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Completeness of classical ϕ^4 theory on two-dimensional lattices

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We formulate a quantum formalism for the statistical mechanical models of discretized field theories on lattices and then show that the discrete version of ϕ^4 theory on 2D square lattice is complete in the sense that the partition function of any other discretized scalar field theory on an arbitrary lattice with arbitrary interactions can be realized as a special case of the partition function of this model. To achieve this, we extend the recently proposed quantum formalism for the Ising model [M. Van den Nest, W. Dur, and H. J. Briegel, Phys. Rev. Lett. **98**, 117207 (2007)] and its completeness property [M. Van den Nest, W. Dur, and H. J. Briegel, Phys. Rev. Lett. **100**, 110501 (2008)] to the continuous variable case.

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I. INTRODUCTION

The understanding that a single partition function can describe different phases of matter is rather recent and, indeed, as late as the 1930s, there was not a consensus among physicists that a partition function can give a sharp phase transition. The works of Kramers, Wannier, Onsager [1–3], and others gradually established beyond doubt that in the thermodynamic limit, singular behavior and phase transition can arise from a single partition function based on a single model Hamiltonian. For example, the Ising Hamiltonian can describe both the ordered phase of a ferromagnet and the disordered phase of a paramagnet. Near the point of second-order phase transition, even the details of the model Hamiltonian do not matter and only some general properties like symmetries are important.

Decades of works on statistical mechanics models, inspired by the above general understanding, has revealed even further fruitful relations between different models. Besides the well-known duality relations between the low- and high-temperature phases of the Ising model, one can also mention the so-called vertex models, which reduce to other models in different limits.

One can ask if there are certain statistical mechanical models that are complete, in the sense that their partition function reduce to the partition function of any other model in a suitable limit? If this turns out to be the case, then we can imagine a very large space of coupling constants and one single Hamiltonian, i.e., the Ising model with inhomogeneous couplings, so that when we move through this space, we meet new phases and new models that at present are thought to be completely unrelated. This will then be another forward in the unification program mentioned above.

It seems that the answer to the above question may be positive. Recent results [4–13] brought about by merging of ideas from statistical mechanics and quantum information theory give positive clues in favor of the above idea. These investigations [4,5] have been made possible by establishing a link between statistical mechanics and quantum information on the one hand and the new paradigm of measurement-based quantum computation (MQC) and the universality of cluster states for MQC on the other [14–23].

In a series of recent works, it has been shown that the Ising model on two-dimensional square lattices with complex inhomogeneous nearest-neighbor interactions is complete in the sense that the partition function of all other discrete models with general k-body interactions on arbitrary lattices can be realized as special cases of the partition function of Ising model on a square lattice, which is polynomially or exponentially larger than the original lattice. The starting point of these developments was the observation in Ref. [4] that the partition function of any given discrete model can be written as a scalar product,

$$Z_G(J) = \langle\alpha|\Psi_G\rangle, \tag{1}$$

where $\langle\alpha|$ is a product state encoding all the coupling constants J and $|\Psi_G\rangle$ is an entangled *graph* state, defined on the vertices and edges of a graph, and encoding the geometry of the lattice. The core concept of completeness is the fact that the 2D cluster state is universal. More concretely, we know that the graph state $|\Phi_G\rangle$ corresponding to a graph G can be obtained from an appropriate cluster state $|\Psi_\square\rangle$ corresponding to a rectangular lattice (denoted by $\square$), through a set of adaptive single-qubit measurements $\mathcal{M}$. The measurements being single-qubit can be formally written as $|\alpha_M\rangle\langle\alpha_M|$, where $|\alpha_M\rangle$ is a product state encoding the qubits the bases and the results which have been measured. Thus the totality of measurements $\mathcal{M}$ transforms the cluster state as follows

$$|\Psi_\square\rangle \longrightarrow |\alpha_M\rangle\langle a_M|\Psi_\square\rangle = |\alpha_M\rangle \otimes |\Psi_G\rangle, \tag{2}$$

which shows that after disregarding the states of the measured qubits $|\alpha_M\rangle$, what is left is an appropriate graph state

$$|\Psi_G\rangle = \langle\alpha_M|\Psi_\square\rangle. \tag{3}$$

Combination of this relation with Eq. (1) leads to the completeness result mentioned above. That is, one writes

$$Z_G(J) \equiv \langle\alpha|\Psi_G\rangle = \langle\alpha,\alpha_M|\Psi_\square\rangle \tag{4}$$

and notes that $\langle\alpha,\alpha_M|$ now encodes a set of generally inhomogeneous pattern of interactions on the cluster state. Therefore, one has

$$Z_G(J) \equiv Z_\square(J,J'). \tag{5}$$

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In this way it has been shown that the 2D Ising model with complex inhomogeneous couplings is complete [5].

Like any completeness result, an interesting question is whether there are other complete models. The situation is reminiscent of results on NP completeness of certain problems in computer science [24]. In this direction it has been shown in Refs. [6,7] that the four-dimensional Z_2 lattice gauge theory, with real couplings, is complete for producing any spin model in any dimension. This result was extended in Ref. [8] to show that the four-dimensional U(1) lattice gauge theory with real couplings can produce, to arbitrary precision, a large number of continuous models (those whose Hamiltonian allow a finite Fourier series). Certainly there may be many other complete models that can be converted to each other. Exploration of the set of complete models certainly will add to our insight and to our power in connecting different models with each other.

In this paper, we show that the discrete form of ϕ^4 field theory, on a two-dimensional rectangular lattice is also complete in the sense that the partition function of any continuous model on any graph with any type of interaction can be obtained, to arbitrary precision, from a ϕ^4 model with inhomogeneous complex couplings on an enlarged 2D lattice.

The structure of this paper is as follows: First we gather the necessary ingredients for our analysis in Sec. II, i.e., elementary facts about continuous variable (CV) states, operators, and measurements. We then reconsider in a new language three types of continuous variable stabilizer states, namely the CV Kitaev states, the CV extended Kitaev states, and the ordinary graph states and the relations between them. In Sec. IV we introduce the quantum formalism for scalar field models on arbitrary graphs and investigate properties of these models, properties that are made transparent by using the quantum formalism and are otherwise not easy to unravel. Then, in Sec. VI, we show that the free field theory on two-dimensional rectangular lattice is complete for free theories in the sense that from its partition function, every other free field theory on any graph can be obtained as a special case. Finally, we show that the ϕ^4 field theory on 2D rectangular lattice is complete, in the sense that its partition function reduces to the partition function of any interacting model on any graph. As in the Ising case, the price that one pays is that the coupling constants of the complete model should be inhomogeneous and complex.

II. PRELIMINARIES

In this section, we collect the preliminary materials necessary for generalization of the quantum formalism to the continuous variable case [25–28]. First we review the definition of Heisenberg-Weyl group and the way a unitary operator can be performed on CV state (a qumode) by measurements of an appropriate graph state. We end this section with a note on decomposition of CV unitary operators.

