Losses in rotating degenerate cavities and a coupled-resonator optical-waveguide rotation sensor

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We present a rigorous theory of rotating degenerate optical cavities and a coupled-resonator optical waveguide (CROW) including the impact of material losses associated with practical cavities and resonators. The losses are modeled as a perturbation of the material's relative permittivity by adding to it a small imaginary component. The Sagnac frequency shift in a single lossy cavity is shown to be lower than that of a lossless one. For CROWs, the rotation-induced gap formed in the center of the transmission function of a lossy device is reduced compared to that of a lossless one. The inclusion of propagation losses in the analysis of the CROW reveals a relatively insensitive region (a dead zone) in the response of a finite device at low rotation rates. A periodic modulation of the resonators' resonant frequencies is shown to be an effective artificial CROW biasing technique to overcome this problem. This biasing does not require any active control.

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I. INTRODUCTION

An electromagnetic wave propagating along a circular path in a mechanically rotating medium accumulates an additional phase shift proportional to the rotation rate of the medium and the loop area [1]. This phenomenon is known as the Sagnac effect [2]. Recently, much interest was shown in studying the effect in resonant microcavities [3,4] and in photonic structures supporting mode degeneracy such as ring and coupled microring resonators [5–7]. It was shown that rotation makes the resonant frequency split into M different frequencies, where M is the order of the stationary system mode degeneracy. This result is general and holds for any type of cavity supporting mode degeneracy, be it a photonic crystal cavity, disk resonator, or others. In another publication the effect of rotation was studied in a coupled-resonator optical-waveguide (CROW) rotation sensor consisting of doubly degenerate resonators such as ring resonators [8]. Both of those works did not account for losses existing in practical devices, such as material and bending losses. Investigating the influence of loss on the rotation-induced resonance splitting is of particular scientific interest. Moreover, there are few publications studying the effect of loss on the performance of the miniature rotation sensors [9,10]. However, these studies employed a simple model for losses assuming it is not affected by the rotation of the device. In this paper, we present a rigorous, *ab initio*, study of the impact of losses on the Sagnac effect in degenerate cavities and the performance of CROW rotation sensor.

The structure of this paper is as follows. In Sec. II we study the resonance splitting in a lossy rotating M-degenerate cavity, as seen in the cavity's rest frame, where the losses are introduced as a perturbation of the relative permittivity of the medium. In Sec. III, we apply the theoretical results to the common particular case of a second-order degenerate lossy ring resonator. In Sec. IV we theoretically study a lossy CROW subjected to rotation *and* lossy conditions using the tight-binding theory. We show two major impacts of loss on the device characteristics. First, a new dispersion relation is found and compared with that of a lossless CROW. Second, we show the response of the CROW to rotation, where formation of a dead zone at low rotation rates is observed and a possible solution is proposed. Concluding remarks are provided in Sec. V.

II. LOSSY ROTATING CAVITY WITH DEGENERATE MODES

Let $\varepsilon_c(\boldsymbol{r}) = \varepsilon_r(\boldsymbol{r}) + i\varepsilon_i(\boldsymbol{r})$ be the complex dielectric permittivity of a stationary medium. The imaginary part of the permittivity represents the losses associated with the medium. We assume a low-loss material with $\varepsilon_i \ll \varepsilon_r$. We also assume that the medium rotates slowly with the angular velocity Ω:

$$\boldsymbol{\Omega} = \hat{z}\Omega. \tag{2.1}$$

The assumption of slow rotation velocity implies that neither relativistic effects nor geometrical transformations take place. Therefore, operators such as ∇ are conserved in the rotating rest frame of reference: $\nabla = \nabla'$, and time is invariant in both systems: $t = t'$. According to a formal structure of electrodynamics, the basic physical laws governing the electromagnetic fields are invariant under any coordinate transformations, including a noninertial one [11,12]. The transformation to a rotating system is manifested only by an appropriate change of the constitutive relations. Therefore, under the slow rotation assumption, Maxwell equations in the rotating frame R are given by

$$\begin{aligned} \nabla \times \boldsymbol{E} &= i\omega B, \quad \nabla \cdot B = 0 \\ \nabla \times H &= -i\omega D, \quad \nabla \cdot D = 0 \end{aligned} \tag{2.2}$$

Assuming the material properties at rest are given by ε, μ, then up to the first order in velocity the constitutive relations in R take on the form [11]

$$\begin{aligned} D &= \varepsilon E - c^{-2}\boldsymbol{\Omega} \times r \times H, \\ B &= \mu H + c^{-2}\boldsymbol{\Omega} \times r \times E. \end{aligned} \tag{2.3}$$

In the above, c is the speed of light in vacuum, ω is the frequency, and a time dependence $\exp(-i\omega t)$ is assumed

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and suppressed. The wave equation, up to first order in Ω, describing the magnetic field in a lossy medium can be derived following the procedure outlined in the previous work [13], and it is given by

$$\Theta_c H_\Omega(r) = k^2 H_\Omega(r) + ikL_\Omega H_\Omega(r), \tag{2.4}$$

where Θ_c is the wave operator,

$$\Theta_c \equiv \nabla \times \frac{1}{\varepsilon_c(r)} \nabla \times \cdots, \tag{2.4a}$$

L_Ω is the rotation-induced operator,

$$L_\Omega H_\Omega(r) = \nabla \times \frac{\boldsymbol{\beta}}{\varepsilon_c(r)} \times H_\Omega(r) + \frac{\boldsymbol{\beta}}{\varepsilon_c(r)} \times \nabla \times H_\Omega(r), \tag{2.4b}$$

and $k = \omega/c$, $\boldsymbol{\beta} = c^{-1}\boldsymbol{\Omega} \times r$. In the development of Eqs. (2.4) and (2.4b), only terms up to the first order in $\boldsymbol{\beta}$ are kept.

