Barycentric measure of quantum entanglement

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Majorana representation of quantum states by a constellation of n "stars" (points on the sphere) can be used to describe any pure state of a simple system of dimension $n + 1$ or a permutation symmetric pure state of a composite system consisting of n qubits. We analyze the variance of the distribution of the stars, which can serve as a measure of the degree of noncoherence for simple systems or an entanglement measure for composed systems. Dynamics of the Majorana points induced by a unitary dynamics of a pure state is investigated.

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I. INTRODUCTION

Nonclassical correlations between composite quantum systems have become a subject of intense current research. A particular type of such correlations, called *quantum entanglement*, attracts special attention of theoretical and experimental physicists; see Ref. [1] and references therein. Entanglement in quantum systems consisting of two subsystems is nowdays relatively well understood, but several significant questions concerning multiparticle systems remain still open.

One of the key issues is to workout a practical entanglement measure, used to quantify the quantum resources of a given state. Although various measures of quantum entanglement are known [1–4], they are usually difficult to compute. An important class of entanglement measures can be formulated within the geometric approach to the problem [3,5]. For a given quantum state, one can study its minimal distance to the set of separable states. Various distances [3] can be used for this purpose and the minimization can be performed with respect to the mixed [6] or pure [7,8] separable states.

Working with pure quantum states of a n-qubit system it is possible to distinguish a class of states symmetric with respect to permutations of all subsystems. This class of symmetric pure quantum states can be identified with the set of all states of a single system described in an $N = n + 1$ dimensional Hilbert space.

When analyzing the space of pure states belonging to the N-dimensional Hilbert space, it is useful to distinguish the class of *spin-coherent* states. These states, $|\theta,\phi\rangle$, labeled by a point on the sphere, can be obtained by action of the Wigner rotation matrix $R_{\theta,\phi}$ on the maximal weight state $|j,j\rangle$, so they are also called SU(2)-coherent states. Here $j = n/2$ is the maximal eigenvalue of the component J_z of the angular-momentum operator. Making use of the stereographic projection, one can map the sphere into the plane. The coherent states are then labeled by a complex number α and their expansion in the eigenbasis of J_z reads [9,10]

$$|\alpha\rangle = \frac{1}{(1 + |\alpha|^2)^{n/2}} \sum_{k=0}^{n} \binom{n}{k} \alpha^k |k\rangle. \qquad (1)$$

Consider now an arbitrary state of an N-level system, $|\psi\rangle = \sum_{k=0}^{n} c_k |k\rangle$. It can be expanded in the coherent-states representation,

$$Q_\psi(\alpha) = |\langle\psi|\alpha\rangle|^2 = \frac{|c_n|^2}{(1 + |\alpha|^2)^n} \prod_{i=1}^{n} |\alpha - z_i|^2. \qquad (2)$$

The function $Q_\psi(\alpha)$ is called the *Husimi function* (or Q function) of the state $|\psi\rangle$ and it can be interpreted as a probability density on the plane. As it can be associated with a polynomial of order n of a complex argument, it is uniquely represented by the set if its n roots z_i, $i = 1, \dots n$, which may be degenerated. With help of the inverse stereographic projection, one can map these points back on the sphere. The collection of these n points on the sphere, corresponding to the zeros of the Husimi function (2), represents uniquely the state $|\psi\rangle \in \mathcal{H}_{n+1}$. This approach of Majorana [11] and Penrose [12] leads to the so-called *stellar representation* of a state, as each *Majorana point* on the sphere corresponding to z_i can be interpreted as a star on the sky.

For any coherent state $|\alpha\rangle = |\theta,\phi\rangle$ all n stars sit in a single point antipodal to the vector pointing in the direction (θ,ϕ). If the Majorana points are located in a neighborhood of a single point, the corresponding state is close to be coherent. In contrast, a generic random state, for which the distribution of stars is uniform on the sphere [13,14] is far from being coherent. The degree of noncoherence of $|\psi\rangle$ can be characterized, e.g., by its Fubini-Study distance to the closest coherent state or by the Monge distance equal to the total geodesic distance on the sphere all stars have to travel to meet in a single point [15].

The stellar representation, originally used for states of a simple system of $n + 1$ levels, can be also used to analyze composite systems consisting of n qubits, under the assumption that the states are symmetric with respect to permutations of all subsystems [16–21]. Due to this symmetry the investigation of such states becomes easier and estimation of their geometric measure of entanglement can be simplified [22].

Note that any separable state of the n-qubit system can be considered as a coherent state with respect to the composed group $[SU(2)]^n$. Thus, entangled states of a composite system correspond to noncoherent states of the simple system with $n + 1$ levels [23], while the degree of entanglement can be identified with the degree of noncoherence.

In this work we propose to characterize pure quantum states by the position of the barycenter of its Majorana representation. We will then describe nonclassical properties of a state by the

average and the variance of the corresponding distribution of the Majorana points. The latter quantity has a simple geometric interpretation as a function of the radius of the barycenter of the stars, which is situated inside the Bloch ball. The same quantity can, thus, be applied to characterize the degree of noncoherence of a pure state of an $(n + 1)$-level system or, simultaneously, as a degree of entanglement for the corresponding permutation symmetric pure state of an n-qubit system.

In a similar way, the measure of quantumness of a state of $n + 1$-dimensional system introduced by Giraud *et al.* [24] can be used to characterize the entanglement of permutationally symmetric pure states of n qubits. We are also going to study time evolution of quantum states and an associated dynamics of stars. In particular, we investigate how the barycentric measure of entanglement varies with time.

This work is organized as follows. In Sec. II, we review the Majorana representation used for symmetric pure states of a multiqubit quantum system. In Sec. III the position of the barycenter of Majorana points and their variance is investigated, as it may serve as a characterization of quantum entanglement. A family of n-qubit states maximally entangled with respect to this measure is studied in Sec. IV. Unitary dynamics of quantum states and the associated dynamics of Majorana points on the sphere is analyzed in Sec. V.

II. MAJORANA REPRESENTATION OF MULTI–QUBIT STATES

Consider an n-qubit pure state $|\psi\rangle$ symmetric with respect to permutation of subsystems:

$$|\psi\rangle = \frac{1}{\sqrt{K}} \sum_{\pi} |\phi_{\pi(1)}\rangle |\phi_{\pi(2)}\rangle \cdots |\phi_{\pi(n)}\rangle. \quad (3)$$

