identical as long as the integration boundaries match each other between the two cases. Similarly, for unification of the energy of a gas with weak attraction, we assume $Q_1 > Q_2$. In

the above equations, Q_1 and Q_2 are determined by

$$\frac{N_\uparrow}{L} = \frac{Q_1}{\pi} - \frac{1}{\pi}\int_{-Q_2}^{Q_2} \rho_2(k) G_+(Q_1,k)dk, \tag{42}$$

$$\frac{N_\downarrow}{L} = \frac{Q_2}{\pi} - \frac{1}{\pi}\int_{|k|>Q_1} \rho_1(k) G_-(k,Q_2)dk, \tag{43}$$

which indeed match the integration boundaries B_1 and B_2 [(27) and (28)] in the region $B_1 > B_2$ and $Q_1 > Q_2$.

Furthermore, substituting (40) and (41) into the ground-state energy (12), we obtain

$$E = \frac{Q_1^2}{3\pi} + \frac{1}{2\pi}\int_{-Q_2}^{Q_2} H(k,Q_1)dk - \int_{-Q_2}^{Q_2}\left[\int_{|k'|>Q_1} K_1(k-k')\rho_1(k')dk'\right] H(k,Q_1)dk. \tag{44}$$

From Eqs. (42) and (43), we find

$$Q_1 = n_\uparrow\pi - \frac{|c|}{4\pi}\ln\frac{4(Q_1+Q_2)^2+c^2}{4(Q_1-Q_2)^2+c^2} + \frac{(Q_1-Q_2)}{\pi}\tan^{-1}\frac{|c|}{2(Q_1-Q_2)} - \frac{(Q_1+Q_2)}{\pi}\tan^{-1}\frac{|c|}{2(Q_1+Q_2)} + O(c^2), \tag{45}$$

$$Q_2 = n_\downarrow\pi - \frac{|c|}{4\pi}\ln\frac{4(Q_1+Q_2)^2+c^2}{4(Q_1-Q_2)^2+c^2} + \frac{(Q_1-Q_2)}{\pi}\tan^{-1}\frac{|c|}{2(Q_1-Q_2)} - \frac{(Q_1+Q_2)}{\pi}\tan^{-1}\frac{|c|}{2(Q_1+Q_2)} + O(c^2). \tag{46}$$

Indeed, by calculating the ground-state energy (44) with the integration boundaries (45) and (46) for a weakly attractive interaction, we do find a similar form of the ground-state energy:

$$E = \tfrac{1}{3}n_\uparrow^3\pi^2 + \tfrac{1}{3}n_\downarrow^3\pi^2 - 2|c|n_\uparrow n_\downarrow + O(c^2). \tag{47}$$

We see that the asymptotic ground-state energies (34) and (47) continuously connect at $c = 0$ for arbitrary polarization; see Figs. 1 and 2.

For the balanced attractive regime, the Fermi boundary $Q_1 = 0$ and Q_2 is finite; the Fredholm equations (9) and (10) [or (40) and (41)] reduce to

$$\rho_2(k) = \frac{1}{\pi} + \int_{-Q_2}^{Q_2} K_2(k-k')\rho_2(k')dk'. \tag{48}$$

By iteration, the Fermi boundary Q_2 is obtained from $n = 2\int_{-Q_2}^{Q_2}\rho_2(k)$ in a straightforward way:

$$Q_2 \approx \frac{n\pi}{2}\left[1 - \frac{|c|}{4\pi Q_2}\ln\frac{4Q_2^2+c^2}{c^2} - \frac{1}{\pi}\tan^{-1}\frac{|c|}{2Q_2}\right], \tag{49}$$

which gives a similar form to that of the Fermi boundary B_1; see (37). After lengthy algebra and iteration, we obtain the ground-state energy

$$E = \frac{2}{3\pi}Q_2^3\left[1 + \frac{3}{4\pi}\left(\frac{|c|}{Q_2}\ln\frac{4Q_2^2+c^2}{c^2} + \frac{4}{3}\tan^{-1}\frac{|c|}{2Q_2} - \frac{8}{3}\frac{|c|}{Q_2}\right)\right] + O(c^2). \tag{50}$$