We assume a dielectric structure with a lossless cavity, which resonates at a resonant frequency ω_0 and supports M-order mode degeneracy. At rest, the magnetic field in the cavity is governed by the wave equation

$$\Theta_r H_0^{(m)}(r) = k_0^2 H_0^{(m)}(r)\ k_0 = \frac{\omega_0}{c}, \quad m = 1,2,\ldots,M, \tag{2.5}$$

where $H_0^{(m)}(r)$ is the mth degenerate mode, and the operator Θ_r is defined as

$$\Theta_r \equiv \nabla \times \frac{1}{\varepsilon_r(r)} \nabla \times \cdots. \tag{2.5a}$$

Our goal now is to express the resonant frequency and resonant field of a lossy cavity under rotation (ω, $\boldsymbol{H}_\Omega$), governed by Eq. (2.4), in terms of the resonant frequency and modes of the lossless system at rest (ω_0, $H_0^{(m)}$). To that end we define the inner product between two vector fields as the volume integration:

$$\langle F,G\rangle \equiv \int F \cdot \bar{G} d^3 r, \tag{2.6}$$

where the overbar denotes the complex conjugate, and the dot product is the standard scalar product between the two vectors $\boldsymbol{F}$ and $\boldsymbol{G}$. Performing an inner product of Eq. (2.5) with $\boldsymbol{H}_\Omega$ and of Eq. (2.4) with each of the degenerate modes $H_0^{(m)}$ we get the following set of equations:

$$\langle \Theta_r H_0^{(m)}, H_\Omega\rangle = k_0^2 \langle H_0^{(m)}, H_\Omega\rangle, \tag{2.7a}$$

$$\langle \Theta_c H_\Omega, H_0^{(m)}\rangle = k^2 \langle H_\Omega, H_0^{(m)}\rangle + ik\langle L_\Omega H_\Omega, H_0^{(m)}\rangle, \tag{2.7b}$$

which hold for $m = 1,2,\ldots,M$. By subtracting from Eq. (2.7b) the complex conjugate of Eq. (2.7a), and using the fact that Θ_r is a self-adjoint operator, we obtain

$$\begin{aligned}(k^2 - k_0^2)\langle H_\Omega, H_0^{(m)}\rangle &+ ik\langle L_\Omega H_\Omega, H_0^{(m)}\rangle \\ &= \langle \Theta_c H_\Omega, H_0^{(m)}\rangle - \langle \Theta_r H_\Omega, H_0^{(m)}\rangle.\end{aligned} \tag{2.8}$$

Since slow rotation may only affect the phase accumulation rate and the resonant frequency, but its effect on the mode shape is completely negligible [14], we express $\boldsymbol{H}_\Omega$ as a linear sum of the stationary modes:

$$H_\Omega(r) = \sum_{n=1}^{M} a_n H_0^{(m)}(r), \tag{2.9}$$

where $n = 1,2,\ldots,M$ and a_n are unknown coefficients. Substituting Eq. (2.9) back into Eq. (2.8) results in the following matrix equation:

$$\begin{aligned}(\omega^2 - \omega_0^2)\sum_{n=1}^{M} a_n A_{mn} &- \omega\omega_0\Omega \sum_{n=1}^{M} a_n \tilde{B}_{mn} \\ &= -\omega_0^2 \sum_{n=1}^{M} a_n \tilde{C}_{mn}\ m = 1,2,\ldots,M,\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}A_{mn} &= \langle H_0^{(n)}, H_0^{(m)}\rangle, \\ \tilde{B}_{mn} &= \varepsilon_0 \langle \hat{z} \times r, e^{i\frac{\varepsilon_i}{\varepsilon_r}}(\bar{H}_0^{(n)} \times E_0^{(m)} + H_0^{(m)} \times \bar{E}_0^{(n)})\rangle, \\ \tilde{C}_{mn} &= ic^2\varepsilon_0^2 \left\langle \frac{\varepsilon_i}{\varepsilon_c} E_0^{(n)}, \varepsilon_r E_0^{(m)}\right\rangle.\end{aligned} \tag{2.10a}$$

In the derivation of the last equation we used $\varepsilon_c \cong |\varepsilon_c| e^{i(\varepsilon_i/\varepsilon_r)}$ and the identity $\nabla \times H_0^{(m)} = -i\omega_0\varepsilon_0\varepsilon_r E_0^{(m)}$ was employed. Equation (2.10) is an eigenvalue problem, where the eigenvector consists of the unknowns a_n with the eigenvalue depending on the operation frequency ω which has been shifted from ω_0 as a result of both rotation and loss. Notice that for the lossless case ($\varepsilon_i = 0$) Eq. (2.10) reduces to the result given in Ref. [7]. In order to simplify Eq. (2.10) and to attain more intuitive understanding, we assume that (a) the modes are highly confined within the cavity; (b) the relative permittivities are constant inside the cavity. Therefore, Eqs. (2.10) and (2.10) can be simplified further as

$$\begin{aligned}\left[\omega^2 - \omega_0^2\left(1 - i\frac{\varepsilon_i}{\varepsilon_c}\right)\right]\sum_{n=1}^{M} a_n A_{mn} \\ = \omega\omega_0\Omega\left(1 - i\frac{\varepsilon_i}{\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right)\sum_{n=1}^{M} a_n B_{mn} \\ m = 1,2,\ldots,M,\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}A_{mn} &= \langle \boldsymbol{H}_0^{(n)}, \boldsymbol{H}_0^{(m)}\rangle, \\ B_{mn} &= \varepsilon_0\langle \hat{z} \times \boldsymbol{r}, (\bar{\boldsymbol{H}}_0^{(n)} \times \boldsymbol{E}_0^{(m)} + \boldsymbol{H}_0^{(m)} \times \bar{\boldsymbol{E}}_0^{(n)})\rangle, \\ \tilde{C}_{mn} &\cong i\frac{\varepsilon_i}{\varepsilon_c}\langle \boldsymbol{H}_0^{(n)}, \boldsymbol{H}_0^{(m)}\rangle = i\frac{\varepsilon_i}{\varepsilon_c} A_{mn}.\end{aligned} \tag{2.11a}$$

