Quantifying correlations via the Wigner-Yanase skew information

Shunlong Luo* and Shuangshuang Fu
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, 100190 Beijing, People's Republic of China

Choo Hiap Oh
Center for Quantum Technologies and Physics Department, National University of Singapore, 3 Science Drive 2, 117543, Singapore
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In order to quantify the information content of quantum states in the presence of conserved quantities, Wigner and Yanase introduced the notion of skew information [Proc. Natl. Acad. Sci. USA **49**, 910, (1963)], which was later identified as a paradigmatic version of quantum Fisher information. The skew information is quite different from, yet deeply connected with, the ubiquitous quantum entropies and has important applications in quantum theory. In this paper, pursuing further the original idea of Wigner and Yanase, we propose a measure for correlations in terms of the skew information, investigate its fundamental properties, and elucidate its characteristics. An appealing feature of this measure for correlations is that its evaluation does not involve any optimization, in sharp contrast to the entanglement and discord measures, and can be straightforwardly calculated. In particular, the algorithm and explicit formulas for general bipartite states are prescribed, and simple analytical expressions for some special states, including arbitrary bipartite pure states, the Bell-diagonal states, and some highly symmetric states such as the Werner states and the isotropic states, are obtained.

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I. INTRODUCTION

With the development of quantum information theory, correlations are playing an increasingly fundamental and important role in the study and exploitation of quantum advantage. In particular, it has been widely recognized, during the past few decades, that correlations (including entanglement, and more generally, quantum correlations) are valuable resources for quantum information processing [1–5]. Thus, it is desirable to characterize and quantify correlations in the quantum context from various perspectives. The celebrated Holevo inequality sets an upper bound for the accessible information (i.e., the correlations between measurement outcomes and the original states) in quantum measurements [6,7]. Many measures for entanglement are widely studied, such as the entanglement of formation, the distillable entanglement, the relative entropy of entanglement, etc. [1–5]. On the other hand, several measures for correlations beyond entanglement, such as the quantum discord [8–10], the information deficit [11], the geometric discord [12,13], the measurement-induced nonlocality [14], are also introduced to capture quantum correlations or nonlocality beyond entanglement.

Although all the above measures are introduced from intuitive motivations and reasonable arguments, they are all notoriously difficult to calculate. For example, the analytical expression for the entanglement of formation is only known for very few special states, such as two-qubit states and some other special states, and for the quantum discord, we even do not have an analytical formula for general two-qubit states [10,15–17].

In contrast to the entropic approach to correlations, we will in this paper introduce an alternative measure for correlations in terms of quantum Fisher information, which is motivated by quantum estimation, and has the advantage that it can be straightforwardly evaluated. This measure is based on the important and significant notion of skew information,

$$I(\rho,X) := -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho},X]^2, \qquad (1)$$

introduced by Wigner and Yanase in 1963 [18]. Here ρ is a quantum state (in general mixed) and X is an observable (a self-adjoint operator, which is often a Hamiltonian served as a conserved quantity), and $[\cdot,\cdot]$ denotes commutator. If ρ is pure, then $I(\rho,X)$ reduces to the variance $V(\rho,X) := \mathrm{tr}\rho X^2 - (\mathrm{tr}\rho X)^2$. The skew information constitutes an alternative measure for the information content of a quantum state ρ skew to an observable X [18] and is quite different from, but deeply connected with, the ubiquitous quantum entropies [18–21]. It is also an important concept in information geometry [20,22,23].

There are at least four closely related interpretations of the skew information:

(i) As the information content of ρ with respect to observables *not commuting* with (i.e., skew to) the conserved quantity X. This is the original meaning of the skew information [18].

(ii) As a measure quantifying the noncommutativity between ρ and X. This is plainly clear from the defining Eq. (1) and is exploited by Connes and Stormer in their elegant proof of the homogeneity of states in type III von Neumann algebras [24]. The subtle and intriguing point here is that the square root $\sqrt{\rho}$, rather than ρ, enters the defining Eq. (1), which is reminiscent of the replacement of a probability density by its square root (wave amplitude) in the wave formulation of quantum theory.

(iii) As a version of quantum Fisher information with respect to the time parameter encoded in the evolution of ρ driven by the conserved quantity (Hamiltonian) X [25,26]. This viewpoint shows that the skew information quantifies the accuracy in parameter estimation and is intrinsically connected with quantum estimation and quantum metrology.

(iv) As a measure of quantum uncertainty of X in the state ρ [27,28], which is in some sense dual to the viewpoint of (i). This meaning has interesting applications in refining the

*luosl@amt.ac.cn

Heisenberg uncertainty relations [28–32] and in characterizing the Bell inequalities [33].

The skew information has many advantages over the variance, and is a remarkable informational quantity with an interesting history and nice properties [18,19,34,35]:

(1) The skew information is nonnegative and reduces to the variance if the state is pure. Moreover, $0 \leqslant I(\rho,X) \leqslant V(\rho,X)$.