The sum is taken over all permutations π, while the normalization constant reads

$$K = n! \sum_{\pi} \prod_{i=1}^{n} |\langle \phi_i | \phi_{\pi(i)} \rangle|.$$

Any one-qubit pure state can be represented as $|\phi_i\rangle = \cos \frac{\theta_i}{2} |0\rangle + e^{i\Phi_i} \sin \frac{\theta_i}{2} |1\rangle$. Making use of the the Dicke states $|S_{n,k}\rangle$ with $k = 0, 1, \ldots, n$ [25],

$$|S_{n,k}\rangle = \binom{n}{k}^{-\frac{1}{2}} \sum_{\rho} P_{\rho} |0\rangle^{\otimes n-k} |1\rangle^{\otimes k}, \quad (4)$$

we can represent the state $|\psi\rangle$ as their superposition,

$$|\psi\rangle = \sum_{k=0}^{n} d_k |S_{n,k}\rangle = \sum_{k=0}^{n} d_k \binom{n}{k}^{-\frac{1}{2}} \sum_{\rho} P_{\rho} |0\rangle^{\otimes n-k} |1\rangle^{\otimes k}, \quad (5)$$

where d_k denote complex coefficients, $\binom{n}{k}$ is the binomial coefficient, and the sum $\sum_{\rho} P_{\rho}$ goes over all states of n-qubits with exactly k qubits in the state $|1\rangle$ and $n - k$ qubits in the state $|0\rangle$. It comes out that quotients of the coefficients $\frac{\cos \frac{\theta_i}{2}}{e^{i\Phi_i} \sin \frac{\theta_i}{2}} = e^{-i\Phi_i} \cot(\theta_i/2)$ determining the one-qubit state $|\phi_i\rangle$ can be

obtained as roots z_i of the polynomial

$$P(z) = \sum_{k=1}^{n} d_k (-1)^k \binom{n}{k}^{\frac{1}{2}} z^k. \quad (6)$$

The polynomial defined above has degree $D \leqslant n$ and, consequently, as many roots. In the case of $D < n$, the remaining $(n - D)$ coefficients $\cos \frac{\theta_i}{2}$ were set to unity [16]. In this way, we can represent any n-qubit state symmetric with respect to permutations as n points on the Bloch sphere related to spin-$\frac{1}{2}$ (one-qubit) states $|\phi_i\rangle = \cos \frac{\theta_i}{2} |0\rangle + e^{i\Phi_i} \sin \frac{\theta_i}{2} |1\rangle$. These n points on the Bloch sphere define a *stellar representation* of a state and are called stars or *Majorana points* (MP), while polynomial (6) is called the *Majorana polynomial* [11].

Note that the same constellation of n stars determines, on the one hand, a pure state of the simple system described in the $N = n + 1$ dimensional Hilbert space [3,9,12]. On the other hand, it describes also a permutationally symmetric pure state of an n-qubit system [17,20,21] and belongs to the Hilbert space of dimension 2^n.

To establish a one-to-one link between both problems it is sufficient to identify bases in both spaces. In the case of a simple system described in the Hilbert space of size $n + 1$ we select the standard eigenbasis of the angular-momentum operator J_z,

$$|j,m\rangle, \quad m = -j, \ldots, j, \quad (7)$$

with $j = n/2$. A state from this basis is represented by a constellation of $n/2 + m$ stars at the north pole and the remaining $n/2 - m$ stars at the south pole, as shown in Fig. 1. Thus, the state $|j,m\rangle \in \mathcal{H}_N$ can be identified with the Dicke state $|S_{n,k}\rangle \in \mathcal{H}_2^{\otimes n}$ defined in Eq. (4) with $k = n/2 - m$. In this way, any constellation of stars can be used to describe the pure quantum state of two different physical systems: an $(n + 1)$-level system and a symmetric state of n qubits. In following sections we show that this link can be extended also for quantum dynamics. Any discrete unitary dynamics of an
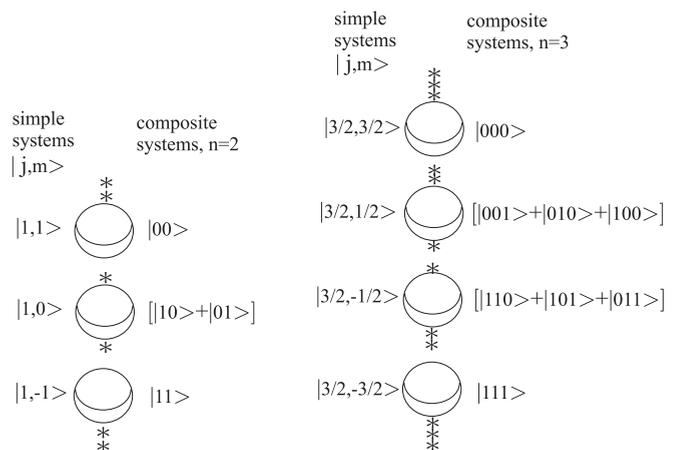
FIG. 1. Stellar representation of the orthogonal basis in the $(n + 1)$-dimensional Hilbert space describing a simple system (states $|j,m\rangle$ with $j = n/2$) with the corresponding basis in the subspace of permutation symmetric states of n-qubit systems plotted for $n = 2$ and $n = 3$.

n-qubit system, which preserves the permutation symmetry, can be represented by an unitary matrix of size 2^n with a block structure in the computational basis. The diagonal block of size $n + 1$ is unitary and it defines the corresponding dynamics of a simple quantum system with $n + 1$ levels.

III. BARYCENTER AS AN ENTANGLEMENT MEASURE

We propose a quantity designed to characterize entanglement for permutation symmetric states. It is based on the distance between the barycenter of the Majorana points representing a permutation-symmetric state and the center of the Bloch ball. We will show that this quantity vanishes on separable states and it does not increase under local operations which preserve the symmetry.

A. Entanglement measure for permutation symmetric states

For any permutation symmetric state [Eq. (3)] with $|\phi_i\rangle = \cos\frac{\theta_i}{2}|0\rangle + e^{i\Phi_i}\sin\frac{\theta_i}{2}|1\rangle$, let us define,

$$E_B = 1 - d^2, \qquad (8)$$

where d is the distance between the barycenter of the Majorana points representing the state and the center of the Bloch ball,

$$d = \left| \frac{1}{n} \sum_{i=1}^{n} (\sin\theta_i \cos\Phi_i, \ \sin\theta_i \sin\Phi_i, \ \cos\theta_i) \right|, \quad (9)$$

where $|x|$ denotes the length of a vector x.

Let us call E_B the *barycentric measure* of entanglement. This quantity can be interpreted as the variance of all Majorana points representing the state. Indeed, since a vector directed toward any point at the unit sphere has the unit length, we have $E_B = D^2 X = \langle X^2 \rangle - \langle X \rangle^2 = 1 - \langle X \rangle^2 = 1 - d^2$. It is easy to see that $E_B \in [0, 1]$.

Note that the quantity E_B can be used to characterize the symmetric states of the n-qubit system or the states of a single system of size $N = n + 1$. For a separable state $|\psi\rangle = |\phi\rangle^{\otimes n}$ [or an SU(2)-coherent state of a single qunit] all stars are located in a single point, so $d = 1$ and $E_B = 0$. Moreover, a generic random state is likely to be highly entangled (or highly noncoherent) as the Majorana points are distributed almost uniformly at the Bloch sphere [13,14], which implies $d \approx 0$ and $E_B \approx 1$.