In the transition from Eq. (2.10) to Eq. (2.11) the matrix elements $\tilde{C}_{mn}$ were expressed by the elements A_{mn} as shown in Eq. (2.11a). The last equation can be rewritten as the eigenvalue problem with $\mathbf{A}$ and $\mathbf{B}$ being square $M \times M$ matrices with elements A_{mn} and B_{mn}, respectively:

$$\Omega\mathbf{A}^{-1}\mathbf{B}\mathbf{a} = \frac{\left[\omega^2 - \omega_0^2\left(1 - i\frac{\varepsilon_i}{\varepsilon_c}\right)\right]}{\omega\omega_0\left(1 - i\frac{\varepsilon_i}{\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right)}\mathbf{a}, \tag{2.12}$$

where $\mathbf{a}$ is a column vector with the elements a_n. We rewrite Eq. (2.12) as

$$\mathbf{Ca} = \frac{\frac{1}{2}\left[\omega^2 - \omega_0^2\left(1 - i\frac{\varepsilon_i}{\varepsilon_c}\right)\right]}{\Omega\omega\omega_0\left(1 - i\frac{\varepsilon_i}{\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right)}\mathbf{a} = \Lambda\mathbf{a}, \quad \mathbf{C} = \frac{1}{2}\mathbf{A}^{-1}\mathbf{B}. \tag{2.12a}$$

Generally, the matrix $\mathbf{C}$ possesses M distinct eigenvalues Λ_j, $j = 1,2,\ldots,M$. Note that Eq. (2.12a) is identical to the eigenvalue problem of a lossless degenerate cavity [7], meaning the eigenvectors are identical and the eigenvalues must be real numbers. Therefore, the expansion in Eq. (2.9) is identical for both lossy and lossless cavities, but the resonant frequencies in each cavity's wave equation are different. Expressing the resonant frequency shift as

$$\delta\omega = \omega - \omega_0, \tag{2.13}$$

and substituting it back into Eq. (2.12a) we can approximately express it in terms of the eigenvalues Λ_j as

$$\delta\omega_j\,(\Omega,\varepsilon_i) = \omega_0\left[\Lambda_j\Omega\left(1 - i\frac{\varepsilon_i}{2\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right) - i\frac{\varepsilon_i}{2\varepsilon_r} - \frac{3\varepsilon_i^2}{8\varepsilon_r^2}\right]. \tag{2.13a}$$

The last expression should be interpreted with caution, since the last two terms in the square brackets are not rotation related and they contribute to the change of the resonant frequency in a stationary cavity as a result of loss only. On the other hand, the other three terms in the inner brackets are due to rotation and are modified by loss. Therefore, we can divide the different contributions as follows:

$$\begin{aligned}
\delta\omega_j(\varepsilon_i)_{\text{stat,re}} &= \delta\omega(\varepsilon_i)_{\text{stat,re}} = -\frac{3\varepsilon_i^2}{8\varepsilon_r^2}\omega_0\forall j,\\
\delta\omega_j(\varepsilon_i)_{\text{stat,im}} &= \delta\omega(\varepsilon_i)_{\text{stat,im}} = -i\frac{\varepsilon_i}{2\varepsilon_r}\omega_0\forall j,\\
\delta\omega_j(\Omega,\varepsilon_i)_{\text{rot,re}} &= \Lambda_j\Omega\left(1 - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right)\omega_0,\\
\delta\omega_j(\Omega,\varepsilon_i)_{\text{rot,im}} &= -i\frac{\varepsilon_i}{2\varepsilon_r}\Lambda_j\Omega\omega_0.
\end{aligned} \tag{2.13b}$$

From the above we can make three major observations: (a) The resonant frequency of a stationary cavity is reduced by $\delta\omega(\varepsilon_i)_{\text{stat,re}}$ having an imaginary part $\delta\omega(\varepsilon_i)_{\text{stat,im}}$ which represents the attenuation coefficient; (b) when that cavity rotates, the resonant frequency splits into M distinct frequencies, where the Sagnac frequency splitting is $\delta\omega_j(\omega,\varepsilon_i)_{\text{rot,re}}$ for the jth mode, and the attenuation coefficient is modified by $\delta\omega_j(\omega,\varepsilon_i)_{\text{rot,im}}$; (c) the Sagnac frequency splitting is reduced by loss, and the specific attenuation coefficient is reduced for the corotating modes or increased for the contrarotating modes, all relatively to the stationary one.