A. The Heisenberg-Weyl Group

The definition of CV stabilizer states starts with generalization of the Pauli group to the continuous setting [29–33]. For one qumode (a term which replaces qubit in the continuous setting) the resulting group is called Heisenberg-Weyl group W, whose algebra of generators is spanned by the coordinate and momentum operators satisfying $[\widehat{Q},\widehat{P}] = iI$. Thus modulo U(1) phases, the group W is the group of unitary operators of the form $w(t,s) = e^{it\widehat{Q}+is\widehat{P}}$. Since the two unitary operators

$$\hat{X}(s) = e^{-is\widehat{P}} \quad \text{and} \quad \hat{Z}(t) = e^{it\widehat{Q}} \tag{6}$$

have the simple relation

$$\hat{Z}(t)\hat{X}(s) = e^{ist}\hat{X}(s)\hat{Z}(t), \tag{7}$$

multiplication of any two elements of W can be recast in the form $X(t)Z(s)$ modulo a phase. The Heisenbergy-Weyl group can be represented on the Hilbert space of one particle, spanned by the basis states $|y\rangle_q$ (eigenstates of $\widehat{Q}$) or $|y\rangle_p$ (eigenstates of $\widehat{P}$). On these states, the operators X and Z act as follows:

$$\begin{aligned}\hat{Z}(t)|y\rangle_q &= e^{ity}|y\rangle_q, \quad \hat{X}(s)|y\rangle_p = e^{-isy}|y\rangle_p,\\ \hat{Z}(t)|y\rangle_p &= |y+t\rangle_p, \quad \hat{X}(s)|y\rangle_q = |y+s\rangle_q.\end{aligned} \tag{8}$$

Remark. Hereafter we denote the states $|y\rangle_q$ simply as $|y\rangle$ and use $|y\rangle_p$ for eigenstates of $\widehat{P}$ as above.

The CV Hadamard operator is a unitary non-Hermitian operator defined as

$$\hat{H} := \int |y\rangle_p\langle y|_q dy \equiv \frac{1}{\sqrt{2\pi}}\int e^{ixy}|x\rangle\langle y|dxdy, \tag{9}$$

from which we obtain

$$\hat{H}\widehat{Q}\hat{H}^{-1} = \widehat{P}, \qquad \hat{H}^{-1}\widehat{P}\hat{H} = \widehat{Q}. \tag{10}$$

Moreover, from Eq. (9) we find

$$\hat{H}^2 := \int dx|-x\rangle\langle x|, \tag{11}$$

which leads to $\hat{H}^4 = I$. Therefore, the square Hadamard operator acts as the parity operator. This means that

$$\hat{H}^2\widehat{Q}\hat{H}^{-2} = -\widehat{Q}, \qquad \hat{H}^{-2}\widehat{P}\hat{H}^2 = -\widehat{P}. \tag{12}$$

The n-mode Heisenberg-Weyl group is the tensor product of n copies of W; i.e., $W_n := W^{\otimes n}$ and all the above properties are naturally and straightforwardly extended to n modes. Of particular interest is the continuous variable $C\hat{Z}$ operator, which is defined as $C\hat{Z}|x,y\rangle = e^{ixy}|x,y\rangle$, with the operator expression

$$C\hat{Z} := e^{i\widehat{Q}\otimes\widehat{Q}}, \tag{13}$$

and satisfying the relation

$$[\hat{X}(t)\otimes I]C\hat{Z} = C\hat{Z}[\hat{X}(t)\otimes\hat{Z}(t)]. \tag{14}$$

We will also need a more general operator $C\hat{Z}$, namely $C\hat{Z}(s) = e^{is\widehat{Q}\otimes\widehat{Q}}$, which has the following relation:

$$[\hat{X}(t)\otimes I]C\hat{Z}(s) = C\hat{Z}(s)[\hat{X}(t)\otimes\hat{Z}(st)]. \tag{15}$$

1. Single mode unitary operators induced by measurements

In this subsection, we review how a CV unitary operator can be induced on a mode by measuring a suitable graph state. Here we restrict ourselves to operators diagonal in the coordinate basis, since this is the only type of operator that we encounter in our analysis. We also use the basic result of MQC that certain states are complete, in the sense that by a suitable

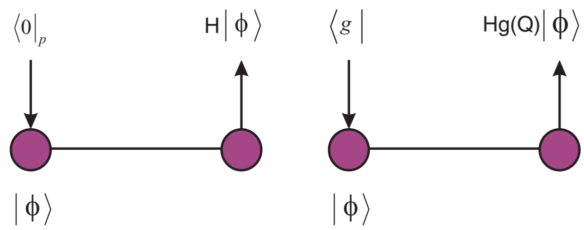

FIG. 1. (Color online) Projection of the left mode on the zero momentum state is equivalent to the action of the Hadamard operator on the right mode. Projection on the state $\langle g|$ is equivalent to the action of the operator $\hat{H}g(\hat{Q})$ on the right mode. A downward arrow means projection on a state; an upward arrow means the resulting state.

sequence of adaptive single-site measurements on them, any other state can be reached.

Consider a very simple two-mode graph state as shown in Fig. 1. The first mode is in an arbitrary state $|\phi\rangle = \int dx\phi(x)|x\rangle$, the second mode is in the state $|0\rangle_p := \frac{1}{\sqrt{2\pi}}\int dy|y\rangle$, the two modes have been joined by a CZ operator (shown by a line in Fig. 1) and so the two mode state is

$$|\Psi\rangle_{12} = (CZ)_{12}|\phi\rangle_1|0_p\rangle_2 = \int dxdye^{ixy}\phi(x)|x,y\rangle_{1,2}, \quad (16)$$

where the indices 1 and 2 refer to the modes from left to right in Fig. 1. Now we project the first mode on the zero momentum state $\langle 0|_p$ [33–39]. The state of the second mode will be

$$|\phi'\rangle_2 = {}_1\langle 0_p|\Psi\rangle_{12} = \frac{1}{\sqrt{2\pi}}\int dye^{ixy}\phi(x)|y\rangle_2 = H|\phi\rangle. \quad (17)$$

Thus, projection of the first mode onto a zero momentum state is equivalent to the action of the Hadamard operator on the state $|\phi\rangle$ and putting it on the second mode. This is shown in Fig. 1, where projection is depicted by a downward arrow and the result is depicted by an upward arrow.

Suppose now that we project mode 1 onto the state $\langle g| := \frac{1}{\sqrt{2\pi}}\int dyg(y)\langle y|$. If we note that

$$\langle g| = \frac{1}{\sqrt{2\pi}}\int dyg(y)\langle y| = \langle 0_p|g(\hat{Q}), \quad (18)$$

and note that $g(\hat{Q}_1)$ commutes with $(CZ)_{12}$, we find that projecting the first mode on the state $\langle g|$ is equivalent to the action of the operator $Hg(\hat{Q})$ on the state $|\phi\rangle$ and putting it on the second mode. This is shown in Fig. 1. We can write this symbolically as

$$P_0 \longrightarrow H, \qquad P_g \longrightarrow Hg(\hat{Q}), \quad (19)$$

where in the left-hand side we show the projections and in the right-hand side we show the resulting action on the state. In order to enact the operator $g(\hat{Q})$, i.e., remove H from $Hg(\hat{Q})$, we need to enact the operator H three times. Thus, using the symbols in Eq. (19), we have

$$P_0P_0P_0P_g \longrightarrow H^3(Hg(\hat{Q})) = g(\hat{Q}), \quad (20)$$

which is shown in Fig. 2.

We are now in a position to state a basic theorem [40–43] in this section.