A useful entanglement measure should not increase under local operations. The barycentric measure E_B is defined only for symmetric states so monotonicity can be checked only for local unitary operations which preserve the permutation symmetry. Consider two n-qubit permutation symmetric states $|\psi\rangle$, $|\phi\rangle$ connected by a local operation. This means that there exist unitary operators A_i such that $|\psi\rangle = A_1 \otimes \cdots \otimes A_n |\phi\rangle$. Mathonet *et al.* proved in Ref. [26] that in fact we may find a single invertible operator A for which $|\psi\rangle = A^{\otimes n}|\phi\rangle$. Geometrically such an operation corresponds to a rotation of each Majorana points around the same axis by the same angle or, equivalently, a rigid rotation of the whole Bloch sphere. Obviously, thus, the radius of the barycenter d, and, consequently, the barycentric measure E_B, do not change. This concludes a proof of Proposition 1.
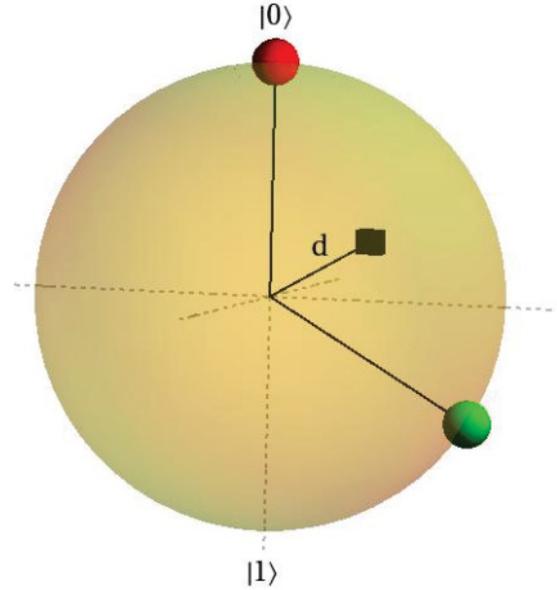
FIG. 2. (Color online) Majorana representation of a symmetric two-qubit state of the form $|\phi_1\rangle = |0\rangle$ and $|\phi_2\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$ with $\theta = \frac{2}{3}\pi$. Majorana points are shown as two balls. The cube denotes the barycenter of MP located inside the Bloch ball and its distance to the center of the Bloch ball is equal to d.

Proposition 1. The barycentric measure E_B is invariant under local operations preserving the permutation symmetry.

B. Examples

We shall calculate the barycentric measure E_B for certain exemplary quantum states and make a comparison with the *geometric measure* of entanglement E_G defined as a function of the distance from the analyzed state ϕ and the set of separable states $\mathcal{H}_{\text{SEP}}$. Such a quantity, first proposed by Brody and Hughston [7] and later used in Ref. [8], can be defined by

$$E_G(|\phi\rangle) = \min_{|\lambda\rangle \in \mathcal{H}_{\text{SEP}}} \log_2 \left(\frac{1}{|\langle\lambda|\phi\rangle|^2} \right), \qquad (10)$$

Let us start with the two-qubit case. Without loss of generality we may set $|\phi_1\rangle = |0\rangle$ and $|\phi_2\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$. The special case $\theta = \frac{2}{3}\pi$ is shown in Fig. 2. Direct calculation gives for this family of states $E_B = 1 - \cos^2\frac{\theta}{2}$. Figure 3 shows how E_B and E_G change with the angle θ. We see that in this case $E_B \geqslant E_G$, but for a product state ($\theta = 0$) and for the maximally entangled state ($\theta = \pi$) both quantities coincide.

Analysis of three-qubit states exhibits a more interesting situation. Consider a one-parameter family of three-qubit states, $|\phi_1\rangle = |0\rangle$, $|\phi_2\rangle = \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle$ and $|\phi_3\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$. The case $\theta = \frac{5}{6}\pi$ is shown in Fig. 4. Using simple geometry we obtain that for such states the barycentric measure reads $E_B = 1 - |\frac{2\cos\theta + 1}{3}|^2$. Figure 5 shows how E_B and E_G change with the angle θ. Note a significant difference between E_B and E_G. The barycentric measure E_B reaches its maximum at $\theta = \frac{2}{3}\pi$, which corresponds to the GHZ state and we have $E_B(|\text{GHZ}_3\rangle) = E_G(|\text{GHZ}_3\rangle) = 1$. However, for $\theta > \frac{2}{3}\pi E_G$ increases and reaches its maximal value at $\theta = \pi$, at the state $|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. Moreover, it is easy to see
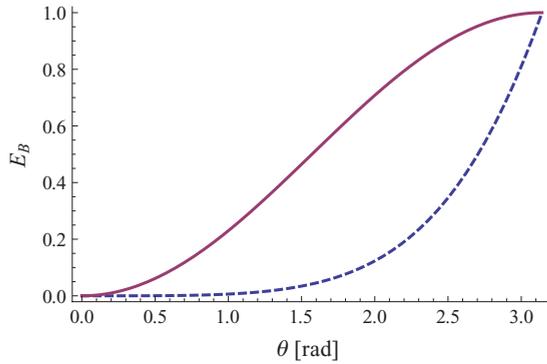
FIG. 3. (Color online) Dependence of entanglement measures for a symmetric two-qubit state determined by $|\phi_1\rangle = |0\rangle$ and $|\phi_2\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$. The solid curve denotes the barycentric measure E_B, while the geometric measure E_G is represented by the dashed curve.

that the state $|GHZ_3\rangle$ is maximally entangled with respect to the barycentric measure E_B: For three points at the sphere their barycenter could be in the center of the sphere if and only if they form an equilateral triangle. As a third example, we consider the Dicke states $|S_{n,k}\rangle$ [Eq. (4)]. They are characterized by k stars at the south pole and the remaining $n - k$ Majorana point occupying the north pole of the Bloch sphere.

The length of the radius of the barycenter is, thus, easily calculated as $d = |\frac{n-2k}{n}|$. We would like to make the comparison with E_G in a slightly different way, employing results of Hayashi *et al.* [20], who showed that the product state closest to $|S_{n,k}\rangle$ reads

$$|\Lambda\rangle = \left(\sqrt{\frac{n-k}{n}}|0\rangle + \sqrt{\frac{k}{n}}|1\rangle\right)^{\otimes n}, \qquad (11)$$
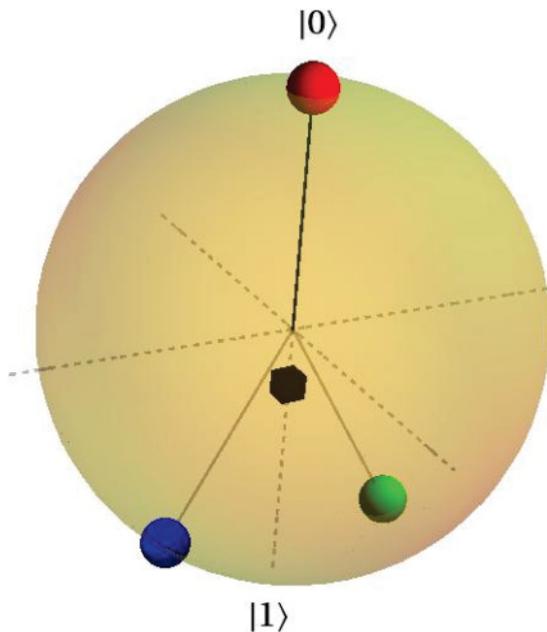
FIG. 4. (Color online) Majorana representation of a symmetric three-qubit state $|\phi_1\rangle = |0\rangle$, $|\phi_2\rangle = \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle$, and $|\phi_3\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$ with $\theta = \frac{5}{6}\pi$. Majorana points are shown as three balls. The black cube denotes the barycenter of MP.
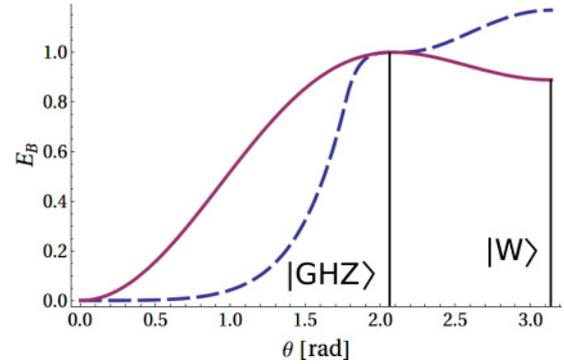
FIG. 5. (Color online) Entanglement measures of symmetric three-qubit state of the form such that $|\phi_1\rangle = 0$, $|\phi_2\rangle = \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle$, and $|\phi_3\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$. (Solid line) E_B; (dashed line) E_G. Result for E_G from Ref. [17].

