Hard and soft excitation regimes of Kerr frequency combs

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We theoretically study the stability conditions and excitation regimes of hyperparametric oscillation and Kerr frequency comb generation in continuously pumped nonlinear optical microresonators possessing an anomalous group velocity dispersion. We show that both hard and soft excitation regimes are possible in the microresonators. Selection between the regimes is achieved via change in the parameters of the pumping light.

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I. INTRODUCTION

The phenomenon of four-wave mixing (FWM) results in hyperparametric oscillation [1,2] and frequency comb generation [3–5] in solid-state optical microresonators. The oscillation is similar to modulation instability lasing [6–9] and parametric oscillation in optical fibers [10–14]. The nonlinear process can be described using externally driven damped nonlinear Schrödinger equation (NLSE) with periodical boundary conditions [15] and, hence, is closely related to the nonlinear phenomena observed in a variety of physical systems [16–19], including dipolar excitations in one-dimensional condensates [20], optical soliton propagation in a dispersive ring cavity in the presence of an input forcing beam [21], a long Josephson junction in a periodic field [22], easy-axis ferromagnets in a rotating magnetic field perpendicular to the easy axis [23,24], and plasmas driven with radio-frequency radiation [26].

Four wave mixing in optical microresonators results in the generation of frequency sidebands in the light leaving the resonator. The sidebands are separated approximately by integer numbers of free spectral range (FSR) of the resonator and are always equidistant ($\tilde{\omega}_+ - \omega = \omega - \tilde{\omega}_-$). This is because the oscillation results from the four-photon process $\hbar\omega + \hbar\omega \rightarrow \hbar\tilde{\omega}_+ + \hbar\tilde{\omega}_-$, where ω, $\tilde{\omega}_+$ and $\tilde{\omega}_-$ are the frequencies of the pump light and the generated sidebands, respectively. Multiple sidebands are generated and the Kerr frequency comb is produced as the power of the pump light is increased [3].

The efficiency of the FWM process depends on the group velocity dispersion (GVD) of the resonator. A resonator with nonzero GVD does not have an equidistant spectrum, since $2\omega_0 - \omega_+ - \omega_- \simeq c\beta_2\omega_{\rm FSR}^2/n_0$, where β_2 is the GVD (by definition, normal dispersion corresponds to $\beta_2 > 0$); n_0 is the refractive index of the material, assumed to be constant in the calculations; c is the speed of light in vacuum; $2\omega_{\rm FSR} \approx \omega_+ - \omega_-$ is the FSR of the resonator; and ω_0, ω_+, and ω_- are the eigenfrequencies of consecutive modes.

In addition to dispersion, the spectrum of the resonator is influenced by self- and cross-phase modulation resulting from the nonlinearity of the material. In the case of anomalous GVD and positive cubic nonlinearity, these effects compensate each other so the spectrum of the resonator becomes locally equidistant for a particular power and wavelength of light within the resonator. The frequency difference between the modes of the pumped resonator influences the FWM and impacts the spectrum and threshold of the hyperparametric oscillation and frequency comb generation.

The Kerr frequency combs excited in optical microresonators are promising for many practical applications since their spectral width can span an octave [27,28], and their frequency stability is extremely high [5]. Understanding the fundamental properties of these combs is necessary to achieve their optimal performance. This is why, in addition to multiple experimental investigations, the resonant FWM and Kerr comb generation have been studied theoretically. For example, it was shown that additive modulational instability ring lasers can operate in both normal and anomalous GVD regimes [7]. This result was adjusted to describe the nonlinear processes in microresonators [29–31]. Theories supporting the idea of preferred generation of the optical Kerr comb in resonators possessing anomalous GVD were developed [32,33]. An analytical expression describing the optical pulses generated in resonators with anomalous GVD was derived [15]. Finally, generation of Kerr frequency combs was investigated via numerical simulations [34,35].