It is clearly seen that up to the oder $O(c^2)$ the ground-state energy of the balanced gas with an attractive interaction also converges as $c \to 0^-$. By substituting Q_2 into (50), we find that the logarithmic term is canceled out. Thus the energy is given by

$$E = \tfrac{1}{12}n^3\pi^2 - \tfrac{1}{2}n^2|c| + O(c^2), \tag{51}$$

which continuously connects to the energy (39) at $c \to 0$. But the identity of the asymptotic expansions may not mean that the energy analytically connects because of the divergence in the small region $c \to i0$ and the mismatch of the Fermi boundaries associated with the two sets of Fredholm equations for both cases. Nevertheless, we see that under a mapping

$$r_1(k) \longleftrightarrow \rho_1(k),\ r_2(k) \longleftrightarrow \rho_2(k),\ c \longleftrightarrow c, \tag{52}$$

the Fredholm equations (23) and (24) with (27) and (28) for the repulsive regime and the Fredholm equations (40) and (41) with (42) and (43) for the attractive regime are identical for $Q_1 > Q_2$ and $B_1 > B_2$. In the above equations $c > 0$ for repulsive interaction and $c < 0$ for attractive interaction are implied. We also see that the ground-state energy of the gas with a weakly repulsive interaction (26) and the gas with a weakly attractive interaction (44) are unified under the mapping (52). This unification leads to continuity of the energy for this polarized gas at vanishing interaction strength, i.e., $c \to 0$. Thus the asymptotic expansions of the energies of repulsive and attractive fermions with nonzero polarization are identical to all orders in the vanishing-interaction-strength limit as long as the conditions $Q_1 > Q_2$ and $B_1 > B_2$ hold.

The analyticity of the the energy at $c = 0$ was discussed by Takahashi [23]. Takahashi's theorem states that (a) the energy function $f(n_\uparrow,n_\downarrow;c)$ is analytic on the real axis of c when $n_\uparrow \neq n_\downarrow$; (b) $f(n_\uparrow,n_\downarrow;c)$ is analytic on the real axis of c except for $c = 0$ when $n_\uparrow = n_\downarrow$. This theorem appears not to be true for the region $B_1 < B_2$ and $Q_1 < Q_2$ in our study. Takahashi's proof of this theorem relies on his Lemma 2, i.e., the function f, density n, and density of spin-down fermions $n_\downarrow$ are analytic as functions of Q, B, and c except for the region $c = 0$ and $Q < B$. Here Q and B are two integration boundaries. Even the identity of the asymptotic expansions of the energy may not mean that the energy analytically connects due to the divergence of the two sets of Fredholm equations in the limit $c \to i0$ and the mismatch of the intervals for the density distribution functions. Although we unified the two sets of Fredholm equations (23),(24) and (40),(41) for arbitrary polarization, the integration boundaries between the two regimes are mismatched for the regions $Q_1 < Q_2$ and

$B_1 < B_2$, i.e.,

$$\frac{N_\uparrow}{L} = \frac{B_1}{\pi} - \frac{1}{\pi}\int_{-B_1}^{B_1} r_2(k)G_+(B_1,k)dk - \int_{B_1<|k|<B_2} r_2(k)\left[1-\frac{1}{\pi}G_-(k,B_1)\right]dk, \quad (53)$$

$$\frac{N_\downarrow}{L} = \frac{B_2}{\pi} - \frac{1}{\pi}\int_{|k|>B_2} r_1(k)G_-(k,B_2)dk - \int_{B_1<|k|<B_2} r_1(k)\left[1-\frac{1}{\pi}G_+(B_2,k)\right]dk \quad (54)$$

for the weakly repulsive regime, and

$$\frac{N_\uparrow}{L} = \frac{Q_1}{\pi} - \frac{1}{\pi}\int_{-Q_1}^{Q_1} \rho_2(k)G_+(Q_1,k)dk + \int_{Q_1<|k|<Q_2} \rho_2(k)\left[1+\frac{1}{\pi}G_-(k,Q_1)\right]dk, \quad (55)$$

$$\frac{N_\downarrow}{L} = \frac{Q_2}{\pi} - \frac{1}{\pi}\int_{|k|>Q_2} \rho_1(k)G_-(k,Q_2)dk + \int_{Q_1<|k|<Q_2} \rho_1(k)\left[1+\frac{1}{\pi}G_+(Q_2,k)\right]dk \quad (56)$$

for the weakly attractive regime. It is obvious that the signs in the last term in each equation are mismatched. In the above equations $c > 0$ for repulsive interaction and $c < 0$ for attractive interaction are implied. This mismatch is clearly seen in the balanced case: $B_1 \to Q_2$, $B_2 \to \infty$, and $Q_1 \to 0$. Thus we see that the Fredholm equations cannot be unified for the region $Q_1 < Q_2$ and $B_1 < B_2$ at vanishing interaction strength.