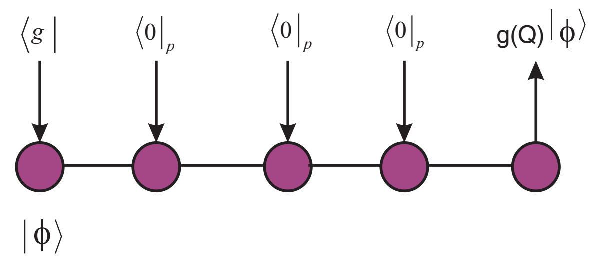

FIG. 2. (Color online) The measurement pattern that enacts the operator $g(\hat{Q})$ on the right-most mode.

Theorem. Let $V(\hat{Q})$ be a polynomial of $\hat{Q}$ with real coefficients and $|\phi\rangle$ be an arbitrary state of an appropriate chain of a cluster state. Then, by projecting the modes of this cluster state on the following three types of states,

$$\langle\beta_1(t)| := \int dye^{-ity}\langle y|, \quad \langle\beta_2(t)| := \int dye^{-ity^2}\langle y|, \quad \langle\beta_4(t)| := \int dye^{-ity^4}\langle y|, \quad (21)$$

we can enact any operator of the form $e^{-iV(\hat{Q})}$ on the state $|\phi\rangle$ to any desired precision. The state will appear on the unprojected modes of the chain.

Proof. First, we note that enacting the operator H comes for free by projecting on the zero momentum state $\langle 0|_p$. Second, we use the operator identity

$$e^{tA}e^{tB}e^{-tA}e^{-tB} \approx e^{t^2[A,B]+o(t^2)}. \quad (22)$$

From projection on the states $\langle\beta_i(t)|$ we find that the operators $e^{-it\hat{Q}}$, $e^{-it\hat{Q}^2}$, and $e^{-it\hat{Q}^4}$ can be obtained. In view of the existence of the Hadamard operator, the algebra of anti-Hermitian operators in the exponential is generated by the set $\{iQ, iQ^2, iQ^4, iP, iP^2, iP^4\}$. It is now easy to see that this algebra contains all monomials of Q. To show this we first note that

$$[iP, iQ^4] \sim iQ^3, \quad (23)$$

where $\sim$ means that we have ignored numerical factors. We then note that

$$[iP,[iP^2,iQ^4]] \sim i(PQ^2 + Q^2P). \quad (24)$$

The latter operator now acts as a raising operator for powers of monomials, since

$$[i(PQ^2 + Q^2P), iQ^n]] \sim iQ^{n+1}, \quad (25)$$

which completes the proof. In this way, we can generate any unitary operator of the form $e^{-iV(Q)}$, where V is a real polynomial of Q.

III. THREE CLASSES OF CONTINUOUS VARIABLE STATES

Let $G = (V,E)$ be a graph, where V and E, respectively, denote the set of vertices and edges. The graph is supposed to admit an orientation. In other words, G is the triangulation of an orientable manifold. This means that all the simplexes of G inherit the orientation of the original manifold in a consistent way. The number of vertices and edges are given by $|V|$ and $|E|$, respectively.

In this section we define three closely related continuous variable states pertaining to a given graph G, which we call the Kitaev state $|K_G\rangle$ [44], the extended Kitaev state $|\overline{K}_G\rangle$, and the graph state $|\Psi_G\rangle$ [45,46]. We will then determine the mutual relationships of these states, which will play an important role in our proof of completeness. These are the generalizations of known states in the qubit case, where they have been possibly named differently in other works. For example, in the qubit case, extended Kitaev states have been called pseudo graph states [5]; however, in view of their explicit construction and stabilizers, we think that the name Kitaev or extended Kitaev states are more appropriate for them.

The crucial difference between the continuous and qubit case is the fact that the operators $\hat{X}(t)$, $\hat{Z}(t)$, CZ, and $\hat{H}$ are not equal to their inverses. Therefore, a consistent and unambiguous description of these states on a graph requires that the graphs be decorated with weights and/or orientations. We emphasize the difference between orientation, which is a Z_2 variable and weight, which is a real variable. We will meet the necessity of each as we go along in our definitions.

A. Kitaev states

Consider an oriented graph $G=(V,E,\sigma)$, where σ means that arbitrary orientations have been assigned to the edges. Any collection of arbitrary orientations on the edges is called a decoration of the graph. We assume that modes live only on the edges E and there are no modes on the vertices V of this graph. The CV Kitaev state is then defined

$$|K_G\rangle = \int d\phi_1 d\phi_2 ... d\phi_N \bigotimes_{e_{ij}} |\phi_i - \phi_j\rangle, \tag{26}$$

where e_{ij} is the edge which goes from the vertex i to the vertex j.

It is easily verified that this state is stabilized by the following set of operators: for each vertex $i \in V$, we have

$$A_v(t) := \prod_{e\in E_v} X_e^{\pm}(t), \tag{27}$$

where E_v denotes the set of edges incident on the vertex v and the $-$ and $+$ signs are used for edges going into and out of a vertex, respectively. The reason that A_v stabilizes the state of Eq. (26) is that it simply shifts the variable ϕ_v, which will be neutralized under the integration. Also for each face of the graph, we have

$$B_f(s) := \prod_{e\in\partial f} Z_e^{\pm}(s), \tag{28}$$

where ∂f denotes the set of edges in the boundary of f and the $+$ and $-$ signs are used, respectively, when the orientation of a link is equal or opposite to us when we traverse a face in the counter-clockwise sense. Note that traversing all the plaquettes in this sense is meaningful for an orientable traingulation. Here also the effect of $B_p(s)$ on the state inside the integral is to multiply it by a unit factor since the phases acquired by all the edges add up to zero for a closed loop.

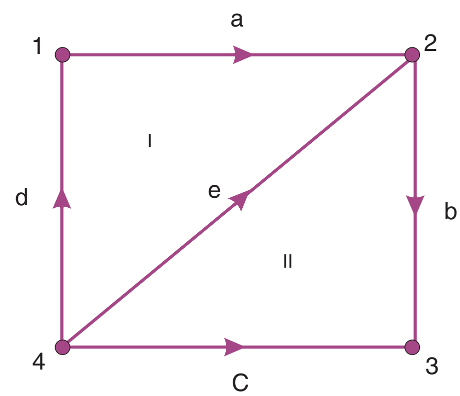

FIG. 3. (Color online) The graph for which the Kitaev state is given in Eq. (29).

As an example, we have for the graph $\mathcal{G}$ in Fig. 3,

$$|K_{\mathcal{G}}\rangle = \int Dx|\phi_1-\phi_2,\phi_2-\phi_3,\phi_4-\phi_3,\phi_4-\phi_1, \phi_4-\phi_2\rangle_{a,b,c,d,e}, \tag{29}$$

where the subscripts a to e determine the position of modes in the state. The stabilizers of this state are then given by Eqs. (27) and (28) as follows:

$$A_1 := X_d^{-1}X_a, \quad A_2 := X_a^{-1}X_e^{-1}X_b, \quad A_3 := X_b^{-1}X_c^{-1}, \quad A_4 := X_cX_dX_e \tag{30}$$

and

$$B_I := Z_eZ_a^{-1}Z_d^{-1}, \quad B_{II} := Z_cZ_b^{-1}Z_e^{-1}. \tag{31}$$

Finally, we note that all Kitaev states on a given graph, corresponding to different decorations are related to each other by local unitary actions. In fact, if we switch the arbitrary orientation on a link e_{ij}, it means that the term $|\ldots\phi_i-\phi_j,\ldots\rangle$ in Eq. (26) changes to $|\ldots\phi_j-\phi_i,\ldots\rangle$, where the remaining parts of the state will remain intact. In view of Eq. (11), this switching is achieved by a local action of the square Hadamard operator H^2 on the edge e. We can thus write

$$|K_{G^{\sigma'}}\rangle = \left(\bigotimes_{e:\sigma(e)\neq\sigma'(e)} H_e^2\right)|K_{G^\sigma}\rangle. \tag{32}$$

Therefore, all the Kitaev states with different decorations belong to the same class of states modulo local actions of square Hadamard operations.

