Polariton condensation with nonlinear photons

Xiaoyong Guo,[1] Zhongzhou Ren,[1,2,*] and Zimeng Chi[3]
[1]*Department of Physics, Nanjing University, Nanjing 210093, China*
[2]*Center of Theoretical Nuclear Physics, National Laboratory of Heavy-Ion Accelerator, Lanzhou 730000, China*
[3]*Department of Basic Courses, Harbin Cambridge College, Harbin 150069, China*
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In this paper we propose a model of polariton condensation with Kerr-type nonlinear photons. We introduce a generalized Dicke Hamiltonian to describe our system. By constructing the partition function as a path integral, the analytical and numerical solutions are presented. On the mean-field level, it is shown that the polariton condensation can occur and the Kerr nonlinearity affects the character of the polariton condensate. As the nonlinear coefficient increases, the condensate evolves from more photon-like to more exciton-like. Although the photon nonlinearity gives rise to a chemical potential greater than the photon energy, the quasiparticle excitation spectrum is still fully gapped. For the condensate collective excitations, the nonlinearity destroys the Goldstone modes and mixes the phase modes with the amplitude modes, resulting four non-zero-frequency collective modes. In addition, the influence of the photon-exciton detuning on the polariton condensate is also discussed.

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I. INTRODUCTION

In the ordinary vacuum the interaction between matter and light is typically very weak. By confining the photons in a small volume (e.g., in an optical cavity), the coupling strength can be enhanced with many orders of magnitude [1]. In this strong coupling regime, coupled oscillations of the material polarization and the electromagnetic field are predominant over the radiative decay. The resulting quasiparticles are known as polaritons [2]. Since they are part matter part light polaritons are bosons, and are good candidates for Bose condensation. Obviously, the polariton condensation and lasing are coherent phenomena involving photons, but they are distinguishable in nature [3]. The lasing is a nonequilibrium weak coupling phenomenon, and is characterized by inverted electronic population generated by pumping-dissipation processes. In a conventional laser the only significant ordering is the coherence of light. However, the polariton itself stems from the strong interaction of light and matter, and the condensation occurs spontaneously without external pumping. Thus the polariton condensation is a strong coupling phenomenon featured by a mixture of the coherent state of photons and material excitations. In the last decade, the investigations of polariton condensation in solid as well as in microcavity systems have attracted enormous attentions [4]. The highlights are many and we only concern ourselves with those that are most relevant to the polariton condensation in a model cavity. Initial works were mainly aimed at studying the quasi-equilibrium condensation of cavity polaritons [5,6] and the crossover between lasing and condensation [7,8]. Subsequent works that dealt with the more complicated cases have addressed the polariton condensation with a propagating photonic mode [9], the BCS-BEC crossover of microcavity polaritons driven by the density fluctuations [10], the finite-size effect as well as the photon statistics near the condensation point [11], the nonequilibrium condensation under incoherent pumping and dissipation [12,13], and the polariton condensation in the microcavity with various geometries [14,15]. Following state-of-the-art experimental techniques, experimental setups with evidence of polariton condensation have been realized in laboratories [16–19].

In a dilute Fermi gas a direct pairwise interaction can be generated and tuned by the technique of Feshbach resonance [20]. A number of new phenomena associated with this tunable interaction are revealed (e.g., the BCS-BEC crossover [21] and the formation of a vortex-antivortex lattice [22]). It is quite natural to surmise that a pairwise interaction of photons in a boson-fermion model will also bring certain new features to a strongly coupled matter-light system. On the other hand, in a conventional theoretical study, the exciton-exciton interaction is assumed to be much stronger than the photon-photon coupling. Therefore it is also interesting to explore the consequences if the converse is true. In a recent work [23], a generalized Dicke model was proposed in which we introduced the repulsive interaction among photons (i.e., the Kerr nonlinearity in the quantum level) and in the mean time the exciton-exciton interaction was omitted. The photons in our model were confined in a cavity and treated as interacting bosons. We have shown that such an additional nonlinearity can manipulate the quantum statistical property of cavity photons. In the present paper we take it one step further. We systematically examine the spontaneous coherence of polaritons in our generalized model. By utilizing the functional integral representation of the partition function [24], an analytical description for the low temperature thermodynamics of the model is obtained. It is shown that a symmetry-breaking ground state with spontaneous condensation of polaritons can be found. The Kerr nonlinearity of the cavity field has the effect of suppressing the photonic component in the polariton condensate. Furthermore, the zero-frequency Goldstone modes vanish and the collective excitations are now characterized by two pairs of non-zero-frequency modes. In the subsequent section, we introduce the model Hamiltonian and implement the path integral approach to derive an effective action. One may argue that in the quantum optics community, the path integral approach is not standard. In the present paper, we show that this approach has the potential advantage to