D. Strong attraction

In recent years, the strongly attractive Fermi gas has received considerable attention in theory and experiment due to the existence of a novel pairing state. For the spin-1/2 Fermi gas with strongly attractive interaction, two fermions with different spin states can form a tightly bound pair. For the ground state, the model has three distinct quantum phases, i.e., a fully paired phase with equal numbers of spin-up and -down fermions, a fully polarized phase of single spin-up fermions, and a partially polarized phase with both pairs and excess fermions. The key features of this $T = 0$ phase diagram of the strongly attractive spin-1/2 Fermi gas were experimentally confirmed using finite-temperature density profiles of trapped fermionic ^{6}Li atoms [7].

Here we calculate the ground-state energy (12) from the Fredholm equations (9) and (10) with the integration boundaries Q_1, Q_2 that characterize the Fermi points of two Fermi seas, i.e., the Fermi seas for excess fermions and pairs, respectively. Therefore, for strong attraction, i.e., $|c|L/N \gg 1$, all integration boundaries are finite, i.e., Q_1 and Q_2 are finite. In this regime, the conditions $c \gg Q_1, Q_2$ hold for arbitrary polarization. From the following calculation, we will see that the conditions $Q_1 > Q_2$ and $Q_1 < Q_2$ do not change the expression of the energy. Therefore, the following result is valid for arbitrary polarization, including the balanced case. In this regime, it is convenient to use the notation $|c|$ instead of a negative value of c. The ground-state energy is calculated in the following way:

$$E = \frac{Q_1^3}{3\pi} + \frac{1}{\pi}\int_{-Q_2}^{Q_2}\rho_2(k)\left[\left(k^2-\frac{c^2}{4}\right)(2\pi - \tan^{-1}\frac{2(Q_1-k)}{|c|} + \tan^{-1}\frac{2(Q_1+k)}{|c|}\right) - Q_1|c| - \frac{1}{2}\lambda|c|\ln\frac{4(k-Q_1)^2+c^2}{4(k+Q_1)^2+c^2}\right]dk. \quad (57)$$

Furthermore, we consider a strong-coupling expansion in the energy (57); here we assume $|c| \gg Q_1, Q_2$. We collect contributions up to the order of $1/|c|^3$, i.e.,

$$E \approx \frac{Q_1^3}{3\pi}\left[1 - \frac{4n_\downarrow}{|c|} + \frac{48Q_1^2 n_\downarrow}{5|c|^3} + \frac{32Q_2^3}{3\pi|c|^3}\right] - \frac{c^2}{2}n_\downarrow + 2\int_{-Q_2}^{Q_2}\rho_2(k)k^2dk. \quad (58)$$

In the last equation of (58), the first part in the square brackets is the kinetic energy of excess single atoms including the marginal interference effect between the single atoms and molecules of two atoms. The second term is the total binding energy of the bound pairs. The last term characterizes the total energy of the molecules of two atoms. We now calculate the Fermi momenta Q_1 and Q_2 and the energy of the molecules of two atoms. For our convenience, we denote

$$E = E_0^u + E_0^b + n_\downarrow\varepsilon_b \quad (59)$$

with

$$E_0^u = \frac{Q_1^3}{3\pi}\left[1 - \frac{4n_\downarrow}{|c|} + \frac{48Q_1^2n_\downarrow}{5|c|^3} + \frac{32Q_2^3}{3\pi|c|^3}\right],$$
$$E_0^b = 2\int_{-Q_2}^{Q_2}\rho_2(k)k^2dk, \qquad \varepsilon_b = -\frac{c^2}{2}. \quad (60)$$

We calculate Q_1 from (11) with the density (9):

$$n_\uparrow - n_\downarrow = \int_{-Q_1}^{Q_1}\left(\frac{1}{2\pi} - \int_{-Q_2}^{Q_2}K_1(k-k')\rho_2(k')\right)dk \approx \frac{Q_1}{\pi}\left[1 - \frac{4n_\downarrow}{|c|} + \frac{16Q_1^2n_\downarrow}{3|c|^3} + \frac{32Q_2^3}{3\pi|c|^3}\right].$$

Then we obtain the Fermi momentum

$$Q_1 \approx (n-n_\downarrow)\pi\left[1 + \frac{4n_\downarrow}{|c|} + \frac{16n_\downarrow^2}{c^2} - \frac{16}{3|c|^3}\left((n_\uparrow-n_\downarrow)^2\pi^2n_\downarrow + \frac{n_\downarrow^3\pi^2}{4} - 12n_\downarrow^3\right)\right]. \quad (61)$$