The aim of this article is to analyze the dependence of the excitation dynamics of the hyperparametric oscillation and Kerr frequency combs on the power and frequency of the external continuous wave pump. The dynamics of Kerr comb formation was studied previously [35] with approximation of low amplitudes of the comb frequency harmonics. This assumption allowed modeling the soft regime of the comb excitation. In this work we solve the problem without such an approximation, analyze the oscillation onset in the microresonators, and show that both hard and soft excitation of the oscillation is possible.

The soft oscillation onset is reached when pump photons are not initially present in the resonator. Here, the growth of the oscillation sidebands occurs adiabatically. The hard onset of the oscillation occurs with a discontinuous jump of the intensity of oscillation sidebands to a certain finite level, at the threshold. The hard excitation occurs only when pump photons are initially present in the resonator. The steady-state solution corresponding to the hard excitation cannot be reached adiabatically. In other words, slow modifications of any parameter of the resonator or the pump cannot bring the system to a stable solution requiring hard excitation. Here slow means a time shorter than the time of the comb growth, which can be much longer compared to the lifetime of the light confined in the resonator [35].

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We solve the set of nonlinear differential equations describing the hyperparametric oscillation [29] and frequency comb generation [35] in the steady state and analyze the stability of the solutions in a microresonator possessing a net anomalous GVD. While the first-order hyperparametric oscillation (only the continuous-wave optical pump and the first pair of sidebands are involved in the process) is described analytically, the higher order oscillations are simulated numerically.

We find that oscillation sidebands can be much smaller than the pump in the case of soft excitation, while they are comparable with the pump in the case of hard excitation. The sidebands generated in the case of hard excitation produce optical pulses that travel in the resonator, while the sidebands generated in the case of soft excitation are too small to form a pulse. We also find that hard excitation is observed when the frequency of the pumping light differs significantly from the frequency of the pumped mode, while soft excitation is observed for the nearly resonant pumping. We compare the results of our numerical simulations with the predictions of the analytical model developed for the comb description [15] and find that the simulation predicts the formation of wider frequency combs and shorter optical pulses relative to the values found analytically. The numerical simulations also show that the comb spectral width grows more slowly than the linearized simulations predict [36]. We show that the frequency comb harmonics form nearly Gaussian optical pulses in the time domain.

II. THREE-MODE MODEL OF HYPERPARAMETRIC OSCILLATION

Let us consider the FWM process that involves only three resonant modes. The evolution of the mode amplitudes is described by the equations [29]

$$\dot{A} + \Gamma_0 A = ig[|A|^2 + 2|B_+|^2 + 2|B_-|^2]A + 2igA^* B_+ B_- + F_0, \quad (1)$$

$$\dot{B}_+ + \Gamma_+ B_+ = ig[2|A|^2 + |B_+|^2 + 2|B_-|^2]B_+ + igB_-^* A^2, \quad (2)$$

$$\dot{B}_- + \Gamma_- B_- = ig[2|A|^2 + 2|B_+|^2 + |B_-|^2]B_- + igB_+^* A^2, \quad (3)$$

where $\Gamma_0 = i(\omega_0 - \omega) + \gamma_0$ and $\Gamma_\pm = i(\omega_\pm - \tilde{\omega}_\pm) + \gamma_\pm$; $g = \hbar\omega_0^2 c n_2/(\mathcal{V} n_0^2)$ is the coupling parameter [29]; $\mathcal{V}$ is the mode volume; n_2 is the nonlinearity; $F_0 = [2\gamma_0 P/(\hbar\omega_0)]^{1/2}$ describes the amplitude of the continuous-wave external pump; P is the pump power; and A, B_+, and B_- are the slow amplitudes of the pump and sidebands, respectively. Decay rates γ_0, γ_+, and γ_- reflect both coupling and intrinsic losses of the modes. We assume that the modes are overloaded; i.e., the loss results primarily from the coupling.