(2) $I(\rho,X)$ is convex in ρ in the sense that [18,34]

$$I\left(\sum_i \lambda_i \rho_i, X\right) \leqslant \sum_i \lambda_i I(\rho_i,X) \tag{2}$$

for any quantum states ρ_i and any constants λ_i satisfying $\sum_i \lambda_i = 1, 0 \leqslant \lambda_i \leqslant 1$. The convexity of the skew information and, more generally, of the Wigner-Yanase-Dyson information $I_\alpha(\rho,X) := -\frac{1}{2}\mathrm{tr}[\rho^\alpha,X][\rho^{1-\alpha},X]$, $0<\alpha<1$, is a celebrated conjecture first proved by Lieb in 1973 [34]. This convexity in turn plays a crucial role in the first proof of the strong subadditivity of the von Neumann entropy (equivalently, the monotonicity of the quantum relative entropy), which is extremely fundamental in quantum information theory [4,19,36]. In contrast, the variance $V(\rho,X)$ is concave in ρ.

(3) For any bipartite state ρ^{ab} of a composite system $H^a \otimes H^b$ with marginal $\rho^a = \mathrm{tr}_b \rho^{ab}$, and any observable X^a on H^a, it holds that [34]

$$I(\rho^{ab}, X^a \otimes \mathbf{1}^b) \geqslant I(\rho^a, X^a). \tag{3}$$

Here $\mathbf{1}^b$ is the identity operator on H^b.

The skew information $I(\rho,X)$ depends on both the state ρ and the observable X. In order to get an intrinsic quantity capturing the information content of the state ρ without any other involvement, Luo introduced the average [35,37]

$$Q(\rho) := \sum_i I(\rho,X_i) \tag{4}$$

as a measure of information content in the quantum state ρ of an m-dimensional system H. Here $\{X_i\}$ is a family of m^2 observables (self-adjoint operators), which constitutes an orthonormal base for the real Hilbert space $L(H)$ of all observables on H with the Hilbert-Schmidt inner product $\langle A|B\rangle := \mathrm{tr}AB$. It has been established that $Q(\rho)$ is independent of the choice of the orthonormal base $\{X_i\}$ [35] and is actually an intrinsic quantity given by

$$Q(\rho) = \sum_{\mu<\nu}(\sqrt{\lambda_\mu} - \sqrt{\lambda_\nu})^2 = m - \left(\sum_\mu \sqrt{\lambda_\mu}\right)^2, \tag{5}$$

where $\{\lambda_\mu\}$ are the eigenvalues of ρ. Therefore, the quantity $Q(\rho)$ is intimately related to the notions of generalized entropies of Rényi and Tsallis [38–40], which are essentially the trace of some powers of the state.

Following the original motivation and reasoning of Wigner and Yanase [18], we call $Q(\rho)$ the information content of ρ. In this context, the average uncertainty of ρ may be quantified by [27]

$$V(\rho) := \sum_i V(\rho,X_i),$$

which can be evaluated as $V(\rho) = m - \mathrm{tr}\rho^2$, and is essentially the Brukner-Zeilinger invariant information [41,42].

Now consider a bipartite state ρ^{ab} of the composite system $H^a \otimes H^b$ with $\dim H^a = m$ and $\dim H^b = n$. In order to capture the correlations in ρ^{ab} from the quantum estimation perspective, we first introduce the global information content of ρ^{ab} with respect to the local observables of H^a as

$$Q_a(\rho^{ab}) := \sum_i I(\rho^{ab}, X_i \otimes \mathbf{1}^b), \tag{6}$$

which will be proved to be independent of the choice of the orthonormal base $\{X_i\}$ for $L(H^a)$. Then the difference

$$F(\rho^{ab}) := Q_a(\rho^{ab}) - Q_a(\rho^a \otimes \rho^b) \tag{7}$$

between the information content of ρ^{ab} and $\rho^a \otimes \rho^b$ with respect to the local observables of H^a may be interpreted as a measure for correlations in ρ^{ab}.

As will be seen from subsequent Eq. (9), the above measure for correlations can also be equivalently expressed as $F(\rho^{ab}) := Q_a(\rho^{ab}) - Q(\rho^a)$, and from this, we reveal an alternative meaning of $F(\rho^{ab})$: since $Q_a(\rho^{ab})$ synthesizes the skew information of the *global* state ρ^{ab} with respect to the local observables of H^a, and $Q(\rho^a)$ synthesizes the skew information of the *local* state ρ^a with respect to the local observables of H^a, the difference between these two quantities captures the correlations in ρ^{ab} that can be probed by local observables of H^a. Actually, in view of the statistical interpretation of the skew information as quantum Fisher information [25,26], $F(\rho^{ab})$ quantifies the advantage (accuracy gained) for quantum estimation based on ρ^{ab} over that based on ρ^a, when the driven Hamiltonian generating the evolution with time parameter is local.

The present paper is devoted to investigating the correlations measure $F(\rho^{ab})$, illustrating its fundamental properties and implications. We will show that indeed $F(\rho^{ab})$ is a nice measure for correlations, and give its explicit evaluation in general cases (Sec. II). We also treat explicitly several examples in Sec. III and conclude with discussion in Sec. IV.

II. CORRELATIONS IN TERMS OF SKEW INFORMATION

In this section, we investigate the basic properties of $F(\rho^{ab})$ and prescribe its explicit evaluation.