Similarly, we calculate Q_2 from (11) with the distributions (9) and (10),

$$Q_2 \approx \frac{n_\downarrow\pi}{2}\left(1 + \frac{2n_\uparrow-n_\downarrow}{|c|} + \frac{(2n_\uparrow-n_\downarrow)^2}{c^2} + \frac{(2n_\uparrow-n_\downarrow)^3}{|c|^3} - \frac{n_\downarrow^2\pi^2(8n_\uparrow-7n_\downarrow)}{12|c|^3} - \frac{\pi^2\left[n_\downarrow^3+32(n_\uparrow-n_\downarrow)^3\right]}{12|c|^3}\right). \quad (62)$$

We observe that the kernels in the Fredholm equations (9) and (10) converge quickly with the distribution functions as the interaction strength $|c|$ increases. This allows one to make a proper Taylor series expansion in the distribution functions. In this way, from Eq. (10) we may obtain

$$\begin{aligned}\rho_2(k) &\approx \frac{1}{\pi}\left[1-\frac{n_\downarrow|c|}{c^2+k^2}+\frac{|c|E_0^b}{2(c^2+k^2)^2}\right]-\frac{1}{2\pi}\left[\frac{|c|(n_\uparrow-n_\downarrow)}{\frac{c^2}{4}+k^2}\right.\\ &\left.+\frac{|c|}{\left(\frac{c^2}{4}+k^2\right)^2}\int_{-Q_1}^{Q_1}\rho_1(k')k'^2dk'\right]\\ &=\frac{1}{\pi}+\frac{2Q_2^3}{3\pi^2|c|^3}+\frac{8Q_1^3}{3\pi^2|c|^3}-\frac{n_\downarrow}{\pi}\frac{|c|}{c^2+k^2}\\ &\quad-\frac{n_\uparrow-n_\downarrow}{2\pi}\frac{|c|}{\frac{c^2}{4}+k^2}. \end{aligned}\tag{63}$$

Substituting (63) into the energy E_0^b (60),

$$\begin{aligned}E_0^b &\approx \frac{4Q_2^3}{3\pi}+\frac{8Q_2^6}{9\pi^2|c|^3}+\frac{32Q_1^3Q_2^3}{9\pi^2|c|^3}\\ &\quad-\frac{2n_\downarrow}{\pi}\int_{-Q_2}^{Q_2}\frac{|c|k^2dk}{c^2+k^2}-\frac{(n_\uparrow-n_\downarrow)}{\pi}\int_{-Q_2}^{Q_2}\frac{|c|k^2dk}{\frac{c^2}{4}+k^2}\\ &\approx\frac{4Q_2^3}{3\pi}\left(1-\frac{2n_\uparrow-n_\downarrow}{|c|}+\frac{2\left(Q_2^3+4Q_1^3\right)}{3\pi|c|^3}\right.\\ &\quad\left.+\frac{3(8n_\uparrow-7n_\downarrow)Q_2^2}{5|c|^3}\right). \end{aligned}\tag{64}$$

Substituting Eqs. (61) and (62) into the ground-state energy (58) and (64), we obtain the ground-state energy of the gas with a strongly attractive interaction and with an arbitrary polarization:

$$\begin{aligned}E_0^u &\approx \frac{(n_\uparrow-n_\downarrow)^3\pi^2}{3}\left[1+\frac{8n_\downarrow}{|c|}+\frac{48n_\downarrow^2}{c^2}\right.\\ &\quad\left.-\frac{8n_\downarrow}{15|c|^3}[12\pi^2(n_\uparrow-n_\downarrow)^2-480n_\downarrow^2+5n_\downarrow^2\pi^2]\right], \end{aligned}\tag{65}$$

$$\begin{aligned}E_0^b &\approx \frac{n_\downarrow^3\pi^2}{6}\left[1+\frac{2(2n_\uparrow-n_\downarrow)}{|c|}+\frac{3(2n_\uparrow-n_\downarrow)^2}{c^2}\right.\\ &\quad-\frac{4}{15|c|^3}\left(180n_\downarrow n_\uparrow^2+20\pi^2n_\uparrow^3-90n_\uparrow n_\downarrow^2-22\pi^2n_\downarrow^3\right.\\ &\quad\left.\left.+15n_\downarrow^3-120n_\uparrow^3+63\pi^2n_\downarrow^2n_\uparrow-60\pi^2n_\downarrow n_\uparrow^2\right)\right]. \end{aligned}\tag{66}$$