In view of the shift invariance $\phi_i \longrightarrow \phi_i+\eta$, the Kitaev state has a hidden multiplicative factor that is, in fact, infinite. This symmetry can be removed by fixing a gauge (in the discrete case this is a finite factor that causes no problem). Therefore, we will define the gauge-fixed Kitaev state, denoted by $|K_G^0\rangle$ as follows:

$$|K_G^0\rangle = \int d\phi_1 d\phi_2 ... d\phi_N \delta(\phi_1+\phi_2+\cdots\phi_N)\bigotimes_{e_{ij}}|\phi_i-\phi_j\rangle. \tag{33}$$

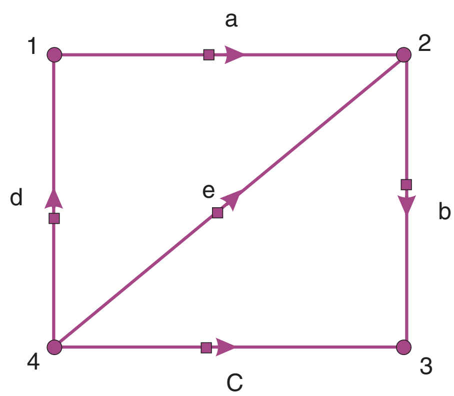

FIG. 4. (Color online) The graph G_0 for which the extended Kitaev state is as in Eq. (35).

This gauge-fixed Kitaev state still has the same set of stabilizers. The same reasoning for the Kitaev state $|K_G\rangle$ also works here.

B. Extended Kitaev states

Let $G = (V,E,\sigma)$ be defined as in the previous subsection. Now, in addition to the modes on the edges, there are also modes on vertices, as in Fig. 4. On such a graph, there are thus two different sets of vertices, which we denote by V (the ones on the nodes) and V_E (the ones on the edges). Then the state $|\overline{K}_G\rangle$ is given by

$$|\overline{K}_G\rangle = \int d\phi_1 d\phi_2 ... d\phi_N \bigotimes_{e_{ij}\in E} |\phi_i - \phi_j\rangle \bigotimes_{i\in V} |\phi_i\rangle, \quad (34)$$

where e_{ij} is the edge that goes from vertex i to j. As an example for the graph in Fig. 4, we have

$$|\overline{K}_\mathcal{G}\rangle = \int D\phi |\phi_1 - \phi_2, \phi_2 - \phi_3, \phi_4 - \phi_3, \phi_4 - \phi_1, \phi_4 - \phi_2\rangle_{a,b,c,d,e} \otimes |\phi_1,\phi_2,\phi_3,\phi_4\rangle_{1,2,3,4}. \quad (35)$$

The stabilizers of this extended Kitaev state are completely different from the simple Kitaev state. In fact, they are: For each vertex $v \in V$,

$$C_v := X_v \prod_{e\in E_v} X_e^{\pm}, \quad (36)$$

where the convention for the $\pm$ signs is the same as in Kitaev state, and for each edge $e \in E$ that goes from v_1 to v_2,

$$D_e := Z_e^{-1} Z_{v_1} Z_{v_2}^{-1}, \quad (37)$$

where again we have suppressed the continuous arguments of these stabilizers for ease of notation. For the example given in Fig. 4, some of the stabilizers are:

$$C_1 = X_1 X_a X_d^{-1}, \quad C_2 = X_2 X_a^{-1} X_e^{-1} X_b,$$
$$D_a = Z_1 Z_a^{-1} Z_2^{-1}, \quad D_b = Z_2 Z_b^{-1} Z_3, \ldots \quad (38)$$

C. Weighted graph states

Finally, we come to the definition of continuous variable weighted graph states. Here as in the qubit case we start with an initial product state of the form $|\Omega\rangle = |0\rangle_p^{\otimes V}$. However, there is an important difference in that on each edge, instead of the simple CZ operator, we can act by the $CZ(J)$ operator, where the real parameter J may depend on the edge. Therefore, we obtain what we call a weighted graph state. Denoting the collection of all weights by J, we have

$$|\Psi_G(J)\rangle = \bigotimes_{e\in E} [CZ(J_e)]|\Omega\rangle, \quad (39)$$

where $|\Omega\rangle = |0\rangle_p^{\otimes V}$. The explicit form of the state will then be given by

$$|\Psi_G(J)\rangle = \int D\phi e^{\sum_{\langle i,j\rangle} J_{ij}\phi_i\phi_j} |\phi_1, \ldots \phi_N\rangle, \quad (40)$$

where N is the number of vertices and $\langle i,j\rangle$ denotes the edge connecting the vertices i and j carrying weight J_{ij}. Note that in contrast to a decorated edge, which is denoted by e_{ij} (going from i to j), a weighted edge is denoted by the symmetric symbol $\langle i,j\rangle$. According to Eq. (15), the stabilizers of this state will be of the form

$$K_i(t) := X_i(t) \prod_{j\in N_i} Z_j(J_{ij}t), \qquad \forall\, i \in V, \quad (41)$$

where in the left-hand side we have suppressed the dependence on the weights for simplicity.

We emphasize that definition of the Kitaev states and extended Kitaev states require the underlying graph to be decorated, while a graph state needs only a weighted graph for its unambiguous definition. We are now left with an important question of whether there is a simple relation between the above three kinds of states or not. The answer turns out to be positive and is explained in the next subsection.

D. Relations between Kitaev, extended Kitaev, and weighted graph states

Consider the extended Kitaev state corresponding to a decorated graph $G = (V,E,\sigma)$. The explicit form of the state is shown in Eq. (34). From Eq. (34) it is clear that if we project all the vertices in V on the zero-momentum basis, we will arrive at the Kitaev state for the same graph. More explicitly we have

$$\langle\Omega|\overline{K}_{G^\sigma}\rangle = \frac{1}{\sqrt{(2\pi)^{|V|}}} |K_{G^\sigma}\rangle, \quad (42)$$

where $|\Omega\rangle = |0\rangle_p^{\otimes V}$, and we have explicitly indicated the decoration σ.

It is instructive to understand this in an alternative way, that is by showing that measurement in the momentum basis actually transforms the stabilizer set of the extended Kitaev state, i.e., $S(|\overline{K}_G^\sigma\rangle)$ to the stabilizer set of the Kitaev state, $S(|K_G^\sigma\rangle)$. From the stabilizer formalism, we know that measurement of a state $|\Psi\rangle$ in the basis of an operator M removes all the operators that do not commute with M from the set $S(|\Psi\rangle)$ and leaves us with a smaller subset. This subset is generated by all the original generators, or their products thereof, that commute with M. With this in mind, it is straightforward to see that measurement in the momentum (the X basis) leaves all the vertex stabilizers C_v intact (except of course removing the vertex X_v from it), hence changing it to

A_v as in Eq. (27). However, since X_v does not commute with Z_v, measurements of all the vertices remove all the generators D_e. The only combinations that survive this elimination will be their product around any faces. These are nothing but the operators B_f for all faces, which are just the right stabilizers of $|K_{G^\sigma}\rangle$.