hence,

$$E_G(S_{n,k}) = \log_2\left(\frac{\left(\frac{n}{k}\right)^k\left(\frac{n}{n-k}\right)^{n-k}}{\binom{n}{k}}\right). \qquad (12)$$

The projection on the z axis of Λ has the length equal $|\frac{n-2k}{n}|$, just like the length of the radius d (see above). For an even n, the state $|S_{n,n/2}\rangle$ is maximally entangled with respect to the measure E_G among all Dicke states; for odd n there are two such states, $|S_{n,(n-1)/2}\rangle$ and $|S_{n,(n+1)/2}\rangle$. The same states are extremal with respect to the barycentric measure.

Moreover, for an even n, the state $|S_{n,n/2}\rangle$ belongs to the class of the maximally entangled n-qubit states as its barycentric measure E_B is equal to unity. A comparison between the measures E_B and E_G is shown in Fig. 6 for Dicke states with $n = 10$ and $n = 11$. The same results hold for the states of the standard $|j,m\rangle$ basis in the space of $N = n + 1$ level systems, with $j = n/2$ and $m = k - n/2$.

IV. FAMILY OF MAXIMALLY ENTANGLED STATES

It is interesting to look for states which are maximally entangled with respect to the barycentric measure E_B. The answer is simple for a small number of qubits. For $n = 2$ a constellation of two stars on the sphere has its barycenter in the center of the sphere if and only if it consists two antipodal points on the sphere. Therefore, the class of maximally entangled two-qubit states with $E_B = 1$ coincides with all states equivalent to the Bell state up to a local unitary transformation.

A. Two and three qubits

For the two-qubit problem, one can establish the following property.

Proposition 2. There is no universal unitary operator $U(\theta)$ which transforms any separable state with both MP at θ_0 into a state with MP at $\theta_0 \pm \theta$ for a given angle θ.

Indeed, assume that A is such an operator. Then, in particular, it would transform the states $|00\rangle$ and $|11\rangle$ into the Bell state $|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, which is impossible since a unitary operator cannot transform two orthogonal states into the same state.
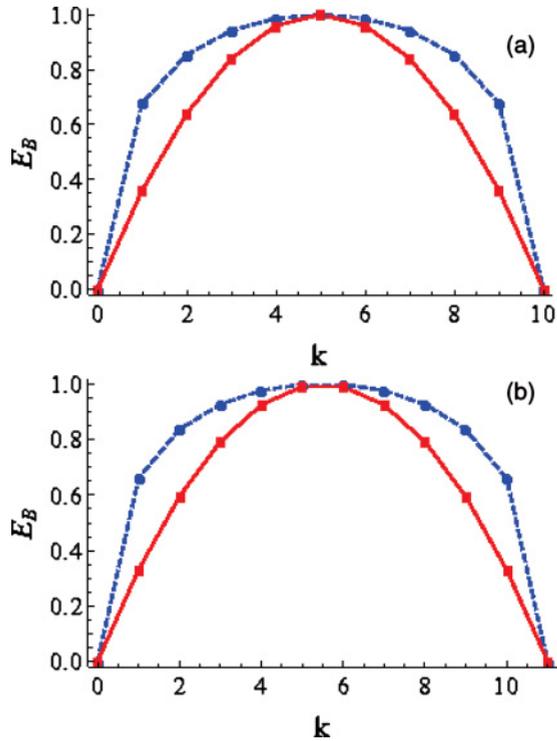
FIG. 6. (Color online) Dependence of entanglement of Dicke states $|S_{n,k}\rangle$, as a function of k for (a) $n = 10$ and (b) $n = 11$. Red squares denote the barycentric measure E_B, and blue circles denote the geometric measure E_G normalized to unity for maximally entangled states; lines are plotted to guide the eye.

The three-qubit case was discussed in the preceding section (Sec. III B) with the conclusion that the class of maximally entangled three-qubit states consist of the GHZ state and the states unitarily equivalent. The case of four-qubit states reduces to a geometric problem of placing four points on the sphere in such a way that their barycenter is located in the center of the sphere. Consider a rectangle inscribed into a sphere, belonging to the XZ plane so ts barycenter is the center of the sphere. Without changing the position of the barycenter we can vary the lengths of its sides by changing the angle θ between one side of the rectangle and its diagonal (see Fig. 7). We can also rotate one of its sides (the bottom side in Fig. 7) varying the angle $\Phi \in [0,\pi]$.

B. Four qubits

Assume now that these four points represent a certain four-qubit permutation symmetric state $|\psi_{\mathrm{rec}}\rangle$. It means that $|\psi_{\mathrm{rec}}\rangle$ is the symmetrization [Eq. (3)] of the tensor product of

$$
\begin{aligned}
|\phi_1\rangle &= \cos\frac{\theta}{2}|0\rangle + e^{i\Phi}\sin\frac{\theta}{2}|1\rangle, \\
|\phi_2\rangle &= \cos\frac{\theta}{2}|0\rangle - e^{i\Phi}\sin\frac{\theta}{2}|1\rangle, \\
|\phi_3\rangle &= \cos\frac{\pi-\theta}{2}|0\rangle + \sin\frac{\pi-\theta}{2}|1\rangle, \\
|\phi_4\rangle &= \cos\frac{\pi-\theta}{2}|0\rangle - \sin\frac{\pi-\theta}{2}|1\rangle,
\end{aligned}
\tag{13}
$$
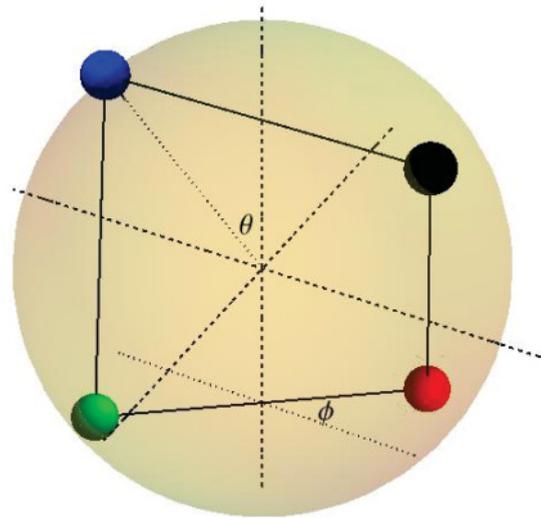
FIG. 7. (Color online) Parameterization of the family of the maximally entangled four-qubit states. The dashed lines are the three axes of the coordinate system. The full lines are the four sides of the rectangle. The upper side is parallel to the X axis, and the bottom side is rotated by an angle Φ. The dotted lines are used to mark the angles: the bottom dotted line is parallel to the X axis, i.e., at the position of the bottom side of rectangle without rotation.