To keep a connection between our mathematical models and the real physical system, we consider oscillations in a magnesium fluoride whispering gallery mode resonator of 340-μm radius (~100-GHz FSR), similar to the resonator studied in [37]. The fundamental TE mode of the resonator is pumped with 1721 nm light. The modes of the resonator have a 200-kHz loaded full width at half-maximum, and the GVD of the modes results in the condition $2\omega_0 - \omega_+ - \omega_- = -\gamma_0$. The cubic nonlinearity is $n_2 = 10^{-16}$ cm^2/W, and the refractive index is $n_0 = 1.38$. The mode volume is $\mathcal{V} = 1.3 \times 10^{-7}$ cm^3, which corresponds to the coupling constant $g = 1.47 \times 10^{-3}$ s^{-1}. By selecting the pump power $P = 0.235$ mW, we obtain $(F_0/\gamma_0)(g/\gamma_0)^{1/2} = 4$ for the normalized pumping constant. It is worth noting that the selected value of the pump power is 16 times larger than the threshold value needed for the hyperparametric oscillation to start.

We solve the set (1)–(3) in the steady state; consider a symmetric case, i.e., put $\gamma_+ = \gamma_- = \gamma_0$, so that $|B_+| = |B_-| = |B|$; and present the slow amplitudes of the fields as $A = |A|(1 + \delta\mathcal{A})\exp[i(\phi_0 + \delta\phi_0)]$ and $B_\pm = |A|(\mathcal{B} + \delta\mathcal{B}_\pm)\exp[i(\phi_\pm + \delta\phi_\pm)]$, where $\delta\mathcal{A}$, $\delta\mathcal{B}_+$, and $\delta\mathcal{B}_-$ stand for the amplitude deviations of the fields related to the drive amplitude, and $\delta\phi_0$, $\delta\phi_+$, and $\delta\phi_-$ stand for the phase deviations of the fields. We substitute these expressions into (1)–(3), linearize the set of equations in the vicinity of the steady-state solution, and study its eigenvalues. The steady-state solution is considered to be stable if all the eigenvalues are negative.

The results of calculations are shown in Figs. 1 and 2. We found that there are two regions where the stable hyperparametric oscillation exists. One solution occurs in the vicinity of the peak of the resonant curve, shifted due to the self-phase modulation effect. The oscillation belonging to this branch can be excited adiabatically if one slowly reduces the frequency of the pump laser approaching the optical resonance. It also can be excited if one fixes the laser frequency at the red wing of the resonance and then reduces the power of the pump (Fig. 2). The oscillation sidebands are much smaller than the pump amplitude for this solution. The numerical solution of (1)–(3) shows that the oscillation is excited when all the initial conditions are 0. Therefore, we conclude that the stability branch corresponds to the case of soft excitation.

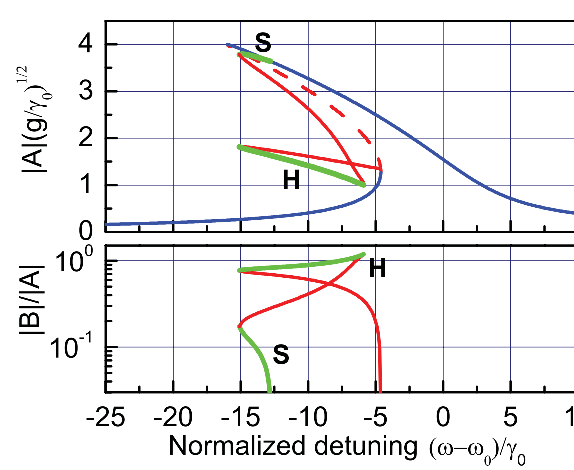

FIG. 1. (Color online) Normalized amplitude of the light within the optically pumped mode and relative amplitude of the oscillation sidebands vs pumping frequency. Stability regions corresponding to soft (S) and hard (H) excitation are shown by solid green lines. Unstable solutions are depicted by solid red lines. The amplitude of the field within the optically pumped mode with no sidebands generated ($B \equiv 0$) is shown by the solid blue line and dashed (red) line. The dashed line represents the unstable solution for the amplitude of the pumped mode when the modulation sidebands are absent.