Let us now study the relation between the extended Kitaev states and graph states. It turns out that there is a simple relation between the two only if the weights of the edges incident on each vertex add up to zero, that is, if

$$\sum_j J_{ij} = 0, \qquad \forall i. \tag{43}$$

Since $J_{ij} = J_{ji}$, this means also that $\sum_i J_{ij} = 0$. In such a case, we can convert an extended Kitaev state to a weighted graph state on the same graph by suitable measurements on the edges. To this end we proceed as follows: Let us project each edge e_{ij} of the extended Kitaev state on the state $\langle\beta_2(t_{ij})|$ defined in Eq. (21).

Let $\langle\beta_2(\mathbf{t})| := \prod_{e_{ij}}\langle\beta_2(t_{ij})|$, then we have

$$\langle\beta_2(\mathbf{t})|\overline{K}_G\rangle = \int D\phi e^{\sum_{e_{ij}} -it_{ij}(\phi_i-\phi_j)^2}|\phi_1,\cdots\phi_N\rangle. \tag{44}$$

If we choose the parameters t_{ij} of quadratures so that $\sum_i t_{ij} = \sum_j t_{ij} = 0$, we find

$$\langle\beta_2(\mathbf{t})|\overline{K}_G\rangle = \int D\phi e^{\sum_{\langle i,j\rangle} it_{ij}\phi_i\phi_j}|\phi_1,\ldots\phi_N\rangle =: |\Psi_G(i\mathbf{t})\rangle, \tag{45}$$

which is a weighted graph state with weights it_{ij} assigned to each edge $\langle i,j\rangle$.

Note that from the Kitaev state for the rectangular lattice $|K_\Box\rangle$, the Kitaev state for any other graph $|K_G\rangle$ can be obtained simply by measurement of the edge modes in the momentum or coordinate bases. In fact, projecting an edge mode on the zero-momentum state $|0\rangle_p$ removes that link from the graph, while projecting it on the zero coordinate state $|0\rangle_q$ merges the two endpoints of that edge. These are shown in Fig. 5 and are proved as follows: Let

$$|\psi\rangle = \int d\phi_1 d\phi_2 D'\phi|\phi_1-\phi_2\rangle\prod_{i\in L}|\phi_1-\phi_i\rangle\prod_{j\in R}|\phi_2-\phi_j\rangle\ldots,$$

where $\cdots$ denotes all the edges that involve neither vertex 1 nor 2. Projecting this state on the state ${}_p\langle 0|$ on the edge $\langle 1,2\rangle$ leaves us with

$$|\psi_p\rangle = \int d\phi_1 d\phi_2 D'\phi\prod_{i\in L}|\phi_1-\phi_i\rangle\prod_{j\in R}|\phi_2-\phi_j\rangle\ldots, \tag{46}$$

which is nothing but the same Kitaev state with the edge $\langle 1,2\rangle$ totally removed. On the other hand, projecting $|\psi\rangle$ on the state ${}_q\langle 0|$ on the edge $\langle 1,2\rangle$, leaves us with

$$\begin{aligned}|\psi_q\rangle &= \int d\phi_1 d\phi_2 D'\phi\,\delta(\phi_1-\phi_2)\prod_{i\in L}|\phi_1-\phi_i\rangle\prod_{j\in R}|\phi_2-\phi_j\rangle\ldots\\ &= \int d\phi_1 D'\phi\prod_{i\in L}|\phi_1-\phi_i\rangle\prod_{j\in R}|\phi_1-\phi_j\rangle\ldots,\end{aligned} \tag{47}$$

which means that the two endpoints of the edge $\langle 1,2\rangle$ have been merged together. With these two simple rules of deleting and merging, one can obtain the Kitaev state of any graph starting from the one on the rectangular lattice. Figure 6 shows an important example in which measurements of some of the edges in the momentum basis (and hence removing them), transforms $|K_\Box\rangle$ to the Kitaev state on the hexagonal lattice $|K_H\rangle$. Measurement of the same edges in the coordinate basis (and hence merging the two endpoints) produces a uniform lattice whose faces are triangles, hence a triangular lattice. This is in accord with the fact that the hexagonal and triangular lattice are dual to each other, a subject which will be explored further in the sequel.

Finally, we use the well-known universality of cluster states proved in the context of measurement-based quantum

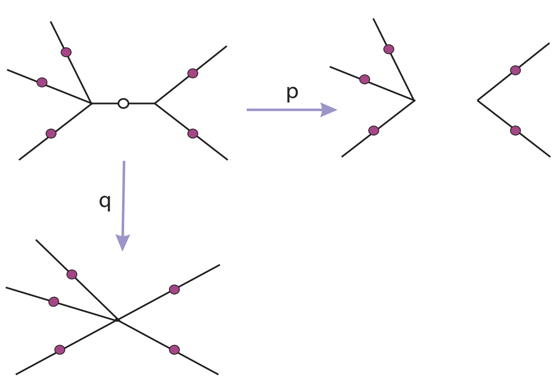

FIG. 5. (Color online) The effect of measurements of an edge in an extended Kitaev state. Measuring in the momentum (X) basis (i.e., projecting onto the ${}_p\langle 0|$) removes the edge, while measurements in the coordinate (Z) basis (i.e., projecting onto the ${}_p\langle 0|$), merges the two end points of that edge.

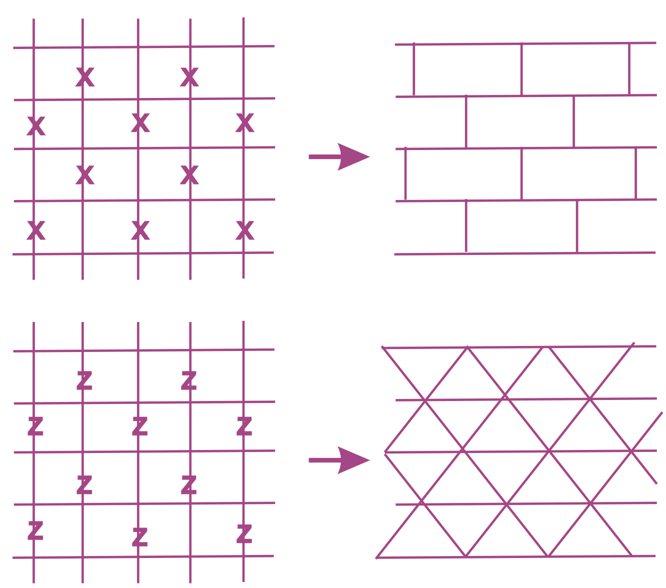

FIG. 6. (Color online) The Kitaev state on the rectangular lattice, when specific edges are measured in the X or Z bases, will turn into the Kitaev state on the hexagonal or triangular lattices.

computation [33–39] to state that both the Kitaev state $|K_G\rangle$ and the extended Kitaev state $|\overline{K}_G\rangle$ can be obtained by Gaussian measurements from a sufficiently large cluster state $|\Psi_\Box\rangle$. The fact that the measurements need only be Gaussian is due to the fact that all these kinds of states are stabilized by subgroups of W_n and hence they can be converted to each other by unitary operators belonging to the Clifford group. Using the well-known fact from MQC, that Clifford operators can be implemented by Gaussian measurements, we arrive at the proof of the above statement.