where $\theta \in [0,\pi/2]$, $\Phi \in [0,\pi]$. Consequently,

$$
\begin{aligned}
|\psi_{\mathrm{rec}}\rangle = \frac{e^{i\Phi}}{\sqrt{K}}\bigg(&6e^{-i\Phi}\sin^2\theta|0000\rangle + 6e^{i\Phi}\sin^2\theta|1111\rangle \\
&+ [4i\cos\theta\sin\Phi - 2(\cos^2\theta + 1)\cos\Phi] \\
&\times \sum_\pi |0011\rangle \bigg),
\end{aligned}
\tag{14}
$$

where K is the normalization factor. The state $|\psi_{\mathrm{rec}}\rangle$ defined above and all locally equivalent states form a class of four-qubit states maximally entangled with respect to the barycentric measure E_B. This class contains the $|W_4\rangle$ state (which we identify with $|S_{4,2}\rangle$), obtained for $\theta = \Phi = 0$, such that all four Majorana points belong to a single line: two stars are localized at the north pole and two other at the south pole. The choice $\theta = \pi/2$ and $\Phi = \frac{\pi}{2}$ produces $|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}[|0000\rangle + |1111\rangle]$ represented by stars in four corners of a square at the equator. Setting $\theta = \pi/4$ and $\Phi = 0$ we obtain a state locally equivalent to $|\mathrm{GHZ}_4\rangle$ with stars situated on a plane perpendicular to the equatorial one. The tetrahedron state $|\psi_{\mathrm{tetr}}\rangle$ corresponds to $\theta = \arccos(1/\sqrt{3})$ and $\Phi = \frac{\pi}{2}$.

To compare both measures of entanglement, following Ref. [17] we computed numerically the geometric measure of entanglement E_G for the entire family [Eq. (14)] of extremal states, for which $E_B = 1$ (see Fig. 8). All states from this class are characterized by $E_G \geqslant 1$, and the minimum is attained for a $|\mathrm{GHZ}_4\rangle$ state, for which $E_G = 1$. For a $|W_4\rangle$ state one has $E_G(|W_4\rangle) = \log_2(8/3) \approx 1.415$ [20], while the maximum is obtained for the tetrahedron state and $E_G(|\psi_{\mathrm{tetr}}\rangle) = \log_2 3 \approx 1.585$ [17]. This observation suggests that both measures are well correlated, so the barycentric measure of entanglement, simple to evaluate, could be used to characterize the state in parallel to the geometric measure.
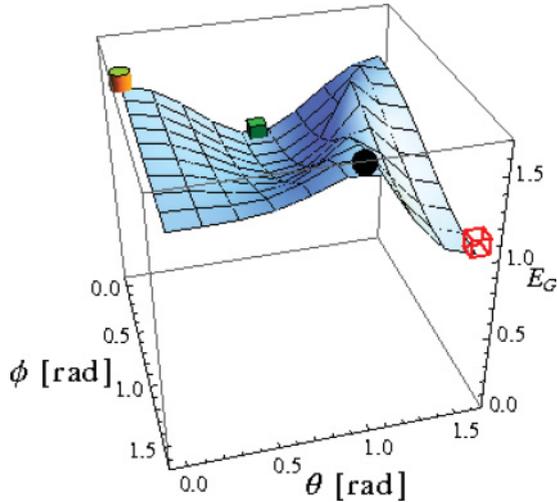
FIG. 8. (Color online) Geometric measure of entanglement for the family [Eq. (14)] of four-qubit states for which $E_B = 1$ parameterized by angles (θ, Φ) is not smaller than 1. The cylinder represents the state $|W_4\rangle$, the ball denotes the tetrahedron state for which E_G is maximal, while two cubes represent two equivalent $|GHZ_4\rangle$ states (empty cube, $|GHZ_4\rangle$ with stars situated on the equatorial plane; full cube, rotated $|GHZ_4\rangle$, situated on a plane perpendicular to the equatiorial one).

Note that the states called "Queens of Quantum" by the authors of [24], ie., the most distant from the set of product states with respect to the Hilbert-Schmidt [24] or Bures [21] distance, are, for small values of N, described by symmetric constellations of the Majorana points, so they are maximally entangled with respect to the barycentric measure E_B.

C. Composition of states

Consider two pure states of a simple, N-level system, $|\psi_1\rangle$ and $|\psi_2\rangle$ in $\mathcal{H}_N$. A *composition* of these states, $|\psi_1\rangle \odot |\psi_2\rangle$, defined in Ref. [3], forms a pure state belonging to a $2N - 1$ symmetric subspace of the tensored Hilbert space, $\mathcal{H}_N \otimes \mathcal{H}_N$. Both initial states are described by $n = N - 1$ stars, and their composition, represented by $2n$ stars, is defined by adding the stars together. In the similar way one can compose the states from two Hilbert spaces of different dimensions. More formally, one can define the composition by its Husimi distribution $Q(\alpha)$, which is given by the product of two Husimi distributions representing individual states,

$$Q_{|\psi_1\rangle \odot |\psi_2\rangle} \propto Q_{|\psi_1\rangle} Q_{|\psi_2\rangle}. \qquad (15)$$

The same idea of composition of two pure states of a simple system described by the stellar representation now can be used for the symmetric states of several qubits. Consider a permutation symmetric n-qubit state $|\psi\rangle \in \mathcal{H}_2^{\otimes n}$ and another m-qubit state $|\phi\rangle \in \mathcal{H}_2^{\otimes m}$. Their composition, written $|\psi\rangle \odot |\phi\rangle \in \mathcal{H}_2^{\otimes n+m}$, represents a state of $n + m$ qubits and is defined by the entire sum of $n + m$ Majorana points on the sphere.

Observe that this notion allows us to write the symmetric Bell state, $|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |1\rangle \odot |0\rangle$. Introducing two, one-qubit superposition states, $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ we see that another Bell state $|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ is

equivalent to the composition $|+\rangle \odot |-\rangle$. On the other hand, any symmetric separable state of n-qubit system can be represented by the composition performed $n - 1$ times, $|\phi, \ldots, \phi\rangle = |\phi\rangle \odot \cdots \odot |\phi\rangle$, as in this case all the stars do coincide.

Take any two figures with the same barycenter. Superposing them, one obtains another figure with the same barycenter. This fact implies a useful proposition.

Proposition 3. Consider two permutation symmetric states of any number of qubits, maximally entangled with respect to the barycentric measure, $E_B(\phi) = E_B(\psi) = 1$. Their composition then is also maximally entangled, $E_B(\phi \odot \psi) = 1$.

To watch this proposition in action, consider the composition of two maximally entangled Bell states, $|\Psi_+\rangle$ and $|\Phi_-\rangle$. Their composition, $|\Psi_+\rangle \odot |\Phi_-\rangle = |0\rangle \odot |1\rangle \odot |+\rangle \odot |-\rangle$, represents the state formed by a square belonging to the plane containing the meridian of the sphere. It is then locally equivalent to the state $|GHZ_4\rangle$, represented by a square inscribed into the equator, also maximally entangled with respect to the barycentric measure.