III. CLASSICAL SAGNAC EFFECT IN A LOSSY RING RESONATOR

The theory developed in the previous section holds for a general lossy cavity that supports mode degeneracy. It, therefore, holds also for the most familiar case: the ring resonator. It has been shown previously [7] that the eigenvalues for a large single lossless ring resonator, calculated from the matrix $\mathbf{C}$ in Eq. (2.12a), are given by

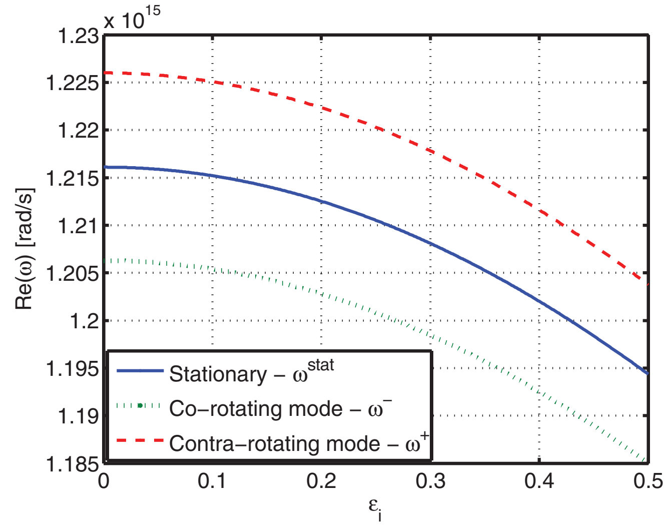

FIG. 1. (Color online) The real part of the resonant frequencies of a lossy ring resonator when stationary and when rotating.

$$\Lambda_{1,2} = \pm\frac{R}{nc}, \tag{3.1}$$

where R is the ring's radius, $n = \sqrt{\varepsilon_r}$, and c is the speed of light in vacuum. Therefore, using this result in Eq. (2.13a), the resonance frequency shift as a result of loss and rotation is

$$\delta\omega_{1,2}(\Omega,\varepsilon_i) = \omega_0\left[\pm\frac{R}{nc}\Omega\left(1 - i\frac{\varepsilon_i}{2\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right) - i\frac{\varepsilon_i}{2\varepsilon_r} - \frac{3\varepsilon_i^2}{8\varepsilon_r^2}\right]. \tag{3.2}$$

For a lossless ring resonator, the above result reduces to the classical Sagnac effect [15]. For a lossy rotating ring resonator the real part of the resonant frequencies is given by

$$\omega_{re}^{\pm} = \omega_0\left[1 - \frac{3\varepsilon_i^2}{8\varepsilon_r^2} \pm \frac{R}{nc}\Omega\left(1 - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right)\right]. \tag{3.3}$$

The real part of the resonant frequencies for a stationary and rotating lossy ring resonator as function of the imaginary part of the relative permittivity is shown in Fig. 1. The parameters used for all the figures in this section are $\varepsilon_r = 2.25$, $\omega_0 = 1.216 \times 10^{15}$ rad/s, $R = 100\ \mu$m, $\omega = 3\text{e–}5\omega_0$. Such a high rotation rate is used solely for illustration purposes, because of the extremely small Sagnac frequency shift for such resonators. We see that the frequency of the corotating mode is lowered by rotation, while the frequency of the contrarotating mode is increased. Note also that as the losses grow, the frequencies decrease.

The imaginary part of the resonant frequencies is

$$\omega_{\text{im}}^{\pm} = -i\frac{\varepsilon_i}{2\varepsilon_r}\left(1 \pm \frac{R}{nc}\Omega\right)\omega_0. \tag{3.4}$$

The result is shown in Fig. 2. Interestingly, the corotating mode, which takes a longer time to complete a round trip in the resonator, has a lower specific attenuation coefficient, as

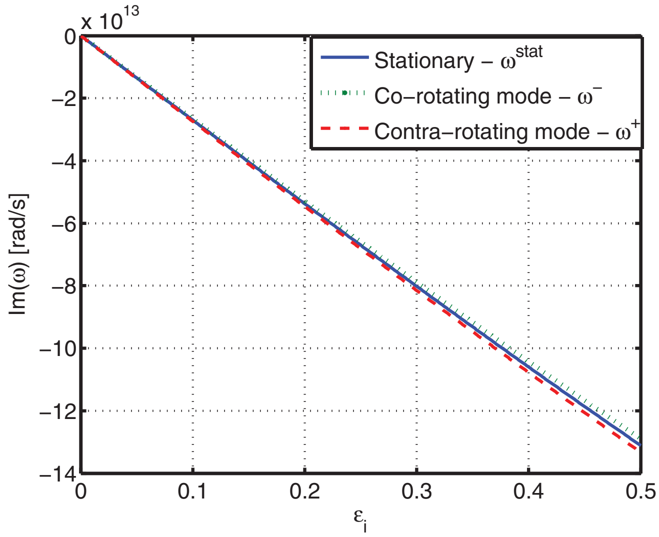

FIG. 2. (Color online) The imaginary part of the resonant frequencies of a lossy ring resonator when stationary and when rotating.