IV. QUANTUM FORMALISM FOR THE PARTITION FUNCTIONS

In this section, we show how the partition function of a classical model defined by a Hamiltonian with continuous variables on an arbitrary graph can be expressed in the quantum formalism. First, consider the case where there is no local term or onsite interaction, that is the Hamiltonian is of the form

$$H = \sum_{\langle i,j\rangle} V_{ij}(\phi_i - \phi_j), \tag{48}$$

where $V_{ij}(x)$ is an arbitrary function. We allow for the function V_{ij} to depend on the edge e_{ij} in order to cover also the inhomogeneous cases. The partition function of this model is

$$\mathcal{Z}(G,\{V\}) = \int D\phi^N e^{-i\sum_{\langle i,j\rangle} V_{ij}(\phi_i-\phi_j)}, \tag{49}$$

where N is the number of vertices and we have absorbed the parameter $\beta \equiv \frac{1}{k_b T}$ in the Hamiltonian. We will do this in all expressions of partition functions that follow.

Remark. We have defined the partition function in the form of Eq. (49), in order to be able to deal with unitary operators in the measurement-based quantum computation. Dealing with nonunitary operators does not pose any problem in the Ising model [5], since states like $|\alpha\rangle = e^{-\beta J}|0\rangle + e^{\beta J}|1\rangle$ are normalizable states. In the continuous case, the analog of the above state may be non-normalizable, rendering the projection to such states problematic. Instead, we resort to partition functions of the type Eq. (70), with the understanding that the final results, like dualities, and completeness can be analytically continued to the whole complex plane.

Due to the shift invariance of the Hamiltonian $\phi_i \longrightarrow \phi_i + \xi$, the above partition function is infinite, so we have to modify the partition function by fixing a gauge, which we will do later on. For the present, we deal with the above partition function as it is. Defining the product state

$$|\alpha\rangle = \bigotimes_{e_{ij}} |\alpha_{ij}\rangle, \tag{50}$$

where

$$|\alpha_{ij}\rangle = \int dx e^{-iV_{ij}(x)}|x\rangle$$

is defined on the edge e_{ij}, one can then write the partition function [Eq. (49)] in the quantum formalism as

$$\mathcal{Z}'(G,\{V\}) = \langle\alpha|K_G\rangle, \tag{51}$$

where $|K_G\rangle$ is the Kitaev state on the graph G [Eq. (26)]. In this way, as in the qubit case, the pattern of interactions is encoded in the entangled Kitaev state and the strength of interactions (including the temperature) are encoded into the product state $\langle\alpha|$. To fix the shift invariance, we define a gauge-fixed partition function as

$$\mathcal{Z}(G,\{V\}) = \int D\phi\delta\left(\sum_i \phi_i\right) e^{-i\sum_{\langle i,j\rangle} V_{ij}(\phi_i-\phi_j)}. \tag{52}$$

Note that other forms of gauge-fixing terms are possible, but we will deal with this simple one. Also, note that the shift invariance is also present in discrete models; however, in those cases the multiplicative factor is finite and not divergent, hence gauge fixing is not necessary. Using the gauge-fixed Kitaev state, we can write this in the quantum formalism as

$$\mathcal{Z}(G,\{V\}) = \langle\alpha|K_G^0\rangle. \tag{53}$$

Let us now consider an edge e and insert the operator $\widehat{Q}_e$ inside the inner product [Eq. (53)]. In view of Eqs. (50) and (53), we will have

$$\frac{\langle\alpha|\hat{Q}_e|K_G^0\rangle}{\langle\alpha|K_G^0\rangle} = \langle\phi_i - \phi_j\rangle, \tag{54}$$

where $\langle\ \rangle$ means the statistical thermal average. Similarly, by acting the momentum operator on the state $\langle\alpha|$, we find

$$\frac{\langle\alpha|\widehat{P}_e|K_G^0\rangle}{\langle\alpha|K_G^0\rangle} = -i\langle V'_{ij}(\phi_i - \phi_j)\rangle, \tag{55}$$

where $'$ means derivative with respect to the argument. We will later see an important application of these equations when they are combined with the topological properties of the Kitaev states.

Consider now the case where there are onsite interactions, then the Hamiltonian will be

$$H = \sum_{\langle i,j\rangle} V_{ij}(\phi_i - \phi_j) + \sum_i W_i(\phi_i), \tag{56}$$

and the above formalism will be extended as follows:

$$\mathcal{Z}(G,\{V\},\{W\}) = \langle\overline{\alpha}|\overline{K}_G\rangle, \tag{57}$$

where

$$|\overline{\alpha}\rangle = \bigotimes_{e_{ij}} |\alpha_{ij}\rangle \bigotimes_{i\in V} |\alpha_i\rangle, \tag{58}$$

in which

$$|\alpha_i\rangle = \int dx e^{-iW_i(x)}|x\rangle.$$

V. APPLICATIONS OF THE QUANTUM FORMALISM

Let us now try to understand some of the properties of a continuous variable statistical model through the quantum formalism. Certainly the results that we will find, like duality, can also be derived by other means, without resorting to the quantum formalism; however, this scheme makes these properties and their derivation much more transparent.

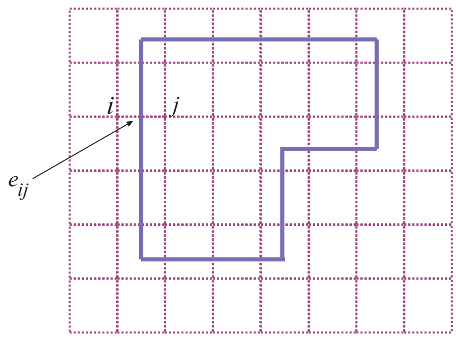

FIG. 7. (Color online) For any closed loop, which is the boundary of a region, equation holds for the correlation functions.

A. Correlation functions

Consider a graph $G = (V,E)$ and a Hamiltonian H_0 defined on it without local potentials, $W_i(\phi) = 0$. Let $W^x_{\widetilde{C}}$ be a closed loop on the dual graph, which is homologically trivial (Fig. 7); i.e., it is the boundary of an area. Since $W^x_{\widetilde{C}}$ can be written as a product of operators A_s, for s inside the loop $\widetilde{C}$, we have

$$W^x_{\widetilde{C}}\big|K^0_G\big\rangle = \big|K^0_G\big\rangle, \tag{59}$$

or, equivalently,

$$\sum_{e\in\widetilde{C}} \widehat{P}_e\big|K^0_G\big\rangle = 0. \tag{60}$$

Then, in view of Eq. (55), this means that

$$\sum_{e_{i,j}\in\widetilde{C}} \langle V'_{ij}(\phi_i - \phi_j)\rangle = 0. \tag{61}$$

This is a general nontrivial relation that is valid for any kind of interaction V, and without using the quantum formalism, it would have been difficult to obtain it. Note that the other kind of loop operator, W^z_C defined as $W^z_C := \prod_{i\in C} Z_i$, where C is a loop in the graph, doesn't lead to a nontrivial relation since in view of Eq. (54), insertion of this operator into the inner product leads to the quantity $\sum_{e_{ij}}(\phi_i - \phi_j)$, which identically vanishes.