In order to establish that $F(\rho^{ab})$ is an intrinsic and reasonable measure, we must first show that $F(\rho^{ab})$ is well-defined; i.e., it is independent of the choice of orthonormal base $\{X_i\}$ for $L(H^a)$. First, note that

$$\begin{aligned} I(\rho^a \otimes \rho^b, X_i \otimes \mathbf{1}^b) &= -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho^a \otimes \rho^b}, X_i \otimes \mathbf{1}^b]^2 \\ &= -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho^a} \otimes \sqrt{\rho^b}, X_i \otimes \mathbf{1}^b]^2 \\ &= -\tfrac{1}{2}\mathrm{tr}([\sqrt{\rho^a}, X_i]^2 \otimes \rho^b) \\ &= -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho^a}, X_i]^2 \\ &= I(\rho^a, X_i), \end{aligned} \tag{8}$$

we conclude that

$$Q_a(\rho^a \otimes \rho^b) = Q(\rho^a),$$

and consequently, $F(\rho^{ab})$, as defined by Eq. (7), can be equivalently expressed as

$$F(\rho^{ab}) = Q_a(\rho^{ab}) - Q(\rho^a). \tag{9}$$

Since we already know that $Q(\rho^a)$, as defined by Eq. (4), is independent of the orthonormal base [35], in order to establish the invariance of $F(\rho^{ab})$, it suffices to show that $Q_a(\rho^{ab})$, as defined by Eq. (6), is independent of the orthonormal base $\{X_i\}$; i.e., if $\{K_j\}$ is another orthonormal base for $L(H^a)$, then

$$\sum_i I(\rho^{ab},X_i \otimes \mathbf{1}^b) = \sum_j I(\rho^{ab},K_j \otimes \mathbf{1}^b).$$

To prove this, noting that since $\{X_i\}$ and $\{K_j\}$ are both orthonormal bases for $L(H^a)$, we may write

$$K_j = \sum_i a_{ij} X_i, \qquad j = 1,2,\cdots,m^2,$$

with (a_{ij}) an $m^2 \times m^2$ real orthogonal matrix, such that

$$\sum_j a_{ij} a_{i'j} = \delta_{ii'}, \qquad i,i' = 1,2,\cdots,m^2.$$

Consequently,

$$\begin{aligned}
&\sum_j I(\rho^{ab},K_j \otimes \mathbf{1}^b)\\
&= -\frac{1}{2}\sum_j \mathrm{tr}\left[\sqrt{\rho^{ab}},\ \sum_i a_{ij} X_i \otimes \mathbf{1}^b\right]^2\\
&= -\frac{1}{2}\sum_j \sum_{ii'} a_{ij} a_{i'j} \mathrm{tr}[\sqrt{\rho^{ab}},X_i \otimes \mathbf{1}^b][\sqrt{\rho^{ab}},X_{i'} \otimes \mathbf{1}^b]\\
&= -\frac{1}{2}\sum_{ii'} \sum_j a_{ij} a_{i'j} \mathrm{tr}[\sqrt{\rho^{ab}},X_i \otimes \mathbf{1}^b][\sqrt{\rho^{ab}},X_{i'} \otimes \mathbf{1}^b]\\
&= -\frac{1}{2}\sum_i \mathrm{tr}[\sqrt{\rho^{ab}},X_i \otimes \mathbf{1}^b]^2\\
&= \sum_i I(\rho^{ab},X_i \otimes \mathbf{1}^b).
\end{aligned}$$

Therefore, $F(\rho^{ab})$ is well defined.

Theorem 1. The measure $F(\rho^{ab})$ for correlations has the following properties.

(1) $F(\rho^{ab}) = 0$ if and only if ρ^{ab} is a product state (i.e., $\rho^{ab} = \rho^a \otimes \rho^b$).

(2) $F(\rho^{ab})$ is locally unitary invariant in the sense that

$$F(\rho^{ab}) = F((U^a \otimes U^b)\rho^{ab}(U^a \otimes U^b)^\dagger)$$

for any unitary operators U^a and U^b on H^a and H^b, respectively.

(3) $F(\rho^{ab})$ is decreasing in the sense that

$$F(\mathcal{I}^a \otimes \mathcal{E}^b(\rho^{ab})) \leqslant F(\rho^{ab}). \tag{10}$$

Here $\mathcal{I}^a$ is the identity operation on the state space of H^a, and $\mathcal{E}^b$ is an operation on the state space of H^b.

To establish property (1), first note that if $\rho^{ab} = \rho^a \otimes \rho^b$ is a product state, then from Eq. (7), we readily see that $F(\rho^{ab}) = 0$. Conversely, if $F(\rho^{ab}) = 0$, then in view of inequality (3), we have

$$I(\rho^{ab},X \otimes \mathbf{1}^b) = I(\rho^a,X) \tag{11}$$

for *any* observable X on H^a, since we can always, up to a constant normalization, take X as an element of $\{X_i\}$ in the definition of $F(\rho^{ab})$. Further, we can always expand $\sqrt{\rho^{ab}}$ as

$$\sqrt{\rho^{ab}} = \sum_j A_j \otimes Y_j, \tag{12}$$

with $\{A_j\}$ observables on H^a and $\{Y_j\}$ an orthonormal base for $L(H^b)$. Then

$$\rho^{ab} = \sum_{jj'} A_j A_{j'} \otimes Y_j Y_{j'}, \quad \rho^a = \mathrm{tr}_b \rho^{ab} = \sum_j A_j^2.$$