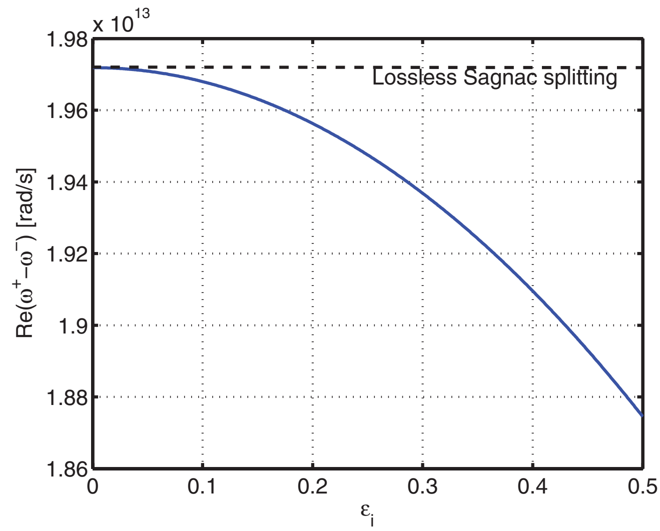

FIG. 4. (Color online) Sagnac frequency splitting of a ring resonator, as a function of the imaginary part of relative permittivity. The dashed line shows the value for a lossless resonator.

opposed to the contrarotating mode. This fact may indicate that the round trip losses of both modes are the same, and are equal to the round trip loss of the stationary ring resonator. In order to explore that, we calculate the ratio between the imaginary part and the real part of the resonant frequency of a mode defined as the quality factor. The quality factors of a rotating resonator normalized to that of a stationary one are shown in Fig. 3. Surprisingly, the quality factors are different, meaning the round trip losses of the rotating modes are not the same. The round trip loss of the corotating mode in the rest frame of the rotating resonator is lower (higher Q factor) than that of the contrarotating mode (lower Q factor).

The Sagnac frequency splitting, which is the difference in the resonant frequencies of the two modes, is shown in Fig. 4. The splitting is getting lower for higher losses, but since it is a second-order effect, it may not impose a practical limitation for low-loss ring resonators.

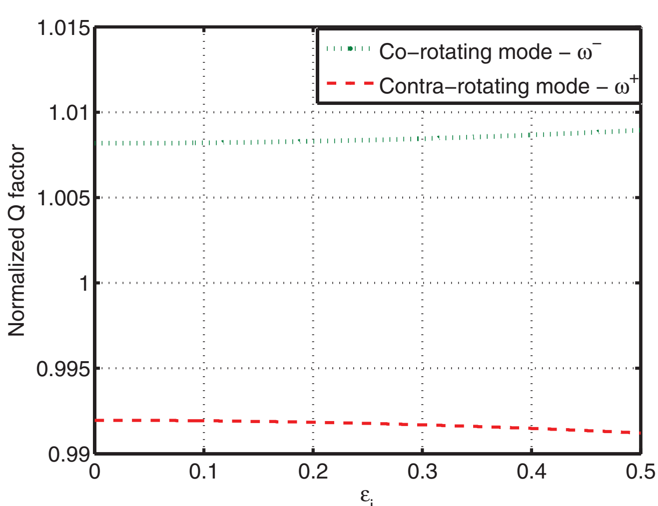

FIG. 3. (Color online) Quality factors of a rotating ring resonator normalized to the quality factor of a stationary resonator.

IV. STUDY OF THE LOSSY ROTATING CROW

In this section we study a rotating CROW consisting of weakly coupled doubly degenerate lossy cavities, such as ring resonators. For a lossless and stationary CROW the counterclockwise (CCW) mode H_0^+ in even-numbered resonators couples to the clockwise (CW) mode H_0^- in odd-numbered resonators, all having the same resonant frequency ω_0, as shown in Fig. 5. When the entire structure is rotating, the resonant frequency of the mth resonator shifts to $\omega_0+(-1)^m\delta\omega(\Omega)$ where $\delta\omega(\Omega)$ is the Sagnac frequency shift of a lossless ring resonator, and the modal fields in each resonator are no longer $H_0^{\pm}$.

Our goal is to perform an analysis of the CROW dispersion relation [8], while taking into consideration the *practical* losses associated with the ring resonators comprising the CROW. To that end, we solve the wave equation given by Eq. (2.4). Since the CROW consists of weakly coupled resonators, the tight-binding approach is a convenient solution technique. Considering the previous observation that a CW rotating mode

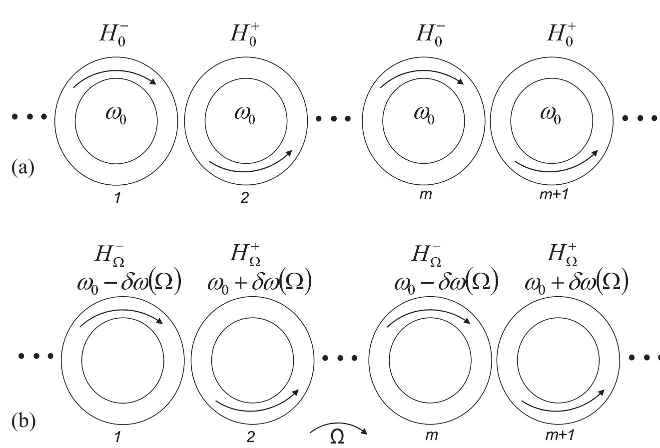

FIG. 5. CROW consisting of doubly degenerate ring resonators: (a) stationary, (b) rotating.

in a given resonator couples only to the CCW rotating mode of its neighbor, we expand the total field of the lossy rotating system with the modes $H_\Omega^\pm$:

$$H_\Omega(r) = \sum_m A_m H_m(r), \quad H_m(r) = \begin{cases} H_\Omega^+(r - r_m), & m \text{ even} \\ H_\Omega^-(r - r_m), & m \text{ odd} \end{cases}, \tag{4.1}$$