*zren@nju.edu.cn

address the coherent phenomenon. In Sec. III we develop a mean-field description. The free energy, the order parameter equation, and the density equation are presented at this stage. In Sec. IV we derive the action for quadratic fluctuations around the saddle point to capture the picture of collective modes which are experimentally measurable. Finally, a summary is given in Sec. V.

II. MODEL

Let us consider an ensemble of N localized excitons strongly coupled to a single mode Kerr-type nonlinear cavity. Meanwhile, the interaction between different exciton sites is omitted. In the rotating-wave approximation, the system is described by the following Hamiltonian (in units of $\hbar$)

$$H = \frac{1}{2}\sum_{n=1}^{N} E_g(b_n^\dagger b_n - a_n^\dagger a_n) + \omega_c \psi^\dagger \psi + \frac{g}{\sqrt{N}}\sum_{n=1}^{N}(b_n^\dagger a_n \psi + \psi^\dagger a_n^\dagger b_n) + U\psi^\dagger\psi^\dagger\psi\psi. \quad (1)$$

Here each of the excitons are described by a two-level oscillator with a pair of fermionic operators a_n and b_n; ψ is the annihilation operator of cavity photons; E_g and ω_c are the ground-state energy of the oscillator and the resonance frequency of the cavity field; and the cavity mode couples to all excitons with a homogeneous coupling constant g. The last term in Eq. (1) denotes the Kerr nonlinearity, which has the form of a pairwise repulsion of photons [25,26] and its strength is characterized by U. For $U = 0$, the collective and cooperative emission in the model (1) has attracted tremendous interest since the model was proposed in a celebrated paper [27]. Remarkably, earlier works [28,29] theoretically predicated a phase transition (i.e., Dicke phase transition) from the normal to the superradiant phase. It is known that the superradiant phase cannot exist in closed equilibrium systems [30]. As a result the Dicke phase transition is forbidden in our investigation. In studies of quantum optics and condensed matter physics a large number of coupled matter-light systems were described by the Dicke or Dicke-type models. For example, the model can be used to describe an ensemble of two-level atoms coupled to a single mode microcavity [31], and also can be used to modify the localized electronic excitations, such as excitons, interacting with photons [32,33]. Moreover, the Dicke Hamiltonian is a basic model of the laser physics [34]. The excitons are created from the vacuum by $S^+ = (1/\sqrt{N})\sum_n b_n^\dagger a_n$, and the exciton number is expressed as $\sigma_z = (1/2)\sum_n (b_n^\dagger b_n - a_n^\dagger a_n)$, which is also the inversion operator for oscillators. The operator S^+ is approximately a bosonic creation operator only when the excitation density is sufficiently low. However, away from the low excitation limit, S^+ is no longer a bosonic operator, and the conventional concept of polariton is generalized to be the quantum of excitations of the Hamiltonian (1) [6]. Since, in our study, the presence or absence of a localized exciton is represented by a two-level oscillator, the anticommutators of a_n and b_n cannot be neglected for all excitation densities. As a result, the Hamiltonian (1) is regarded to be a coupled boson-fermion model [i.e., a single bosonic mode (light) coupled with two fermionic modes (matter)], and the role of these modes is not equivalent and interchangeable. Moreover, the interaction between polaritons is introduced by an additional photonic nonlinearity, thus the polaritons should be treated as interacting bosons. The polariton number is the total number of photons and excited two-level oscillators

$$N_{\rm pol} = L + N/2 = \psi^\dagger\psi + \frac{1}{2}\sum_{n=1}^{N}(b_n^\dagger b_n - a_n^\dagger a_n) + N/2. \quad (2)$$

Here the operator L is the excitation number, and the corresponding excitation density is defined as $\rho_{\rm ex} = \langle L\rangle/N$.