Now let X be any spectral projection of (or any observable commuting with) ρ^a in Eq. (11), we obtain $I(\rho^{ab},X \otimes \mathbf{1}^b) = 0$. From

$$\begin{aligned}
&I(\rho^{ab},X \otimes \mathbf{1}^b)\\
&= -\frac{1}{2}\mathrm{tr}\left[\sum_j A_j \otimes Y_j, X \otimes \mathbf{1}^b\right]^2\\
&= -\frac{1}{2}\mathrm{tr}\left(\sum_j [A_j,X] \otimes Y_j\right)^2\\
&= -\frac{1}{2}\sum_{jj'} \mathrm{tr}([A_j,X][A_{j'},X] \otimes Y_j Y_{j'})\\
&= -\frac{1}{2}\sum_j \mathrm{tr}[A_j,X]^2,
\end{aligned}$$

we conclude that

$$[A_j,X] = 0$$

for any j since each term $-[A_j,X]^2$ is a nonnegative self-adjoint operator. Noting that X could be any spectral projection of ρ^a, we conclude that $\{A_j\}$ is a commuting family of observables and thus can be diagonalized under a common orthonormal base $\{|\mu\rangle\}$ of H^a. Consequently, $\sqrt{\rho^{ab}}$, as defined by Eq. (12), can be expressed as

$$\sqrt{\rho^{ab}} = \sum_\mu \alpha_\mu |\mu\rangle\langle\mu| \otimes \sqrt{\rho_\mu^b},$$

with $\{\rho_\mu^b\}$ a set of states of H^b. This means that

$$\rho^{ab} = \sum_\mu \alpha_\mu^2 |\mu\rangle\langle\mu| \otimes \rho_\mu^b$$

is a classical-quantum state. For such a state, it can be readily evaluated that

$$F(\rho^{ab}) = \sum_{\mu<\nu} 2|\alpha_\mu \alpha_\nu|\big(1 - \mathrm{tr}\sqrt{\rho_\mu^b}\sqrt{\rho_\nu^b}\big).$$

Clearly, this quantity vanishes if and only if $\mathrm{tr}\sqrt{\rho_\mu^b}\sqrt{\rho_\nu^b} = 1$ for any μ,ν, and this can happen if and only if $\rho_\mu^b = \rho_\nu^b$. Therefore, ρ^{ab} is a product state.

Property (2) follows from $Q(U^a \rho^a U^{a\dagger}) = Q(\rho^a)$ and

$$\begin{aligned}
&Q_a((U^a \otimes U^b)\rho^{ab}(U^a \otimes U^b)^\dagger)\\
&= \sum_j I((U^a \otimes U^b)\rho^{ab}(U^a \otimes U^b)^\dagger, X_j \otimes \mathbf{1}^b)\\
&= \sum_j I(\rho^{ab},(U^a \otimes U^b)^\dagger (X_j \otimes \mathbf{1}^b)(U^a \otimes U^b))
\end{aligned}$$

$$= \sum_j I(\rho^{ab},(U^{a\dagger}X_jU^a)\otimes \mathbf{1}^b)$$
$$= Q_a(\rho^{ab}). \tag{13}$$

The last equality follows from the fact that $\{U^{a\dagger}X_iU^a\}$ is still an orthonormal base for $L(H^a)$.

To establish inequality (10), first note that the operation $\mathcal{E}^b$ can always be expressed as

$$\mathcal{E}^b(\rho^b) = \mathrm{tr}_c[U(\rho^b\otimes\rho^c)U^\dagger],$$

where U is a unitary operator on $H^b\otimes H^c$ with H^c an ancillary system and ρ^c a state of H^c. Consequently, for any observable X^a of system H^a, in view of inequality (3) for the skew information under partial trace, we have

$$\begin{aligned}
&I(\mathcal{I}^a\otimes\mathcal{E}^b(\rho^{ab}),X^a\otimes\mathbf{1}^b)\\
&\quad = I(\mathrm{tr}_c[(\mathbf{1}^a\otimes U)(\rho^{ab}\otimes\rho^c)(\mathbf{1}^a\otimes U)^\dagger],X^a\otimes\mathbf{1}^b)\\
&\quad \leqslant I((\mathbf{1}^a\otimes U)(\rho^{ab}\otimes\rho^c)(\mathbf{1}^a\otimes U)^\dagger,X^a\otimes\mathbf{1}^b\otimes\mathbf{1}^c)\\
&\quad = I(\rho^{ab}\otimes\rho^c,(\mathbf{1}^a\otimes U)^\dagger(X^a\otimes\mathbf{1}^b\otimes\mathbf{1}^c)(\mathbf{1}^a\otimes U))\\
&\quad = I(\rho^{ab}\otimes\rho^c,X^a\otimes\mathbf{1}^b\otimes\mathbf{1}^c)\\
&\quad = I(\rho^{ab},X^a\otimes\mathbf{1}^b),
\end{aligned}$$

and the desired result, inequality (10), follows.