The notion of a composition allows us two write down a generic random state of an n-qubits system,

$$|\psi_{\rm rand}\rangle := |\phi_1\rangle \odot |\phi_2\rangle \odot \cdots \odot |\phi_n\rangle, \qquad (16)$$

where $\phi_i, i = 1, \ldots n$, denotes a one-qubit state, generated by a vector taken randomly with respect to the uniform measure on the sphere. For large n, such a generic state is highly entangled with respect to E_B, as the barycenter of the points will be located close to the center of the sphere.

Making use of proposition 3, one may also design a random state for which the barycentric measure is equal to unity. It is sufficient to compose several maximally entangled Bell states represented by a random collection of pairs of antipodal points,

$$|\psi'_{\rm rand}\rangle := |\phi_1\rangle \odot |\bar{\phi}_1\rangle \odot \cdots \odot |\phi_{n/2}\rangle \odot |\bar{\phi}_{n/2}\rangle. \qquad (17)$$

The number n of qubits is assumed to be even and the directions labeled by ϕ_i and $\bar{\phi}_i$ are antipodal, so the state $|\phi_i\rangle \odot |\bar{\phi}_i\rangle$ is maximally entangled and locally equivalent to the Bell state. Their composition produces the state $|\psi'_{\rm rand}\rangle$, which is characterized by the maximal possible value of the barycentric measure, $E_B = 1$.

V. DYNAMICS OF MAJORANA POINTS

A. Interpolation between states

In this section we analyze the parametric dynamics of the Majorana points when a state changes with some parameter β. A related problem was analyzed by Prosen [27], who analyzed the parametric dynamics of stars on the complex plane representing a quantum state under a change of paremeters of the system. We will allow the pure state to evolve in time, so the parameter β can be interpreted as time.

As a first example, consider the state $|\psi_1\rangle = \cos\beta|00\rangle + \frac{\sin\beta}{\sqrt{2}}(|01\rangle + |10\rangle)$ with $\beta \in [0, \pi]$. It can be written in the form of Eq. (3) with the help of the one-qubit states,

$$|\phi_1\rangle = |0\rangle$$
$$|\phi_2\rangle = \frac{\sqrt{K(\beta)}}{2}\cos\beta|0\rangle + \sqrt{\frac{K(\beta)}{2}}\sin\beta|1\rangle, \qquad (18)$$
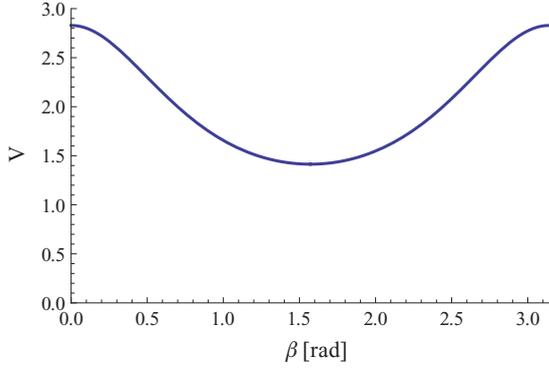
FIG. 9. (Color online) Dependence of the velocity $V = \partial_\beta \theta$ of the star traveling across the sphere defined as a function of the phase β for the state $|\psi_1\rangle = \cos\beta|00\rangle + \frac{\sin\beta}{\sqrt{2}}(|01\rangle + |10\rangle)$. The minimal V is achieved for a Bell state, $\beta = \pi/2$, while the maximal velocity is obtained for product states, $\beta = 0$ and $\beta = \pi$.

where $K(\beta) = \frac{4}{2-\cos^2\beta}$. A unitary operator U which generates such a dynamics is not uniquely determined. If we assume that the operator U transforming $|00\rangle$ into $|\psi_1\rangle$ has the form $U = \mathbb{I} \otimes M_1 + M_2 \otimes \mathbb{I}$, we arrive at two possibilities,

$$U_1 = \cos\beta \, \mathbb{I} \otimes \sigma_z - i\frac{\sin\beta}{\sqrt{2}}\sigma_y \otimes \mathbb{I} + \frac{\sin\beta}{\sqrt{2}}\mathbb{I} \otimes \sigma_x,$$
$$U_2 = \cos\beta \, \sigma_z \otimes \mathbb{I} - i\frac{\sin\beta}{\sqrt{2}}\mathbb{I} \otimes \sigma_y + \frac{\sin\beta}{\sqrt{2}}\sigma_x \otimes \mathbb{I}. \quad (19)$$

The operators U_1 and U_2 are similar: The only difference is the order of the operators in the tensor product. Operating on a permutation symmetric state, we can act either on the first or on the second subsystem to obtain the same outcome.

Equation (18) shows that the dynamics of the Majorana points for such a family is relatively simple. A single star representing $|\phi_1\rangle$ stays at the north pole all the time, while the second star makes a loop around the Bloch sphere. In particular, for $\beta = 0$ or π, we have both points at the north pole and for $\beta = \pi/2$ we have the Bell state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ with one point at the north and the second at the south pole. As shown in Fig. 9 the velocity of this second Majorana point is highly nonlinear. In the case considered the velocity is anticorrelated with the amount of entanglement. For the maximally entangled Bell state which emerges at $\beta = \pi/2$, the velocity reaches its minimum while the maximum is attained for the separable state $|00\rangle$.

In general, the dynamics of the stars defining the stellar representation of a state is not as simple. For instance, for a family $\frac{1}{\sqrt{K(\beta)}}(\cos\beta|000\rangle + \sin\beta|GHZ\rangle)$, with $K(\beta)$ a suitable normalization factor, one obtains the motion of three points, shown in Fig. 10.

B. Permutation-invariant Hamiltonians

Consider a more general dynamics defined by a one-parameter subgroup of the unitary group,

$$U = e^{-i\beta H}, \quad (20)$$

where H is a Hermitian Hamiltonian. Of interest are only transformations preserving the permutation symmetry of
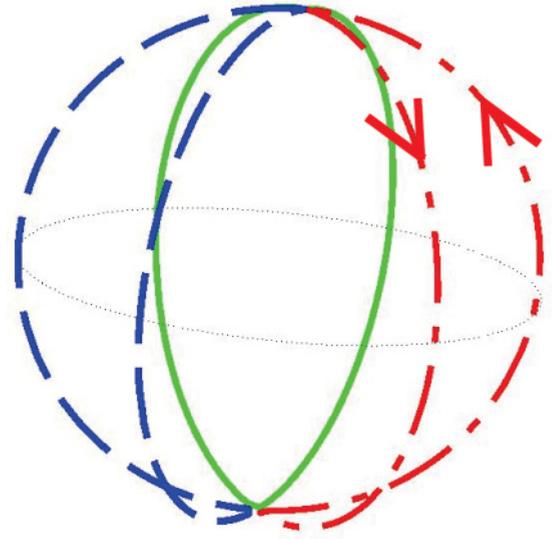
FIG. 10. (Color online) Trajectories of Majorana points for the state $\frac{1}{\sqrt{K}}(\cos\beta|000\rangle + \sin\beta|GHZ\rangle)$. Regular green, dashed blue, and dot-dashed red lines (with arrows) represent trajectories of three MP for $\beta \in [0,\pi]$; the thin black dashed line represents the equator. Three stars travel from the north pole to the south pole and, at the equator, for $\beta = \pi/2$, they form a $|GHZ\rangle$ state. Arrows at the red dot-dashed trajectory show the movement direction of the corresponding Majorana point.