In the nonlinear optics, the Kerr nonlinearity is often generated by certain optical medium with nonzero third-order nonlinear polarizability, such as doped semiconductor glass [26]. Unfortunately, most of these nonlinear materials lose their function at the quantum level. However, the recent advances in the study of the quantum simulator (see Ref. [35] and references therein) pave the way to consider the situation in reverse (i.e., photon nonlinearity dominant and exciton-exciton interaction negligible). To this end, a more feasible realization of the Hamiltonian (1) may be a hybrid system consisting of a semiconductor quantum dot (QD) surrounded by laser driven four-level atoms and imbedded within a single mode cavity. Figure 1 schematically illustrates a typical structure of this proposal. Here the QD contains an ensemble of localized excitons. For a small cavity volume, the strong exciton-photon coupling and a high cooperativity of four-level atoms can be realized simultaneously. To maintain sufficient cavity finesse, the cavity can be fabricated into a photonic crystal platform where the photon localization effect guarantees a very high cavity Q factor. Concerning the atomic gas, the configuration and the driven pattern of four-level atoms are in the same manner as in the electromagnetically induced transparency [35]. By modulating the detuning between external laser and exciton resonance the driven field can act only on the atomic gas, in the mean time the QD excitons do not feel the presence of the external laser. In certain parameter regimes, one can integrate out the degrees of freedom of four-level atoms and then the remaining photon field will possess a quartic interaction [36]. Parameters of the external laser and

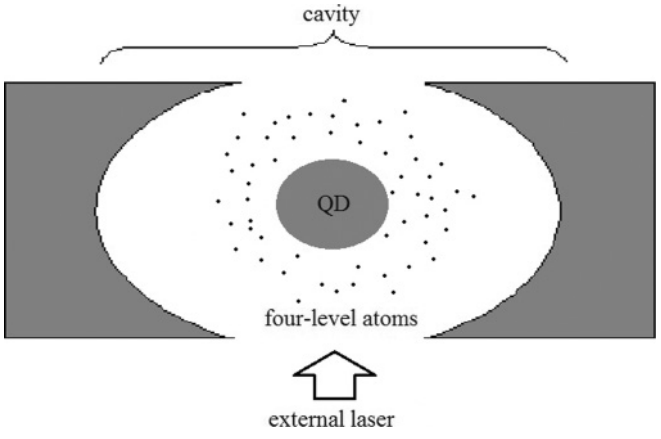

FIG. 1. Sketch of a typical realization of our proposed model. The cavity consists of a semiconductor QD surrounded by laser-driven four-level atoms (black spots). To maintain sufficient cavity finesse, the cavity system can be fabricated into a photonic crystal platform.

the four-level atomic gas are encapsulated into the constant U. Together with the Hamiltonian of the QD excitons, a system containing an ensemble of localized excitons with adjustable photon-photon repulsion has been achieved. In Ref. [23] we proposed a similar setup where a two-level atomic ensemble plays the role of QD excitons.