In this context, it seems reasonable to make the following conjecture: $F(\rho^{ab})$ is decreasing in the sense that

$$F(\mathcal{E}^a\otimes\mathcal{E}^b(\rho^{ab}))\leqslant F(\rho^{ab}), \tag{14}$$

for any local operations $\mathcal{E}^a$ and $\mathcal{E}^b$ on the state spaces of H^a and H^b, respectively.

This conjecture is true at least when ρ^a is the maximally mixed state and $\mathcal{E}^a$ is a random unitary channel (operation) in the sense that [44,45]

$$\mathcal{E}^a(\rho^a) = \sum_i p_iU_i\rho^aU_i^\dagger, \tag{15}$$

where U_i are unitary operators on H^a, and p_i are constants such that $\sum_i p_i = 1,\ 0\leqslant p_i\leqslant 1$. In order to establish inequality (14) under the above conditions, noting that

$$\mathcal{E}^a\otimes\mathcal{E}^b = (\mathcal{E}^a\otimes\mathcal{I}^b)(\mathcal{I}^a\otimes\mathcal{E}^b)$$

and inequality (10) ($\mathcal{I}^a$ and $\mathcal{I}^b$ are the identity operations on the state spaces of H^a and H^b, respectively), it suffices to prove

$$F(\mathcal{E}^a\otimes\mathcal{I}^b(\rho^{ab}))\leqslant F(\rho^{ab}) \tag{16}$$

when $\mathcal{E}^a$ is of the form of Eq. (15) and $\rho^a = \mathbf{1}^a/m$. But from $\rho^a = \mathbf{1}^a/m$ and $\mathcal{E}^a(\rho^a) = \sum_i p_iU_i\rho^aU_i = \mathbf{1}^a/m$, we have

$$Q(\rho^a)=0,\quad Q(\mathcal{E}^a(\rho^a))=0,$$

which in turn imply that

$$F(\rho^{ab}) = Q_a(\rho^{ab}),\quad F((\mathcal{E}^a\otimes\mathcal{I}^b)\rho^{ab}) = Q_a((\mathcal{E}^a\otimes\mathcal{I}^b)\rho^{ab}).$$

Now by the convexity inequality (2), we have

$$\begin{aligned}
&Q_a((\mathcal{E}^a\otimes\mathcal{I}^b)(\rho^{ab}))\\
&\quad = Q_a\left(\sum_i p_i(U_i\otimes\mathbf{1}^b)\rho^{ab}(U_i\otimes\mathbf{1}^b)^\dagger\right)\\
&\quad \leqslant \sum_i p_iQ_a((U_i\otimes\mathbf{1}^b)\rho^{ab}(U_i\otimes\mathbf{1}^b)^\dagger)\\
&\quad = \sum_i p_iQ_a(\rho^{ab})\quad \text{[by Eq. (13)]}\\
&\quad = Q_a(\rho^{ab}),
\end{aligned}$$

from which inequality (16) follows.

Furthermore, noting that any unital (identity preserving) operation on a qubit system is a random unitary channel [44,45], we conclude that the above conjecture is true when $\dim H^a = 2$ and $\mathcal{E}^a$ is any unital operation.

From Theorem 1, we see that $F(\rho^{ab})$ cannot be regarded as a measure for classical or quantum correlations in the framework of Refs. [8,9,43]. Rather, it is a kind of measure for total correlations, or nonlocality, and resembles in some sense the accessible information.

From the defining Eqs. (5), (6), and (9), we see that $F(\rho^{ab})$ can be calculated rather straightforwardly. More precisely, the evaluation may proceed as follows. Let $\{X_i\}$ and $\{Y_j\}$ be sets of observables that constitute orthonormal bases for $L(H^a)$ and $L(H^b)$, respectively, then $\{X_i\otimes Y_j\}$ constitutes an orthonormal base for $L(H^a\otimes H^b)$. In particular, the local orthonormal base $\{X_i\}$ may be taken in the following canonical way. Let $\{|\mu\rangle\}$ be an orthonormal base for H^a, and put

$$A_{\mu\nu} := \frac{1}{\sqrt{2}}(|\mu\rangle\langle\nu|+|\nu\rangle\langle\mu|),$$
$$B_{\mu\nu} := \frac{i}{\sqrt{2}}(|\mu\rangle\langle\nu|-|\nu\rangle\langle\mu|),$$

then

$$\{|\mu\rangle\langle\mu|\}\cup\{A_{\mu\nu}:\mu<\nu\}\cup\{B_{\mu\nu}:\mu<\nu\} \tag{17}$$

constitutes an orthonormal base for $L(H^a)$, and we may use this base as $\{X_i\}$ for evaluating $F(\rho^{ab})$.

Now, for any bipartite state ρ^{ab} of $H^a\otimes H^b$, its square root $\sqrt{\rho^{ab}}$ can always be expressed as

$$\sqrt{\rho^{ab}} = \sum_{ij} c_{ij}X_i\otimes Y_j, \tag{18}$$

with $c_{ij} := \mathrm{tr}\sqrt{\rho^{ab}}(X_i\otimes Y_j)$.