states. We are going to study an exemplary family of two-qubit Hamiltonians, parametrized by indices $i,j = 0, \ldots 3$ with $i \neq j$,

$$H_{ij} = \tfrac{1}{2}(\sigma_i \otimes \sigma_j + \sigma_j \otimes \sigma_i), \quad (21)$$

where σ_i represents the Pauli matrix and $\sigma_0 = \mathbb{I}$. Using $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \mathbb{I}\delta_{ij}$, we obtain

$$U = -\tfrac{1}{2}[\cos\beta(\mathbb{I} \otimes \mathbb{I} + \sigma_k \otimes \sigma_k)$$
$$+ i\sin\beta(\sigma_i \otimes \sigma_j + \sigma_j \otimes \sigma_i) + (\mathbb{I} \otimes \mathbb{I} - \sigma_k \otimes \sigma_k)]. \quad (22)$$

Clearly U preserves the permutation symmetry as it acts on both particles in the same way. This is also true for any linear combination of generators H_{ij} of the form (21). Taking, thus,

$$H = \frac{H_{23} + H_{03}}{\sqrt{2}} = -\frac{1}{2\sqrt{2}}[\sigma_2 \otimes (\sigma_0 + \sigma_3) + (\sigma_0 + \sigma_3) \otimes \sigma_2],$$

we obtain,

$$U = \begin{bmatrix} \cos\beta & -\frac{\sin\beta}{\sqrt{2}} & -\frac{\sin\beta}{\sqrt{2}} & 0 \\ \frac{\sin\beta}{\sqrt{2}} & \frac{1}{2}(1+\cos\beta) & \frac{1}{2}(-1+\cos\beta) & 0 \\ \frac{\sin\beta}{\sqrt{2}} & \frac{1}{2}(-1+\cos\beta) & \frac{1}{2}(1+\cos\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (23)$$

which transforms the state $|00\rangle$ into $|\psi_1\rangle = \cos\beta|00\rangle + \frac{\sin\beta}{\sqrt{2}}(|01\rangle + |10\rangle)$, as in the example considered previously. Note that the operator U which generates the same dynamics of stars differs from the operators U_1 and U_2 defined in Eq. (19).

C. Unitary dynamics of symmetric states of $(N-1)$ qubits and corresponding dynamics of a single qunit

As discussed in previous sections, any symmetric pure state of an n-qubit system can be associated with a state of a simple system of size $N = n + 1$. Thus, any unitary dynamics of the composite system, which preserves the permutations symmetry, induces a certain dynamics in $\mathcal{H}_N$. To make such a link explicit, consider a permutation symmetric state $\psi \in \mathcal{H}_2^{\otimes n}$ which can be represented in the standard basis, $\{|0\rangle^{\otimes k} \otimes |1\rangle^{\otimes(n-k)}\}_{0\leqslant k\leqslant n}$. Define a a unitary transition matrix T that transforms this basis into an orthonormal basis with first $n + 1$ permutation symmetric vectors and remaining $2^n - (n + 1)$ vectors, which span a basis in the orthogonal subspace. Denote $|\psi'\rangle = T^{-1}|\psi\rangle = \binom{\phi}{0}$, where $|\phi\rangle \in \mathcal{H}_{n+1}$, $|0\rangle \in \mathcal{H}_{2^n-n-1}$ and take any unitary operator U, acting on $\mathcal{H}_2^{\otimes n}$, which preserves the permutational symmetry. We then have

$$|\tilde{\psi}\rangle = U|\psi\rangle,$$
$$T^{-1}|\tilde{\psi}\rangle = T^{-1}UTT^{-1}|\psi\rangle, \quad (24)$$
$$|\tilde{\psi}'\rangle = U'|\psi'\rangle,$$

where $U' = T^{-1}UT$ and $|\tilde{\psi}'\rangle = T^{-1}|\tilde{\psi}\rangle$. As U preserves the permutation symmetry condition, Eq. (25) implies the following block structure of the matrix U':

$$U' = \left(\begin{array}{c|c} V & 0 \\ \hline 0 & W \end{array}\right). \quad (25)$$

Here V and W denote unitary matrices of size $n + 1$ and $2^n - (n + 1)$, respectively. The matrix V acts on the $(n + 1)$–dimensional subspace of permutation symmetric vectors. These vectors are in one-to-one correspondence with vectors

spanning an orthonormal basis in $\mathcal{H}_{n+1}$; see, e.g., Ref. [28]. In other words, the matrix U representing a permutation symmetry preserving dynamics has to be reducible. The unitary block V of the rotated matrix, $U' = T^{-1}UT$, defines, thus, a unitary dynamics of the one qunit system, associated with the n-qubit dynamics U, which preserves the permutational symmetry.

D. A three-qubit example

To illustrate the above reasoning, let us analyze a three-qubit example. We use the natural basis, $(|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle)$, and the unitary transition matrix T reads,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (26)$$

Using formula (20) we construct permutation symmetry preserve operator U, taking $H = \sum_\pi \sigma_1 \otimes \sigma_3 \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $\sum_\pi$ stands for the sum over all permutations. The matrix U obtained in this way has no special structure. However, the transformation T brings it to the matrix $U' = T^{-1}UT$ which enjoys the block structure as in Eq. (25),

$$V = \begin{bmatrix} \frac{1}{4}(1 + 3\cos(4\beta)) & -\frac{1}{2}i\sqrt{3}\sin(4\beta) & 2\sqrt{3}\cos(\beta)^2\sin(\beta)^2 & 0 \\ -\frac{1}{2}i\sqrt{3}\sin(4\beta) & \cos(4\beta) & \frac{1}{2}i\sin(4\beta) & 0 \\ 2\sqrt{3}\cos(\beta)^2\sin(\beta)^2 & \frac{1}{2}i\sin(4\beta) & \frac{1}{4}(3 + \cos(4\beta)) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix V determines, thus, the corresponding dynamics of a simple system of size $n + 1 = 4$, while the remaining part,

$$W = \begin{bmatrix} \cos(\beta) & 0 & -\frac{1}{2}i\sin(\beta) & \frac{1}{2}i\sqrt{3}\sin(\beta) \\ 0 & \cos(\beta) & \frac{1}{2}i\sqrt{3}\sin(\beta) & \frac{1}{2}i\sin(\beta) \\ -\frac{1}{2}i\sin(\beta) & \frac{1}{2}i\sqrt{3}\sin(\beta) & \cos(\beta) & 0 \\ \frac{1}{2}i\sqrt{3}\sin(\beta) & \frac{1}{2}i\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}.$$

is irrelevant for the corresponding dynamics in $\mathcal{H}_{n+1}$.

E. Dynamics for two qubit case

From Eq. (20) we obtain a differential equations for the dynamics,

$$i\partial_\beta|\psi\rangle = H|\psi\rangle, \quad (27)$$

which can be also written in terms of the angles θ_i and the phases Φ_i, $i = 1,2$ characterizing the positions of the

Majorana points. In the general case such equations are not easy to solve. Therefore, we will restrict here our attention to the special case of $n = 2$ qubits.