We consider the polariton condensation in thermal equilibrium, and work in the grand-canonical ensemble $\tilde{H} = H - \mu L$ with chemical potential μ. Although the current experiments may not have reached equilibrium, there are many sound reasons to deal with the polaritons as quasi-equilibrium particles [10]. Expressed in the coherent state path integral, the quantum partition function takes the form

$$Z = \int D\psi \prod_n D\eta_n e^{-S}, \tag{3}$$

with the action

$$S = \int_0^\beta d\tau [\bar{\psi}(\partial_\tau + \tilde{\omega}_c + U|\psi|^2)\psi + \sum_n \bar{\eta}_n M \eta_n], \tag{4}$$

where $\tilde{\omega}_c = \omega_c - \mu$; $\bar{\eta}_n = (\bar{b}_n \; \bar{a}_n)$ are the Nambu spinors for each exciton site and the matrix M is given by

$$M = \begin{pmatrix} \partial_\tau + \tilde{\varepsilon} & g\psi/\sqrt{N} \\ g\bar{\psi}/\sqrt{N} & \partial_\tau - \tilde{\varepsilon} \end{pmatrix}, \tag{5}$$

with $\tilde{\varepsilon} = (E_g - \mu)/2$. In Eq. (3) the integration over the fermionic field can be taken exactly by the Gaussian integral formula. After rescaling the field $\psi \to \sqrt{N}\psi$, it yields an effective action for photons

$$S_{\text{eff}} = \int_0^\beta d\tau \bar{\psi}(\partial_\tau + \tilde{\omega}_c + \tilde{U}|\psi|^2)\psi - \text{Tr} \ln M. \tag{6}$$

Here $\tilde{U} = NU$ is termed as the nonlinear coefficient. In the next two sections, we employ the action (6) to examine the condensation of polaritons with a nonlinear light field. We intend to address the properties that cannot be obtained for the linear cavity polaritons.

III. MEAN-FIELD DESCRIPTION

In this section, we develop a mean-field description to investigate the properties of the condensate. Before subjecting the action (6) to a saddle-point analysis, it is helpful to derive the free energy to acquire some intuition of how the nonlinear photons affect the polariton condensation. Making the static approximation $\psi(\tau) = \psi_0$ and introducing the fermionic Matsubara frequencies ω_n, the matrix M has two eigenvalues $-i\omega_n \pm E$ for each ω_n, where $E = \sqrt{\tilde{\varepsilon}^2 + g^2|\psi_0|^2}$. Thus $\det M$ can be calculated in terms of the product of these eigenvalues

$$\det M = \prod_{\omega_n} (-i\omega_n + E) \prod_{\omega_n} (-i\omega_n - E) = \cosh^2\left(\frac{1}{2}\beta E\right). \tag{7}$$

Here the fermionic nature of a_n and b_n is fully embodied into the infinite product over ω_n. Then the action (6) yields that

$$S_{\text{eff}}[\psi_0] = \beta(\tilde{\omega}_c|\psi_0|^2 + \tilde{U}|\psi_0|^4) - 2\ln\cosh\left(\tfrac{1}{2}\beta E\right). \tag{8}$$

It is known that a static approximation in a field integral amounts to replacing a quantum degree of freedom by its classical approximation [32]. Thus this approximation gives rise to an action of the coherent field ψ_0, which will have the physical significance of the order parameter. The free energy is then given as

$$F = \tilde{\omega}_c|\psi_0|^2 + \tilde{U}|\psi_0|^4 - \frac{2}{\beta}\ln\cosh\left(\frac{1}{2}\beta E\right). \tag{9}$$

For the Kerr nonlinearity, the constant $\tilde{U}$ is positive defined. Since in the condensed state $\psi_0 \neq 0$, the quartic field term has a positive contribution to the free energy. This contribution will prevent the system achieving minimal energy, thus it has a demolished effect on the photonic component of the condensate. In turn, when the free energy reaches its minimum, the energy contribution from the Kerr effect will suppress the coherent field amplitude ψ_0. In short a small magnitude of ψ_0 is energetically favorable for the equilibrium state. In the normal state $\psi_0 = 0$, the contribution of nonlinear term vanishes. Thus on the mean-field level, the normal state, the transition temperature, and the phase boundary are essentially unaffected by the nonlinear photons. This allows us to concentrate on discussing the properties of the polariton condensate. One could note that when the first two terms of the free energy (9) dominates we obtain a textbook example of Ginzburg-Landau theory. In this regime, the system is trivial and the ground state should be a Fock state (i.e., there is no coherence at all). However, this is not the case since the third term contributes significantly for all temperature scales.