Theorem 2. For any bipartite state ρ^{ab} with square root expressed as Eq. (18), we have

$$F(\rho^{ab}) = \frac{1}{2}\sum_{ii'}x_{ii'}\sum_j c_{ij}c_{i'j} + \left(\sum_\mu\sqrt{\lambda_\mu}\right)^2 - m. \tag{19}$$

Here $x_{ii'} := \sum_k \mathrm{tr}([X_i,X_k][X_k,X_{i'}])$ which may be interpreted as the structural constants of the base $\{X_i\}$, and $\{\lambda_\mu\}$ are the eigenvalues of $\rho^a = \mathrm{tr}_b\rho^{ab} = \sum_{ii'}\sum_k c_{ik}c_{i'k}X_iX_{i'}$.

To prove this, noting that $\mathrm{tr}Y_jY_{j'} = \delta_{jj'}$ and

$$\begin{aligned}
&I(\rho^{ab},X_k\otimes\mathbf{1}^b)\\
&\quad = -\frac{1}{2}\mathrm{tr}\left[\sum_{ij}c_{ij}X_i\otimes Y_j,X_k\otimes\mathbf{1}^b\right]^2\\
&\quad = -\frac{1}{2}\mathrm{tr}\left(\sum_{ij}c_{ij}[X_i,X_k]\otimes Y_j\right)^2
\end{aligned}$$

$$= -\frac{1}{2}\sum_{iji'j'} c_{ij}c_{i'j'}\mathrm{tr}([X_i,X_k][X_{i'},X_k]\otimes Y_jY_{j'})$$

$$= -\frac{1}{2}\sum_{iji'j'} c_{ij}c_{i'j'}\mathrm{tr}([X_i,X_k][X_{i'},X_k])\mathrm{tr}Y_jY_{j'}$$

$$= \frac{1}{2}\sum_{ii'}\sum_{j} c_{ij}c_{i'j}\mathrm{tr}([X_i,X_k][X_k,X_{i'}]),$$

then Eq. (19) follows by taking the sum with respect to k, and subtracting $Q(\rho^a)$.

Specifying to the two-qubit case, up to local unitary equivalence, we may always expand the square root of any two-qubit state ρ^{ab} as

$$\sqrt{\rho^{ab}} = h\left(\mathbf{1}^{ab} + \vec{a}\cdot\vec{\sigma}\otimes\mathbf{1}^b + \mathbf{1}^a\otimes\vec{b}\cdot\vec{\sigma} + \sum_{j=1}^{3} c_j\sigma_j\otimes\sigma_j\right). \tag{20}$$

Here $\{\sigma_j\}$ are the Pauli matrices, $\vec{a}\cdot\vec{\sigma} := \sum_j a_j\sigma_j$, $\vec{b}\cdot\vec{\sigma} := \sum_j b_j\sigma_j$,

$$h := \frac{1}{2\sqrt{1+\sum_j\left(a_j^2+b_j^2+c_j^2\right)}}$$

is a normalization constant to ensure $\mathrm{tr}\rho^{ab} = 1$.

By direct evaluation, we have

$$Q_a(\rho^{ab}) = \frac{2\sum_j\left(a_j^2+c_j^2\right)}{1+\sum_j\left(a_j^2+b_j^2+c_j^2\right)}.$$

On the other hand,

$$\rho^a = \frac{1}{2}\left[\mathbf{1}^a + \frac{\sum_j(a_j+b_jc_j)\sigma_j}{1+\sum_j\left(a_j^2+b_j^2+c_j^2\right)}\right],$$

and therefore

$$Q(\rho^a) = 1 - \sqrt{1 - \frac{\sum_j(a_j+b_jc_j)^2}{\left[1+\sum_j\left(a_j^2+b_j^2+c_j^2\right)\right]^2}}.$$

Consequently,

$$F(\rho^{ab}) = \frac{2\sum_j\left(a_j^2+c_j^2\right)}{1+\sum_j\left(a_j^2+b_j^2+c_j^2\right)} + \sqrt{1 - \frac{\sum_j(a_j+b_jc_j)^2}{\left[1+\sum_j\left(a_j^2+b_j^2+c_j^2\right)\right]^2}} - 1.$$

Furthermore, if $\vec{a} = \vec{b} = 0$, then the state ρ^{ab} defined via Eq. (20) is actually a Bell-diagonal state:

$$\rho^{ab} = \frac{1}{4}\left[\mathbf{1}^{ab} + \frac{2}{1+\sum_j c_j^2}\sum_j\left(c_j - \frac{c_1c_2c_3}{c_j}\right)\sigma_j\otimes\sigma_j\right],$$

and in this instance, we have

$$F(\rho^{ab}) = \frac{2\sum_j c_j^2}{1+\sum_j c_j^2}. \tag{21}$$

Theorem 3. Let $\mathcal{C}(\rho^a) = \{\rho^{ab} : \mathrm{tr}_b\rho^{ab} = \rho^a\}$ be the set of all bipartite states on a composite system with the fixed marginal ρ^a, then $F(\rho^{ab})$ is convex on this set.

Clearly, $\mathcal{C}(\rho^a)$ is a convex set, and the result follows from the convexity, inequality (2), of the skew information $I(\rho,X)$ with respect to ρ.