B. Duality

Denote the dual graph by $\widetilde{G}$. The vertices, edges, and faces of G are in one-to-one correspondence with the faces, edges, and vertices of $\widetilde{G}$, respectively. For an oriented graph, we should also choose a convention for choosing the orientations. We choose the convention that for each edge e the dual $\widetilde{e}$ be such that the pair $(e,\widetilde{e})$ form a right-handed frame, as shown in Fig. 8. In view of the form of stabilizers of the Kitaev states [Eqs. (27) and (28)] and the relations [Eq. (10)], and the normalization of the state $|K_G\rangle$, we see that

$$|K_{\widetilde{G}}\rangle = (2\pi)^{\frac{|E|}{2}} H^{\otimes E}|K_G\rangle. \tag{62}$$

Note that contrary to the qubit case the duality relation is not an involution, that is, as shown in Fig. 8, the dual of the dual of an oriented graph is not the original graph but the original graph with all the orientations reversed. This is in accord with the fact that $H^2 \neq I$ and, indeed, the action of H^2 on all edges reverses their orientations. Consider now the partition function on G, with βH as in Eq. (48). We have

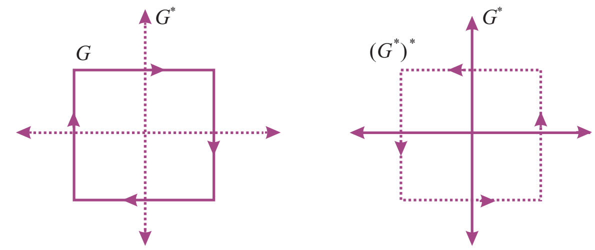

FIG. 8. (Color online) The dual of the dual of a graph is the same graph with all the orientations reversed.

$$Z(G,\{V\}) = \langle\alpha|K_G\rangle = \frac{1}{(2\pi)^{\frac{|E|}{2}}}\langle\alpha|H^{\dagger\otimes E}|K_{\widetilde{G}}\rangle$$
$$= \frac{1}{(2\pi)^{\frac{|E|}{2}}}\langle\widetilde{\alpha}|K_{\widetilde{G}}\rangle, \tag{63}$$

where

$$|\widetilde{\alpha}\rangle = H|\alpha\rangle = H\int dxe^{-iV(x)}|x\rangle = \frac{1}{\sqrt{2\pi}}\int dxdye^{ixy-iV(x)}|y\rangle$$
$$= \int dye^{-i\widetilde{V}(y)}|y\rangle, \tag{64}$$

where

$$e^{-i\widetilde{V}(y)} = \frac{1}{\sqrt{2\pi}}\int dxe^{ixy-iV(x)}. \tag{65}$$

This gives the following duality relation

$$Z(G,\{V\}) = \frac{1}{(2\pi)^{\frac{|E|}{2}}} Z(\widetilde{G},\{\widetilde{V}\}), \tag{66}$$

where $e^{-i\widetilde{V}}$ is the Fourier transform of e^{-iV}, as given in Eq. (65). An example of interest is when

$$V_{ij}(x) = \tfrac{1}{2}k_{ij}x^2, \tag{67}$$

which leads to the following duality relation

$$Z(G,\{k_{ij}\}) = \frac{1}{(2\pi)^{\frac{|E|}{2}}}\frac{1}{\sqrt{\prod_{ij}k_{ij}}} Z\left(\widetilde{G},\left\{\frac{1}{k_{ij}}\right\}\right). \tag{68}$$

VI. COMPLETENESS OF TWO-DIMENSIONAL ϕ^4 MODEL FOR ALL DISCRETE SCALAR FIELD THEORIES

In this section, we use quantum formalism to show that the discrete form of two-dimensional ϕ^4 field theory is complete. Before proceeding, we should make precise the meaning of the above statement. By the two-dimensional discrete ϕ^4 theory, we mean the following Hamiltonian on a two-dimensional square lattice with periodic boundary conditions

$$H_c = \sum_{\langle r,s\rangle} K_{r,s}(\phi_r - \phi_s)^2 + \sum_r h_r\phi_r + m_r\phi_r^2 + q_r\phi_r^4, \tag{69}$$

where $K_{r,s} \in \{i,-i\}$, $i = \sqrt{-1}$, and the real parameters h_r, m_r, and q_r denote, respectively, the inhomogeneous external field, the quadratic (mass term), and quartic coupling strengths. The linear terms $\{h_r\}$ are also necessary for completeness. Denote the partition function of this model by

$$Z(\Box,\{h\},\{m\},\{q\}) := \int D\phi e^{-iH_c}, \tag{70}$$

where $\Box$ is a 2D square lattice of size N, and $\{h\}$, $\{m\}$, and $\{q\}$ denote the totality of all the inhomogeneous coupling strengths. By completeness, we mean that the partition function of any other model

$$H = \sum_{r,s} V_{r,s}(\phi_r - \phi_s) + \sum_r W_r(\phi_r) \tag{71}$$

on any graph $G = (V,E)$ is equal to the partition function for the Hamiltonian H_c on the 2D square lattice, for some specifically chosen sets $\{h\}$, $\{m\}$, and $\{q\}$.

Note that the interaction terms in the Hamiltonian H are not necessarily of nearest neighbor type. In fact, as we will show in the proof, H can even contain k-body interactions, as long as k is bounded above by a finite constant independent of $|V|$.

Consider a general Hamiltonian of the form of Eq. (71). The partition function of this model can be written in the quantum formalism as

$$Z(G,\{V\},\{W\}) = \langle\overline{\alpha}|\overline{K}_G\rangle, \tag{72}$$

where

$$\langle\overline{\alpha}| = \bigotimes_{e_{i,j}}\langle\alpha_{ij}|\bigotimes_i\langle\alpha_i| \tag{73}$$

and

$$\langle\alpha_{ij}| = \int dy e^{-iV_{ij}(y)}\langle y|, \quad \langle\alpha_i| = \int dy e^{-iW_i(y)}\langle y|. \tag{74}$$

Note that $\langle\alpha_{ij}|$ lives on the edge vertex $v(e_{ij})$ and $\langle\alpha_i|$ lives on the vertex i. Rewriting the above states in the form

$$\begin{aligned}\langle\alpha_{ij}| &= \int dy\langle y|e^{-iV_{ij}(\widehat{Q})} = \langle 0_p|e^{-iV_{ij}(\widehat{Q})},\\ \langle\alpha_i| &= \int dy\langle y|e^{-iW_i(\widehat{Q})} = \langle 0_p|e^{-iW_i(\widehat{Q})},\end{aligned} \tag{75}$$

where $\langle 0_p|$ is the zero momentum eigenstate, we find an equivalent form for the partition function, namely

$$Z(G,\{V\},\{W\}) = \langle\mathbf{0_p}|\bigotimes_{e_{ij}} e^{-iV_{ij}(\widehat{Q})}\bigotimes_i e^{-iW_i(\widehat{Q})}|\overline{K}_G\rangle, \tag{76}$$