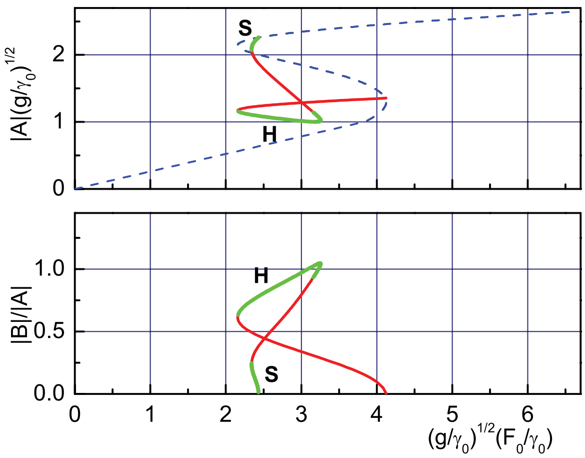

FIG. 2. (Color online) Normalized amplitude of the light within the optically pumped mode and the relative amplitude of the oscillation sidebands vs normalized amplitude of the pumping light. The detuning of the pumping light from the corresponding optical mode is fixed at $\omega = \omega_0 - 4.7\gamma_0$ (the mode is pumped at its red wing). The dashed (blue) line represents the undisturbed solution for the light accumulated in the pumped mode, $B \equiv 0$. Red and green solid lines describe unstable and stable-steady state solutions respectively.

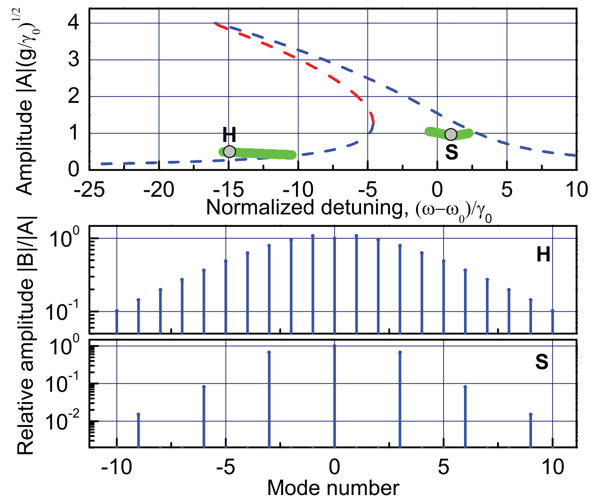

FIG. 3. (Color online) Normalized amplitude of the optically pumped mode and relative amplitude of the oscillation sidebands vs pumping frequency found for the same conditions as used in Fig. 1. Frequency combs shown were generated when the parameters of the system corresponding to points H and S were selected.

The other stable solution exists when the oscillation sidebands have fixed nonzero power. The stability region is localized in the space of parameters and does not cross the region of stable solution for $B \equiv 0$. Only nonadiabatic change of the parameters of the system allows reaching the localized stability region. A solution of the initial value problem, (1)–(3), in the approximation of zero initial conditions does not reveal the stability region. Hence, this stable branch describes the oscillation with hard excitation. The localized stable attractor, corresponding to the hard excitation regime, cannot be discovered if the original set of equations describing the oscillation is solved in the approximation of small sidebands.

The analysis that involves only two generated optical sidebands is not entirely valid for the description of hyperparametric oscillations in realistic resonators, since a nonlinear interaction of multiple modes should be taken into account. Only the soft excitation regime that produces small oscillation sidebands can be simulated using a limited number of interacting modes, as the higher order oscillation sidebands are expected to have an even lower power. The hard excitation regime, in which the oscillation sidebands can be more powerful than the light confined in the pumping mode, cannot be strictly understood from the three-mode model.