In general, $F(\rho^{ab})$ is neither convex nor concave. To see this, note that for any separable state $\rho^{ab} = \sum_i p_i\rho_i^a\otimes\rho_i^b$ which is not a product state, we have $F(\rho^{ab}) > 0$, but $\sum_i p_iF(\rho_i^a\otimes\rho_i^b) = 0$. Consequently, $F(\rho^{ab})$ cannot be convex in general. On the other hand, from Theorem 3 we know that $F(\rho^{ab})$ cannot be concave in general.

III. EXAMPLES

In this section, we illustrate the measure $F(\rho^{ab})$ by several typical examples, and thus show that it indeed captures correlations in a novel way.

Example 1. Let $\rho^{ab} = |\Psi\rangle\langle\Psi|$ be a bipartite pure state of $H^a\otimes H^b$ with the Schmidt decomposition $|\Psi\rangle = \sum_{\mu=1}^{m}\sqrt{\lambda_\mu}|\mu\rangle\otimes|b_\mu\rangle$, then

$$F(\rho^{ab}) = \left(\sum_\mu\sqrt{\lambda_\mu}\right)^2 - \sum_\mu\lambda_\mu^2. \tag{22}$$

Here $m = \dim H^a$.

To derive the above formula, we use the base given by (17) to evaluate $Q_a(\rho^{ab})$:

$$I(\rho^{ab},|\mu\rangle\langle\mu|\otimes\mathbf{1}^b) = \lambda_\mu - \lambda_\mu^2,$$
$$I(\rho^{ab},A_{\mu\nu}\otimes\mathbf{1}^b) = \tfrac{1}{2}(\lambda_\mu+\lambda_\nu),$$
$$I(\rho^{ab},B_{\mu\nu}\otimes\mathbf{1}^b) = \tfrac{1}{2}(\lambda_\mu+\lambda_\nu).$$

Summing up, we have

$$Q_a(\rho^{ab}) = m - \sum_\mu\lambda_\mu^2.$$

On the other hand, from $\rho^a = \sum_\mu\lambda_\mu|\mu\rangle\langle\mu|$ and Eq. (5), we obtain

$$Q(\rho^a) = m - \left(\sum_\mu\sqrt{\lambda_\mu}\right)^2,$$

from which the desired result follows.

Since $Q(\rho^a)\geqslant 0$, and $\sum_{\mu=1}^m\lambda_\mu^2\geqslant\frac{1}{m}$, it follows that

$$F(\rho^{ab})\leqslant m - \frac{1}{m},$$

with the equality if and only if all λ_μ are equal. Thus, $F(\rho^{ab})$ reaches its maximum for maximally entangled pure states.

Example 2. For the Bell-diagonal states with the spectral decomposition

$$\rho^{ab} = \lambda_1|\Psi^+\rangle\langle\Psi^+| + \lambda_2|\Psi^-\rangle\langle\Psi^-| + \lambda_3|\Phi^+\rangle\langle\Phi^+| + \lambda_4|\Phi^-\rangle\langle\Phi^-|,$$

where $|\Psi^\pm\rangle = (|00\rangle\pm|11\rangle)/\sqrt{2}$, $|\Phi^\pm\rangle = (|01\rangle\pm|10\rangle)/\sqrt{2}$, $F(\rho^{ab})$ can be evaluated as

$$F(\rho^{ab}) = 2 - \frac{1}{2}\left(\sum_\mu\sqrt{\lambda_\mu}\right)^2.$$

To see this, from the spectral decomposition of ρ^{ab}, we get

$$\sqrt{\rho^{ab}} = \sqrt{\lambda_1}|\Psi^+\rangle\langle\Psi^+| + \sqrt{\lambda_2}|\Psi^-\rangle\langle\Psi^-| + \sqrt{\lambda_3}|\Phi^+\rangle\langle\Phi^+| + \sqrt{\lambda_4}|\Phi^-\rangle\langle\Phi^-|,$$

By direct calculation, we have

$$Q_a(\rho^{ab}) = 2 - \frac{1}{2}\left(\sum_\mu \sqrt{\lambda_\mu}\right)^2,$$

since $\rho^a = \mathbf{1}^a/2$, we have $Q(\rho^a) = 0$, and the desired result follows. It can be readily checked that the above formula is consistent with that given by Eq. (21).

Example 3. For the $m \times m$ dimensional Werner state

$$\rho^{ab} = \frac{m-x}{m^3-m}\mathbf{1} + \frac{mx-1}{m^3-m}S, \qquad x \in [-1,1]$$

with $S := \sum_{\mu\nu} |\mu\rangle\langle\nu| \otimes |\nu\rangle\langle\mu|$, we have

$$F(\rho^{ab}) = \frac{1}{2}[m - x - \sqrt{(m^2-1)(1-x^2)}].$$

To establish this, noting that

$$\sqrt{\rho^{ab}} = \frac{1}{2}(m_1+m_2)\sum_{\mu\neq\nu} |\mu\rangle\langle\mu| \otimes |\nu\rangle\langle\nu| + m_1 \sum_\mu |\mu\rangle\langle\mu| \otimes |\mu\rangle\langle\mu| + \frac{1}{2}(m_1-m_2)\sum_{\mu\neq\nu} |\mu\rangle\langle\nu| \otimes |\nu\rangle\langle\mu|,$$

with

$$m_1 := \sqrt{\frac{1+x}{m^2+m}}, \quad m_2 := \sqrt{\frac{1-x}{m^2-m}}.$$

By straightforward and tedious calculation, we obtain

$$Q_a(\rho^{ab}) = \frac{1}{2}[m - x - \sqrt{(m^2-1)(1-x^2)}].$$

Since ρ^a is the maximally mixed state, we have $Q(\rho^a) = 0$, and the desired result follows. From this expression, we see that $F(\rho^{ab}) = 0$ if and only if $x = 1/m$, which coincides with the zero point of quantum discord.