where $\boldsymbol{r}_m$ is the location of the mth resonator center, and $H_\Omega^\pm$ are the rotation eigenmodes of a single lossless rotating resonator comprised of a linear combination of the degenerate modes of a stationary lossless resonator as given by Eq. (2.9). For a thorough discussion and overview of their properties the reader may refer to the previous works [3,4]. In this representation of the solution, the modal fields $H_\Omega^\pm$ are used as mere building blocks, and the fact that they also satisfy Eq. (2.5), since they form a combination of stationary modes, will be exploited. The intercavity coupling, as well as the effect of loss and rotation, will be manifested through the expansion coefficients A_m. We decompose now the relative permittivity of the entire structure $\varepsilon_c(\boldsymbol{r}) = \varepsilon_r(\boldsymbol{r}) + i\varepsilon_i(\boldsymbol{r})$ to that of the background structure $\varepsilon_b(\boldsymbol{r})$ (without the resonators) as

$$\frac{1}{\varepsilon_c(r)} = \frac{1}{\varepsilon_b(r)} + \sum_k d(r,r_k), \tag{4.2a}$$

where $d(\boldsymbol{r},\boldsymbol{r}_k)$ represents the variation in $1/\varepsilon_c(\boldsymbol{r})$ introduced by the kth resonator:

$$d(r,r_k) = \frac{1}{\varepsilon_d(r - r_k)} - \frac{1}{\varepsilon_b(r)}. \tag{4.2b}$$

Here $\varepsilon_d(\boldsymbol{r}) = \varepsilon_{dr}(\boldsymbol{r}) + i\varepsilon_{di}(\boldsymbol{r})$ represents the perfect background with a single lossy resonator located at the origin. With this decomposition the operator Θ_c can be decomposed into a series of operators representing the contribution of the background structure and of each of the resonators separately:

$$\Theta_c = \Theta_b + \sum_k \Theta_k, \tag{4.3a}$$

where

$$\Theta_b = \nabla \times \frac{1}{\varepsilon_b(\boldsymbol{r})} \nabla \times , \; \Theta_k = \nabla \times d(\boldsymbol{r},\boldsymbol{r}_k) \nabla \times . \tag{4.3b}$$

Moreover, each of the summed modes in Eq. (4.1) satisfies the wave equation for a lossless resonator:

$$(\Theta_b + \Theta_{rm}) H_m = \left(\frac{\omega_0}{c}\right)^2 H_m, \tag{4.4a}$$

where

$$\Theta_{rm} = \nabla \times \frac{1}{\varepsilon_{dr}(r - r_m)} \nabla \times . \tag{4.4b}$$

In order to obtain a rotating lossy CROW solution for the coefficients A_m, we substitute the expansion in Eq. (4.1) into the wave equation (2.4) and perform an inner product of the resulting equation with $\boldsymbol{H}_n$, with n roaming over all the resonators involved. The result is the following algebraic set of equations for the coefficients A_m:

$$\sum_m A_m \langle \Theta_c \boldsymbol{H}_m, \boldsymbol{H}_n \rangle = k^2 \sum_m A_m \langle \boldsymbol{H}_m, \boldsymbol{H}_n \rangle + ik \sum_m A_m \langle L_\Omega \boldsymbol{H}_m, \boldsymbol{H}_n \rangle. \tag{4.5}$$

Since the indices m, n indicate resonator locations, and exploiting the same assumptions (a) and (b) from Sec. II, regarding the high confinement and the constant permittivity inside the cavities, the last equation can be approximated [up to the second order in $\varepsilon_i(\boldsymbol{r})/\varepsilon_r(\boldsymbol{r})$] as

$$\left[k_0^2\left(1 - i\frac{\varepsilon_i}{\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right) - k^2\right] \left\|\boldsymbol{H}_\Omega^+\right\|^2 A_n + \sum_m \tau_{m-n} A_m - ik \sum_m A_m \langle L_\Omega \boldsymbol{H}_m, \boldsymbol{H}_n \rangle = 0, \tag{4.6}$$

where τ_{m-n} is given by

$$\tau_{m-n} = \left\langle \sum_{k \neq m} \Theta_k \boldsymbol{H}_m, \boldsymbol{H}_n \right\rangle. \tag{4.6a}$$

For $m - n \neq \pm 1$ these elements are exponentially small and negligible compared to the dominant elements $\tau_1 = \tau_{-1}$ [16]. The third term in Eq. (4.6), containing the inner product, can be simplified the same way as Eq. (2.8) in Sec. II, so that the Eq. (4.6) reads as

$$\left[\omega_0^2\left(1 - i\frac{\varepsilon_i}{\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right) - \omega^2\right] \left\|\boldsymbol{H}_\Omega^+\right\|^2 A_n + \tau_1 A_{n+1} + \tau_{-1} A_{n-1} + 2(-1)^n \delta\omega(\Omega)\omega \left\|\boldsymbol{H}_\Omega^+\right\|^2 \left(1 - i\frac{\varepsilon_i}{\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right) = 0, \tag{4.7}$$

where $\delta\omega(\Omega)$ is the Sagnac frequency shift of a single lossless rotating resonator, and the elements τ_1, τ_{-1} are given by [8]

$$\tau_1 = \tau_{-1} = \frac{\Delta\omega \left\|\boldsymbol{H}_\Omega^+\right\|^2 \omega_0}{c^2}, \tag{4.7a}$$

with $\Delta\omega$ being the stationary CROW bandwidth. Substituting Eq. (4.7a) into Eq. (4.7) and rearranging, we get the following eigenvector and eigenvalue problem for the vector coefficients A_n:

$$\frac{1}{2}(A_{n+1} + A_{n-1}) + (-1)^n \frac{\delta\omega(\Omega)\omega}{\omega_0 \Delta\omega}\left(1 - i\frac{\varepsilon_i}{\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right) = \frac{\left[\omega^2 - \omega_0^2\left(1 - i\frac{\varepsilon_i}{\varepsilon_r} - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right)\right]}{2\omega_0 \Delta\omega} A_n. \tag{4.8}$$