III. MULTI-MODE MODEL OF HYPERPARAMETRIC OSCILLATION

To handle this problem we have analyzed the multimode regime numerically, deriving a set of equations, similar to (1)–(3), for many interacting modes [35,38] and solving this set in the steady state. Generation of optical frequency combs with fewer than 21 mutually interacting optical modes can be described with our computing capability. To take the GVD into account we assumed that, for any pair of sideband modes symmetric with respect to the optically pumped mode ($\omega_0 - \omega_- \approx \omega_+ - \omega_0$), the mode unequidistance is defined by $2\omega_0 - \omega_+ - \omega_- = c\beta_2(\omega_+ - \omega_-)^2/4n_0$. The numerical solution revealed excitation regimes similar to those found for the case of two optical sidebands (see Figs. 3 and 4). A broad frequency comb is generated in the case of the hard excitation regime. The comb harmonics have spacings equal to a single FSR of the resonator. The frequency comb resulting from soft excitation has harmonics separated by three FSRs. There is no single-FSR comb characterized with soft excitation for the selected pump power. It is possible to generate the frequency comb with harmonics separated by double or single FSR if a lower optical power is selected.

There is a difference between the three-mode hyperparametric oscillation and the frequency comb generation. Soft excitation of the comb occurs in the region of parameters where

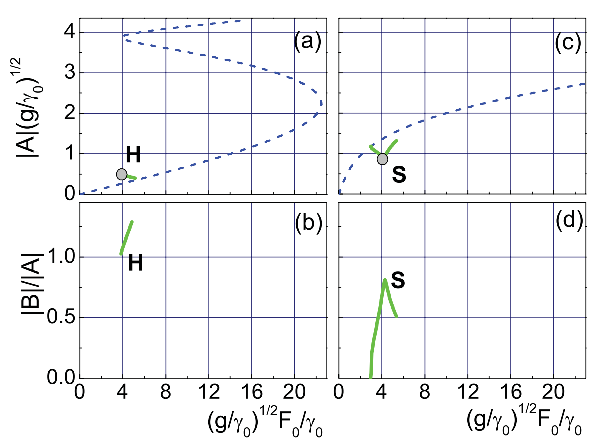

FIG. 4. (Color online) Normalized amplitude of the light within the optically pumped mode and relative amplitude of the oscillation sidebands vs normalized amplitude of the pumping light. The detuning is (a, b) $\omega = \omega_0 - 15\gamma_0$ (hard excitation) and (c, d) $\omega = \omega_0 + 0.92\gamma_0$ (soft excitation). The power for the first sideband is taken from the carrier.

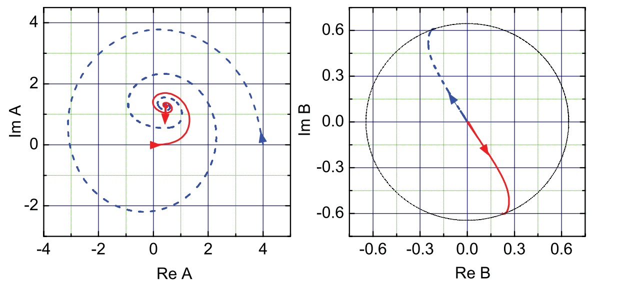

FIG. 5. (Color online) Phase diagram for the pump and the first oscillation sideband in the case of soft excitation. The dashed (blue) line corresponds to nonzero initial conditions, and the solid (red) line to zero initial conditions. The system converges to the same steady-state solution with different phases of the oscillation sideband.

the pumped mode is dynamically stable, while hard excitation can be achieved in the multistability region of the pumped mode. The hyperparametric oscillation, with both hard and soft excitation, occurs only in the region of multistability for the pumped mode.

To demonstrate the hard and soft excitation regimes of the oscillation we studied the dynamical behavior of the light in the optical modes for zero and nonzero initial conditions (see Figs. 5 and 6). The outcome of the calculation (the steady-state solution) is the same for the case of soft excitation (Fig. 5). The oscillation characterized with the hard excitation regime occurs only for nonzero initial conditions (Fig. 6), for both abrupt and adiabatic switching of the pump.