We define the polarization $P=(N_\uparrow-N_\downarrow)/N=(n_\uparrow-n_\downarrow)/n$; then the energy in terms of polarization is given by

$$\begin{aligned}E &\approx \frac{\hbar^2n^3}{2m}\left\{-\frac{(1-P)\gamma^2}{4}+\frac{\pi^2(1-3P+3P^2+15P^3)}{48}\right.\\ &\quad+\frac{\pi^2(1-P)(1+P-5P^2+67P^3)}{48|\gamma|}\\ &\quad\left.+\frac{\pi^2(1-P)^2(1+5P+3P^2+247P^3)}{64\gamma^2}\right.\\ &\quad-\frac{\pi^2(1-P)}{1440|\gamma|^3}\left[-15+31\,125P^4+1861\pi^2P^5\right.\\ &\quad-15\,765P^5-659\pi^2P^4+346\pi^2P^3-14\pi^2P^2\\ &\quad\left.\left.+\pi^2P+\pi^2-105P-150P^2-15\,090P^3\right]\right\}, \end{aligned}\tag{67}$$

which agrees with the result derived from dressed-energy equations [10,11]. This result is highly accurate as can be seen in Figs. 1 and 2. From the energies (65) and (66), we see that the bound pairs have tails and they interfere with each other. But it is impossible to separate the intermolecular forces from the interference between molecules and single fermions. If we consider $n_\downarrow \gg x=n_\uparrow-n_\downarrow$, the single atoms are repelled by the molecules, i.e.,

$$E(n_\downarrow,x)\approx\frac{E(n_\downarrow,0)}{L}+\frac{1}{6}n_\downarrow^3\pi^2\left[\frac{4x}{|c|}+\frac{12x(x+n_\downarrow)}{c^2}\right],$$

where

$$E(n_\downarrow,0)\approx\frac{1}{6}n_\downarrow^3\pi^2\left(1+\frac{2n_\downarrow}{|c|}+\frac{3n_\downarrow^2}{c^2}\right)+\varepsilon_b.$$

In addition, the phase diagram and magnetism can be worked out directly from the relations (8) with the ground-state energy for the four regimes. The phase boundaries of the full phase diagrams may be analytically and numerically obtained by imposing the conditions $s_z=0,0.5$ in the conditions (8), as has been discussed in the literature [8–10,18].

IV. CONCLUSION

In conclusion, we have presented a systematic method to derived the first few terms of the asymptotic expansion of the Fredholm equations for the spin-1/2 Fermi gas with repulsive and attractive δ-function interactions in four regimes: the strongly repulsive, weakly repulsive, weakly attractive, and strongly attractive regimes. We have obtained explicitly the ground-state energy of the Fermi gas with polarization in these regimes; see the key results (18), (22), (34), (47), (65), and (66). By numerical checking, these asymptotic ground-state energies are seen to be highly accurate in the four regimes. In the weakly attractive and repulsive regimes, the ground-state energies, integration boundary relations, and associated two sets of Fredholm equations have been unified. The two sets of Fredholm equations can be identical as long as the associated integration boundaries match each other between the two cases. This suggests that the asymptotic expansions of the energies of the repulsive and attractive fermions are identical to all orders in this region as $c\to 0$. The identity of the asymptotic expansions may not mean that the energy analytically connects, due to the divergence of the two sets of Fredholm equations in the limit $c\to i0$ and the mismatch of the associated integration boundaries between the two cases at some intervals, e.g., $B_1<B_2$ and $Q_1<Q_2$.

Moreover, the explicit results obtained for the ground-state energy provides facilities to study the universal nature of many-body phenomena. The local pair correlation for opposite spins can be calculated directly from the ground-state energy by

$$g_{\uparrow,\downarrow}^{(2)}(0)=\frac{1}{2n_\uparrow n_\downarrow}\partial E(n,s_z)/\partial c.$$

This naturally gives the 1D analog of Tan's adiabatic theorem [24] through the relation

$$\mathcal{C} = \frac{4}{a_{1D}^2} n_\uparrow n_\downarrow g^{(2)}_{\uparrow,\downarrow}(0),$$

where $\mathcal{C}$ is called the universal contact, measuring the probability that two fermions with opposite spin stay together. It was shown [24] that the momentum distribution exhibits universal $\mathcal{C}/k^4$ decay as the momentum tends to infinity. The significant feature of Tan's universal contact is that it can be applied to any many-body system of interacting bosons and fermions in 1D, 2D, and 3D [24,25]. In addition, the explicit forms of the ground-state energies in the four regimes can be used to determine the magnetism and phase diagram of the system in the grand-canonical ensemble. It can help to evaluate quantum-statistical effects by a comparison between the ground-state energies of 1D δ-function-interacting fermions and spinless bosons. These provide a precise understanding of many-body correlations and quantum magnetism in the context of cold atoms. The method which we have developed in this paper can be generalized to study ground-state properties of 1D multicomponent Fermi and Bose gases with δ-function interaction. We consider this in the companion paper [19].

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