Expressing the eigenvector elements A_n as

$$A_n = A_0 e^{i\beta n}, \tag{4.9}$$

and inserting it into Eq. (4.8), we can find the transmission function of a CROW as a function of β. For example, we solve Eq. (4.8) for the lossless case ($\varepsilon_i = 0$) and for a lossy case ($\varepsilon_i = 0.2$) with the following set of parameters: $\varepsilon_r = 2.25$, $\Delta\omega = 10^{13}$ rad/s, $\omega_0 = 10^{14}$ rad/s, $\delta\omega(\Omega) = 0.2\Delta\omega$. The calculated transmission functions are shown in Fig. 6. The curve of the lossy CROW (lower) is shifted towards lower frequencies compared to that of the lossless CROW (upper) and the rotation-induced gap (RIG) is smaller by 0.8%. These results are consistent with those of a single lossy ring resonator shown in Sec. III, where we have shown that losses reduce both the resonant frequency of a resonator and the Sagnac frequency shift. However, it should be emphasized that the

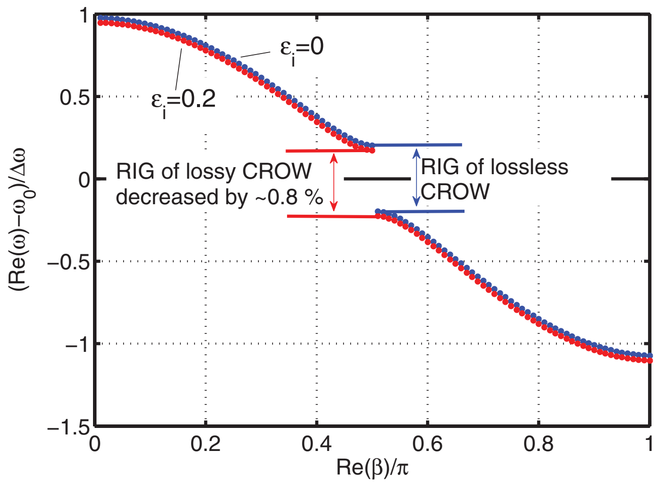

FIG. 6. (Color online) Transmission function of the rotating lossless and lossy CROWs.

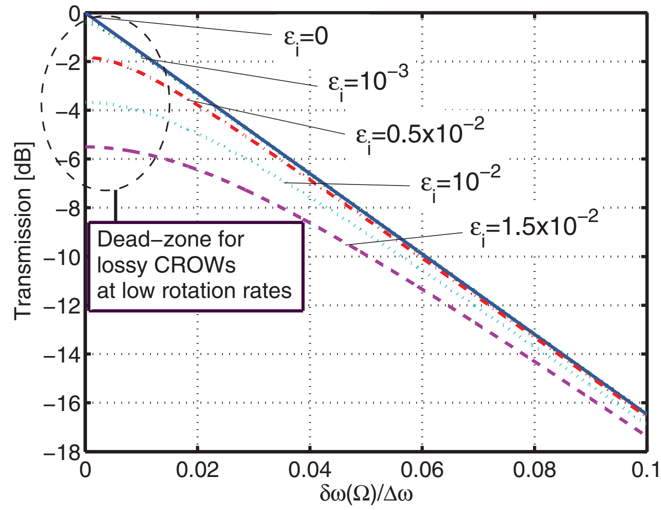

FIG. 7. (Color online) Transmission of a 20-resonator CROW for different loss values as a function of a normalized Sagnac frequency shift.

rotation-induced gap is barely affected even by such substantial losses.

It can be shown that the bandwidth of rotation-induced gap is given by

$$\Delta\omega_r = 2\delta\omega(\Omega)\left(1 - \frac{\varepsilon_i^2}{\varepsilon_r^2}\right). \tag{4.10}$$

This expression is consistent with the example above when putting into it the values of the relative permittivity. We get a reduction of 0.8% in the RIG bandwidth.

Next, we examine the transmission of a finite-length CROW at the center of the rotation-induced gap as a function of the Sagnac shift $\delta\omega(\Omega)$. By substituting $\omega = \omega_0$ into Eq. (4.8) and solving for A_n, we find the amplitudes of the fields in each resonator of the CROW. Particularly, we are interested in the amplitude in the last resonator, which represents the rotation-induced attenuation of the entire CROW. For example, assuming a CROW consisting of 20 resonators, with the following set of parameters: $\varepsilon_r = 2.25$, $\Delta\omega = 10^{13}$ rad/s, $\omega_0 = 10^{14}$ rad/s, we get the transmission as a function of a normalized Sagnac frequency shift (of a single lossless resonator, linearly proportional to the rotation rate) for a number of values of ε_i as shown in Fig. 7. For the lossless case (upper curve), the transmission is linear, indicating exponential attenuation. As the losses grow, we observe a formation of a dead zone around zero rotation rates. At that region the response becomes flat, meaning the device is insensitive to rotation. But as the rotation rate increases, the transmission of a lossy device becomes very close to that of a lossless one. The last observation means that if a lossy device is biased at a nonzero rotation rate, the achieved performance in terms of sensitivity can be very close to that of a lossless device.