Example 4. For the $m \times m$ dimensional isotropic state

$$\rho^{ab} = \frac{1-x}{m^2-1}\mathbf{1} + \frac{m^2x-1}{m^2-1}|\Psi\rangle\langle\Psi|, \qquad x \in [0,1]$$

with $|\Psi\rangle := \frac{1}{\sqrt{m}}\sum_{\mu=1}^m |\mu\mu\rangle$, we have

$$F(\rho^{ab}) = \frac{1}{m}[m^2x - 2x + 1 - 2\sqrt{x(1-x)(m^2-1)}].$$

This can be easily derived by noting that

$$\sqrt{\rho^{ab}} = \sqrt{\frac{1-x}{m^2-1}}\mathbf{1} + \left(\sqrt{x} - \sqrt{\frac{1-x}{m^2-1}}\right)|\Psi\rangle\langle\Psi|$$

and $Q(\rho^a) = 0$. We see that $F(\rho^{ab}) = 0$ if and only if $x = 1/m^2$; in this case, the quantum discord is also zero.

IV. DISCUSSION

Based on skew information, we have introduced a measure for correlations in bipartite states. This measure can be straightforwardly calculated, in sharp contrast to other measures for entanglement and quantum correlations, which involve formidable optimizations and are intractable in general. The informational meaning of this measure is well grounded on the significant and basic notion of skew information, which plays a fundamental role in quantum estimation theory. We may also interpret the correlations measure $F(\rho^{ab})$ as the advantage in quantum estimation based on the global state ρ^{ab} over that based on the local state ρ^a, which in turn is intrinsically due to the correlations in ρ^{ab}. We have worked out explicitly several typical examples and have obtained simple analytical expressions. From these expressions, we see that $F(\rho)$ indeed characterizes the correlations from an informational perspective.

Being based on the skew information, $F(\rho^{ab})$ is quite different from the quantum mutual information, which is based on the von Neumann entropy. While the latter is well established as a measure for correlations from the communication perspective, the former arises naturally from quantum estimation. It would be interesting to investigate their relationships, as well as similar measures based on other quantum Fisher information such as that defined by the symmetric logarithmic derivative [49,50].

It should be noted that $F(\rho^{ab})$ is asymmetric with respect to a and b since we are only taking average on system H^a. It will be desirable to construct some natural symmetric modifications of such a measure for correlations. An interesting candidate might be

$$\begin{aligned} G(\rho^{ab}) &:= \max \sum_{ij}[I(\rho^{ab}, X_i \otimes \mathbf{1}^b + \mathbf{1}^a \otimes Y_j) \\ &\quad - I(\rho^a \otimes \rho^b, X_i \otimes \mathbf{1}^b + \mathbf{1}^a \otimes Y_j)] \\ &= \max \sum_{ij}[I(\rho^{ab}, X_i \otimes \mathbf{1}^b + \mathbf{1}^a \otimes Y_j) \\ &\quad - I(\rho^a, X_i) - I(\rho^b, Y_j)]. \end{aligned}$$

Here the maximum is taken over all local orthonormal bases $\{X_i\}$ and $\{Y_j\}$ of observables for $L(H^a)$ and $L(H^b)$, respectively, and $\rho^a = \mathrm{tr}_b\rho^{ab}$, $\rho^b = \mathrm{tr}_a\rho^{ab}$. Although the superadditivity property,

$$I(\rho^{ab}, X_i \otimes \mathbf{1}^b + \mathbf{1}^a \otimes Y_j) \geqslant I(\rho^a, X_i) + I(\rho^b, Y_j),$$

fails in general [46–48], the weak superadditivity [48]

$$I(\rho^{ab}, X_i \otimes \mathbf{1}^b + \mathbf{1}^a \otimes Y_j) + I(\rho^{ab}, X_i \otimes \mathbf{1}^b - \mathbf{1}^a \otimes Y_j) \geqslant 2[I(\rho^a, X_i) + I(\rho^b, Y_j)]$$

still ensures that $G(\rho^{ab}) \geqslant 0$, since whenever $\{X_i\}$ is an orthonormal base for $L(H^a)$, the sign changes in any elements in $\{X_i\}$ still lead to an orthonormal base. However, due to the involvement of maximization, it seems difficult to evaluate $G(\rho^{ab})$ explicitly.

Although the quantity $F(\rho^{ab})$ inherits intuitive meaning from the skew information, which is a basic and significant quantity, the operational meaning of $F(\rho^{ab})$ is still not clear. It will be desirable to reveal its operational meaning, relate it to certain (communication or estimation) capacity issues, apply it to concrete physical problems, and investigate to what extent it can capture and reveal the physical properties of quantum systems. Finally, it is an interesting issue to consider efficient experimental schemes for determining $F(\rho^{ab})$.

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