where $\langle\mathbf{0_p}| = \bigotimes_{e_{ij}\in E}\langle 0_p|\bigotimes_{i\in V}\langle 0_p|$ is the product of all zero-momentum states on the edge vertices and ordinary vertices of the graph G. We now note that according to Eq. (21), we can approximate the unitary operators $e^{-iV_{ij}(\widehat{Q})}$ and $e^{-iW_i(\widehat{Q})}$ to any degree of precision by a product of the operators H (the Hadamard), $e^{-it\widehat{Q}}$, $e^{-it\widehat{Q}^2}$, and $e^{-it\widehat{Q}^4}$. As explained in (Sec. II A 1), implementation of these operators on a state is effected by suitable possibly non-Gaussian measurements [i.e., projections of vertex modes on the states $\langle\beta_i(t)|(i = 1,2,4)$] of an appropriate enlargement of that state; i.e., one simply adds necessary nodes, glues them by CZ operators, and measures the additional nodes as exemplified in Fig. 2 to affect a desired unitary gate on the original qumode of the lattice. Let us denote this intermediate graph by G', its associated state by $|\Psi_{G'}\rangle$, and the collection of all necessary measurements on it by $\langle\beta_{1,2,4}|$, then we will have

$$\bigotimes_{e_{ij}} e^{-iV_{ij}(\widehat{Q})}\bigotimes_i e^{-iW_i(\widehat{Q})}|\overline{K}_G\rangle = \langle\beta_{1,2,4}|\Psi_{G'}\rangle. \tag{77}$$

The configuration of the graph G' may be complicated, but the important point is that $|\Psi_{G'}\rangle$ is nothing but a stabilizer state and hence, in principle, it can be obtained from a cluster state by Gaussian measurements $\langle\beta_i(t)|(i = 1,2)$. Therefore, we have

$$|\Psi_{G'}\rangle = \langle\beta_{1,2}|\Psi_\Box\rangle. \tag{78}$$

Note that up to now all the projections have been performed on the vertices of the cluster state. The cluster state $|\Psi_\Box\rangle$ can be a weighted cluster state where the weights of all edges are ± 1 and for each vertex the weights of all edges add up to 0. Such a cluster state can be obtained from an extended Kitaev state on the rectangular lattice by projecting all the edge-vertices on the states $\langle\beta_2(\pm 1)| = \int dy e^{\mp iy^2}\langle y|$ according to whether the weights of the edges are $+1$ or -1.

This is the only place where projections are made on the edge vertices and in fixed directions (i.e., eigenstates of $XZ^{\pm 1}$). We write this symbolically in the form

$$|\Psi_\Box\rangle = \langle\pm|\overline{K}_\Box\rangle. \tag{79}$$

Combining Eqs. (77)–(79), we finally arrive at

$$Z(G,\{V\},\{W\}) = \langle\mathbf{0_p}|\langle\beta_{1,2,4},\pm_e|\overline{K}_\Box\rangle, \tag{80}$$

where $\pm_e$ encapsulates all the measurements $XZ^\pm$ that are performed on the edges of the extended Kitaev state, and $\langle\beta_{1,2,4}|$ represents all the projections $\langle\beta_i|$ for $i = 1,2,4$ on the vertices. Putting all this together, we finally arrive at the result that

$$Z(G,\{V\},\{W\}) = Z(\Box,\{h\},\{m\},\{q\}). \tag{81}$$

It is important to note that since the edge vertices are measured in the $ZX^\pm$ bases and the resulting edge is projected on the eigenstates $|\pm\rangle| \propto \int e^{\pm iy^2}|y\rangle dy$, the interactions between neighboring vertices in the complete model is restricted to be of the type $\pm i(\phi_i - \phi_j)^2$. In this way, all the couplings in the original model have been transferred to the mass and potential terms on the vertices.

As a byproduct, this argument shows that when the original model has only quadratic couplings (i.e., it is free field), then there is no linear or quartic coupling in the model on the rectangular lattice, which reduces to this model by Gaussian measurements. This means that the free field theory on the 2D rectangular lattice is complete and can produce any other field theory on any lattice.

It is a simple matter to show that the ϕ^4 theory can also reproduce models with k-body interactions. We know from Ref. [8] that the 4D U(1) lattice gauge theory is complete. Therefore, it is enough to show that ϕ^4 theory can reduce to 4D U(1) lattice gauge theory. The Hamiltonian of the latter model is given by

$$H = -\sum_p J_p\cos(\phi_1 - \phi_2 - \phi_3 + \phi_4), \tag{82}$$

where p denotes a plaquette, J_p denotes the coupling constant on p, and ϕ_i's are the continuous variables around p. The indices 1,2,3, and 4 denotes the edges of p when traversed in clockwise direction. The point is that the partition function of such a model can again be written as a scalar product $Z = \langle\alpha|G\rangle =$, with $\langle\alpha|$ a product state over all plaquettes,

$$\langle\alpha| = \otimes_p \langle\alpha|_p, \qquad \langle\alpha|_p := \int dy e^{-iJ_p \cos y}\langle y| \qquad (83)$$

and $|G\rangle$ a new stabilizer state,

$$|G\rangle = \int Dx|\ldots,(x_1 - x_2 - x_3 + x_4)_p,\ldots\rangle. \qquad (84)$$

If we now note that $\langle\alpha|_p$ can be written as $\langle\alpha|_p = \langle 0|_p e^{-iJ_p\cos\hat{Q}}$ and the latter operator can indeed be expanded to any desired accuracy in terms of $e^{-it\hat{Q}^2}$ and $e^{-it\hat{Q}^4}$, the assertion will be proved along the same line as indicated above.

Efficiency: It is shown in Ref. [4] that simulating the partition function of any Ising or Potts type model on an arbitrary graph can be done on the Ising model on a square lattice with only a polynomial overhead in the number of spins. For more general models, however, an exponential overhead may be needed. A similar statement is true also in our case, namely for simulating the partition function of a model with nearest neighbor interactions on a graph with N vertices, we need a ϕ^4 model on a cluster state with $P(N)$ vertices, where $P(N)$ is a polynomial of N. This result is a combination of the universality result of cluster states which by only a polynomial overhead can produce any other quantum state and the fact that any quantum unitary $e^{iV(\hat{q})}$ can be decomposed to a product of polynomial number of unitaries of the form $e^{it\hat{q}^n}$, with $n = 1,2,4$ to any degree of precision. This result is also true when each site interacts with a finite number k of its neighbors, where k is independent of N. For more general models, an exponential overhead in the number of sites will be necessary, like the case of Ising model.

VII. DISCUSSION

The concept of completeness of certain statistical models is fascinating. The idea that in principle a single complete model, like the 2D Ising model in its rich phase structure, various phases of all the other models, regardless of their lattice structure, type of order statistical variables and order parameter, and the interactions, is an idea that needs much exploration in the future. One of the basic questions is that what other types of models are complete. In this paper we have shown that the ϕ^4 field theory is a complete model. Like the case of Ising model [5] or the U(1) lattice gauge theory [8], our proof is an existence proof at present. The next step in such a program will be to show how other specific models can be obtained from these complete models and what insight about them can be obtained. Like the existence proof itself, this step relies heavily on techniques from quantum information theory, notably the measurement-based quantum computation. In particular, it will be desirable to formulate an algorithmic approach for deriving any specific model (its graph structure and coupling strengths) from a complete model. In this way apparently unrelated models will be linked to each other and the insight gained from this approach will have far-reaching consequences in statistical mechanics, exactly solvable models, and critical phenomena.

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