Let us take $H = -\sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1$. Using Eq. (20), we obtain the operator U which generates dynamics of the MP of state $|\psi_2\rangle = \cos\beta|00\rangle - \sin\beta|11\rangle$. The corresponding dynamics of the stars on the sphere with respect to the time β is shown in Fig. 11. For simplicity, we restrict our analysis to
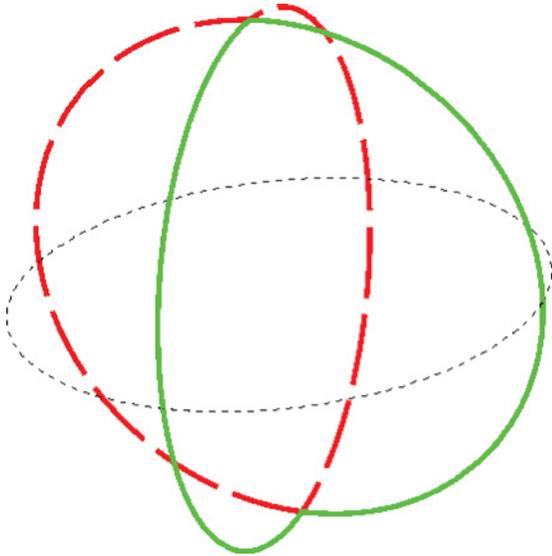
FIG. 11. (Color online) Trajectories of MP for the state $|\psi_2\rangle = \cos\beta|00\rangle - \sin\beta|11\rangle$. Red dashed and green solid lines denote trajectories of two points for $\beta \in [0,\pi]$, and the thin black dashed line represents the equator. Both stars travel from the north to the south pole, where they meet at $\beta = \pi/2$. They later return to the north pole.

$\beta \in [0,\pi/2]$, where $\theta_1 = \theta_2$, $\Phi_1 = \Phi_2 + \pi$ and the phases Φ_1 and Φ_2 do not change. Using Eq. (27), we obtain a differential equation for the polar angle $\theta = \theta_1 = \theta_2$,

$$V = \dot{\theta} = \frac{3 + \cos 3\theta}{2 \sin \theta}. \tag{28}$$

As shown in Fig. 12 the velocity diverges when θ approaches zero or π and both stars are located in a single point. The velocity V reaches its minimum for a Bell state obtained for $\theta = \pi/2$.

It is instructive to compare the velocity of stars during the unitary dynamics of a state parametrized by the the phase β with changes of the barycentric measure E_B; see Fig. 13. Observe an anticorrelation between entanglement measure and V—the maximal velocity occurs for the minimal entanglement and vice versa. This observation implies that product states
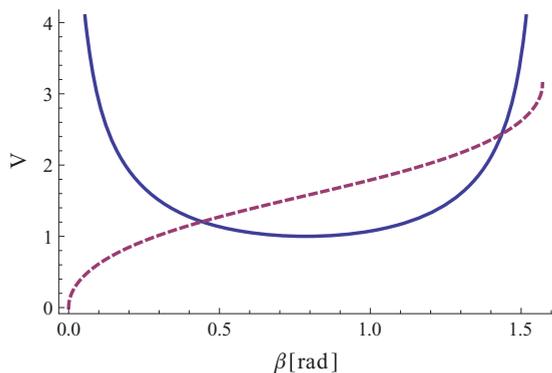
FIG. 12. (Color online) Velocity $V = \partial_\beta\theta$ for the state $|\psi_2\rangle = \cos\beta|00\rangle - \sin\beta|11\rangle$. Full line is V in the function of β, dashed line is $\theta(\beta)$.
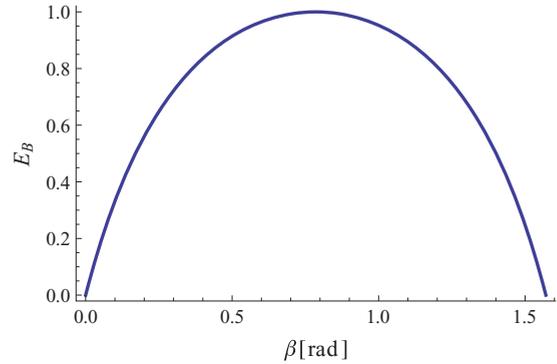
FIG. 13. (Color online) Dependence of the barycentric entanglement measure E_B on the phase β for the state $|\psi_2\rangle = \cos\beta|00\rangle - \sin\beta|11\rangle$.

tend to gain entanglement during a relatively small interaction time.

VI. CONCLUSIONS

In this work we presented a homogeneous approach to study the structure of pure quantum states describing two physical problems: a simple system consisting of $n + 1$ levels and the class of states of an n-qubit system, symmetric with respect to permutations of all subsystems. Making use of the Majorana-Penrose representation, one can find a direct link between these two cases as any constellation of n stars on the sphere determines a quantum state in both setups. In particular, product states of the multiqubit system, for which all Majorana points coalesce in a single point, correspond to spin-coherent states of the simple system. To find a direct relation between two problems one can identify the orthogonal basis of the eigenstates of the angular-momentum operator J_z acting on $\mathcal{H}_{n+1}$ with the set of $(n + 1)$ Dicke states which span the complete basis in the subspace of symmetric states of the composite system.

Physical properties of a given pure state can be, thus, related to the distribution of the corresponding collection of n points on the sphere. The variance of this distribution, related to the radius of the barycenter inside the ball, can be, thus, used to characterize the degree of non spin coherence of the states of a simple system or the degree of entanglement for the composite systems. The proposed barycentric measure of quantum entanglement achieves its maximum for these states, for which the barycenter of the corresponding Majorana points is located at the center of the ball. This class of states includes the Bell state of a two-qubit system, the GHZ state of a three-qubit system, and several states distinguished by being most distant from the set of separable states and are the most nonclassical quantum states [24]. In the case of four-qubit states we have explicitly described a two-parameter class of extremal symmetric states for which the barycentric measure achieves the maximal value, $E_B = 1$. All these states are also highly entangled with respect to the geometric measure E_G, which, for them, belongs to the interval $[1, \log_2 3]$.

It is convenient to define the composition of two permutation symmetric states of an arbitrary number n and m qubits, each of them described by n and m Majorana points, respectively. The composition is characterized by

the collection of $n + m$ stars on the sphere and represents a symmetric state of $m + n$ qubits. Making use of these notions we show that the state $|\text{GHZ}_4\rangle$ can be interpreted as a composition of two two-qubit Bell states and construct a family of maximally entangled multiqubit pure states with $E_B = 1$.

Any unitary dynamics acting on the n-qubit system is described by a matrix U of order 2^n. Assuming that the dynamics does not break the permutation symmetry, the matrix U is reducible and can be written as a direct sum of two unitary matrices, $U = V \oplus W$. The matrix V or order $n + 1$ describes the unitary dynamics in the subspace of symmetric states of the composed system or the corresponding unitary dynamics in the space of all states of the $(n + 1)$-level system. A unitary dynamics of a quantum pure state leads to a

nonlinear dynamics of the corresponding stars of its Majorana representation. In a simple model evolution investigated for a two-qubit system the velocity of stars is small if they are far apart, which corresponds to the highly entangled states, and it increases as the stars get together and the state is close to being separable.

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