A simple solution for the dead-zone formation is to apply a constant artificial biasing (without the actual physical rotation) to the CROW, by tuning the odd- and even-numbered resonators in the chain to different resonance frequencies. This biasing basically mimics the effect of rotation and opens a band gap in the transmission function even in the absence of rotation. As a result the operation point of the CROW is shifted away from the problematic range and the dead-zone formation is eliminated. The simplest constant modulation can be done during the fabrication process of the device. For example, a slight variation in the effective perimeters of the odd-numbered resonators from those of the even-numbered resonators gives the desired result. In case a dynamic modulation is required, it is possible to fabricate a thermoelectric heater on top of each resonator [17]. By periodically changing the refractive index of the material through the thermoelectric effect, one can "bias" the CROW as required to achieve the best possible performance. We demonstrate this concept in the following example. Consider a CROW consisting of 20 ring resonators with a radius of 100 μm, with a Q factor of 10^4, made of a dielectric material with $\varepsilon_r = 2.25$, and an intercavity power coupling of 0.01. The CROW input and output terminals consist of dielectric waveguides coupled to the first and the last rings, both with power coupling coefficients of 0.1. The odd-numbered rings resonate at $\omega_0 = 12.152 \times 10^{14}$ rad/s, which corresponds to the angular mode 608 with a

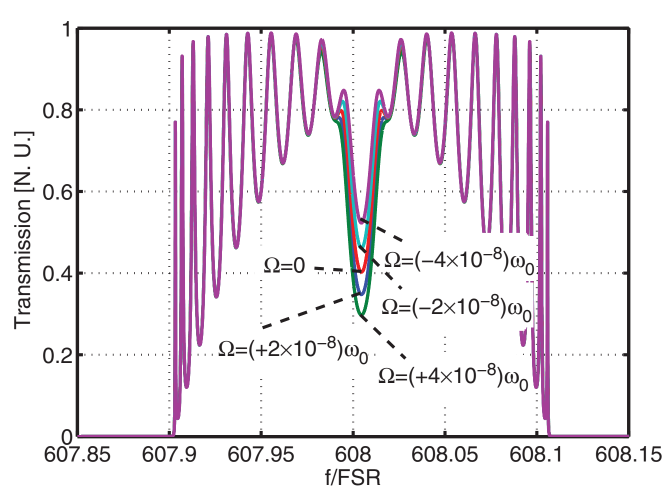

FIG. 8. (Color online) Transmission (normalized units) of a "biased" CROW as a function of frequency (normalized to the free spectral range of a single resonator with ω_0) for various rotation rates.

vacuum wavelength of 1.55 μm, while the even-numbered rings resonate at $\omega_0 \times (1 + 15 \times 10^{-6})$. This resonant frequency shift corresponds to a virtual rotation rate of 3.37×10^7 rad/s. We calculated the CROW transmission as a function of frequency for various values of rotation rate using the transfer matrix method [9]. The result is shown in Fig. 8. A band gap is formed near ω_0 for a stationary CROW (middle) as a result of periodic modulation of the resonant frequencies of the rings. When the CROW rotates, the gap becomes deeper or shallower, depending on the rotation direction expressed as the sign of Ω. Therefore, biasing the CROW makes it also possible to detect the direction of rotation.

V. CONCLUSIONS

We theoretically studied a slowly rotating lossy microcavity with mode degeneracy, and a lossy CROW subjected to rotation. It was shown that the Sagnac frequency splitting is reduced as a result of loss, and that reduction is a second-order effect in the material's imaginary part of the permittivity. Also, each mode's specific attenuation coefficient splits into two different values. In the example of a rotating ring resonator, the quality factor of the contrarotating mode is lower than that of the corotating mode. The study of a lossy CROW has shown that the rotation-induced gap is reduced and the dispersion curve is lowered in frequency. The transmission of the rotating CROW was shown to have a dead zone for low rotation rates and that region, which is insensitive to rotation, becomes wider for higher losses. In order to avoid this performance degradation, we have shown that a virtual biasing of the device by periodic modulation of the resonators' resonant frequencies can be a simple, yet effective solution.

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[1] R. Bergh, H. Lefevre, and H. Shaw, J. Lightwave Technol. **2**, 91 (1984).
[2] G. Sagnac, C. R. Acad. Sci. **95**, 708 (1913).
[3] S. Sunada and T. Harayama, Phys. Rev. A **74**, 021801(R) (2006).
[4] D. Hah and D. Zhang, Opt. Express **18**, 18200 (2010).
[5] Y. Zhang, H. Tian, X. Zhang, N. Wang, J. Zhang, H. Wu, and P. Yuan, Opt. Lett. **35**, 691 (2010).
[6] C. Ciminelli, C. E. Campanella, and M. N. Armenise, J. Lightwave Technol. **27**, 2658 (2009).
[7] B. Z. Steinberg and A. Boag, J. Opt. Soc. Am. B **24**, 1 (2007).
[8] B. Z. Steinberg, J. Scheuer, and A. Boag, J. Opt. Soc. Am. B **24**, 5 (2007).
[9] J. Scheuer and A. Yariv, Phys. Rev. Lett. **96**, 053901 (2006).
[10] M. Terrel, M. Digonnet, and S. Fan, Laser Photonics Rev. **3**, 452 (2009).
[11] T. Shiozawa, Proc. IEEE **61**, 1694 (1973).
[12] J. L. Anderson and J. W. Ryon, Phys. Rev. **181**, 1765 (1969).
[13] B. Z. Steinberg, Phys. Rev. E **71**, 056621 (2005).
[14] H. J. Arditty and H. C. Lefevre, Opt. Lett. **6**, 401 (1981).
[15] E. J. Post, Rev. Mod. Phys. **39**, 475 (1967).
[16] A. Boag and B. Z. Steinberg, J. Opt. Soc. Am. A **18**, 2799 (2001).
[17] R. Amatya, C. W. Holzwarth, H. I. Smith, and R. J. Ram, Photon. Tech. Lett. IEEE **20**, 1739 (2008).