Since we can find the phase and the amplitude of the oscillation sidebands, we are able to calculate the temporal behavior of the field within the resonator. We do this for a broad-frequency comb and compare the result with predictions of the analytical method described in [15] (Fig. 7). As expected, the generated harmonics create optical pulses traveling within the resonator. The numerical modeling and

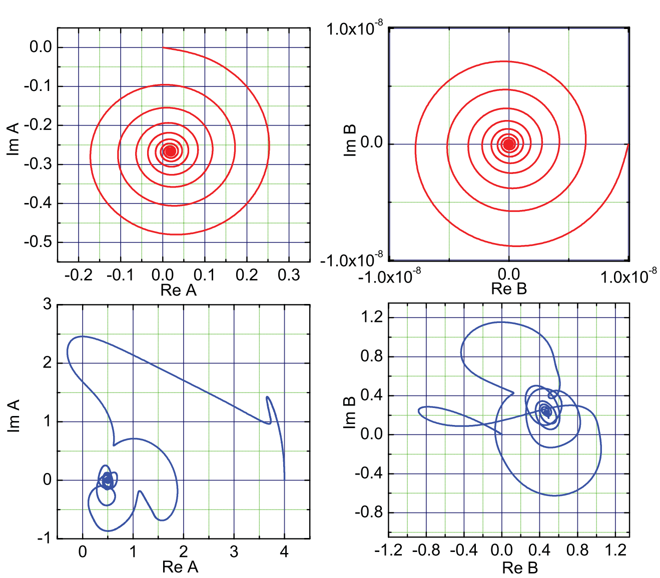

FIG. 6. (Color online) Phase diagram for the pump and the first oscillation sideband in the case of hard excitation. The system does not oscillate if the initial field in the pumped mode is 0 (red line). The oscillation is excited if the initial value of the pump is not zero.

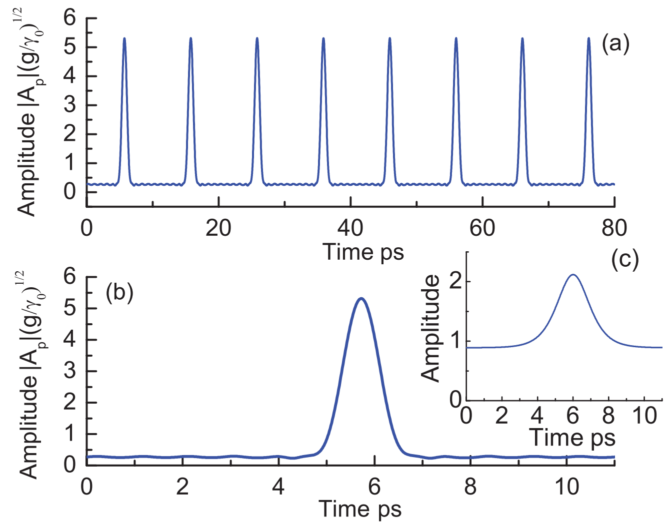

FIG. 7. (Color online) Temporal behavior of the normalized optical field within the resonator. Subpicosecond optical pulses are formed. Inset: Optical pulse envelope found from the analytical expression presented in [15].

the analytical theory predict slightly different behaviors for the system. The reason is that the analytical model does not take into account the finite length of the path of the pulse in the resonator and requires that the pulse should be generated at the specific detuning $\omega = \omega_0 - 5.3\gamma_0$. The numerical solution is unstable at this point and, instead, is stable in a neighboring region of detuning values (Fig. 3).

The solutions analyzed here have a certain correspondence with the solutions of the driven damped NLSE with periodic boundary conditions. The soft excitation regime corresponds to the localized stability windows in the unstable region of parameters of the NLSE [16]. The stability windows occur due to the presence of the boundary conditions [17], while the hard excitation regime is related to the stable mode-locked regime of the driven damped NLSE on an infinite line [17]. Further mathematical analysis is required to map the whole region of parameters and to find both local and global attractors for the equation.

IV. CONCLUSION

We have developed an analytical model and numerical simulations to study the influence of parameters of the pump light on the dynamics of Kerr frequency comb generation. We have found that hyperparametric oscillation and optical frequency comb generation in optical nonlinear resonators can have both hard and soft excitation regimes. We present several examples of such behavior and show that the hard excitation regime leads to the formation of short optical pulses in the resonator.

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