We then proceed by minimizing the effective action to obtain the order parameter equation. This equation is qualified by the condition of vanishing functional derivative. For the action (8) it gives that

$$(\tilde{\omega}_c + 2\tilde{U}|\psi_0|^2)\psi_0 = \frac{g^2\psi_0}{2E}\tanh\left(\frac{1}{2}\beta E\right). \tag{10}$$

Here it relates the coherent photon field with the polarization of excitons strongly modified by the presence of such a coherent field. Physically, it means that the coherent field is sustained by a coherent polarization in the exciton system, which in turn is generated by the presence of the coherent field. In this respect the field ψ_0 has the status of the order parameter of condensation. Analogous to the BCS gap equation, we identify $E = \sqrt{\tilde{\varepsilon}^2 + g^2|\psi_0|^2}$ as the quasiparticle excitation energy. These excitations are the coupled mode of excitons with photons. In the means of the Hopfield they are polaritons [37]. In spite of its innocent appearance, Eq. (10) reveals certain new features of our extension. In the original Dicke model where $\tilde{U} = 0$ the condensed solution exists provided that $\mu < \omega_c$ [5]. A glimpse over Eq. (10) tells us that in our model the condition for being a condensate is related to the order parameter as $\tilde{\omega}_c + 2\tilde{U}|\psi_0|^2 > 0$, and the condensed state can be found even for the case of $\mu > \omega_c$. It seems that the quasiparticle excitation energy will be gapless in a certain parameter regime. We leave this important issue momentarily, and discuss it later with the numerical solution of Eq. (10).

Besides the mean-field Eq. (10), an additional equation relating the excitation density ρ_{ex} to the chemical potential μ

is required. Taking the derivative of free energy with respect to the chemical potential gives that

$$\rho_{\rm ex} = |\psi_0|^2 - \frac{\tilde{\varepsilon}}{2E}\tanh\left(\frac{1}{2}\beta E\right) = \left(1 - \frac{2\tilde{\varepsilon}\tilde{U}}{g^2}\right)|\psi_0|^2 - \frac{\tilde{\varepsilon}\tilde{\omega}_c}{g^2}. \quad (11)$$

Here the last equality follows with the help of Eq. (10). Recalling the definition of the excitation density in Sec. II, it is straightforward to obtain the exciton density as

$$\frac{\langle\sigma_z\rangle}{N} = -\frac{\tilde{\varepsilon}}{g^2}(\tilde{\omega}_c + 2\tilde{U}|\psi_0|^2). \quad (12)$$

Equations (10)–(12) are the main results of our mean-field description. Below we numerically solve these equations at zero temperature and the solution is shown in Fig. 2 for various system parameters.

At zero temperature there are only three parameters remaining: the dimensionless detuning $\Delta = (\omega_c - E_g)/g$, the excitation density $\rho_{\rm ex}$, and the nonlinear coefficient $\tilde{U}$. In Fig. 2 we plot the coherent field density $|\psi_0|^2$, the exciton density $\langle\sigma_z\rangle/N$, and the dimensionless chemical potential $(\mu - \omega_c)/g$ as a function of the excitation density $\rho_{\rm ex}$ at zero temperature limit $\beta E \to \infty$. In the upper part of Fig. 2, excitons are tuned on resonance with photons $\Delta = 0$, while the nonresonance case is illustrated in the lower part for $\Delta = 1$. In units of g we take the values of $\tilde{U}$ as $\tilde{U} = 0.1g$ (small), $\tilde{U} = g$ (moderate), and $\tilde{U} = 10g$ (large) cases. The aim is to see how the photon nonlinearity affects the polariton condensate. The value of $\tilde{U}/g$ chosen above is in agreement with the physical realization of our model. For example, to generate the Kerr nonlinearity, the Rabi frequency of the external laser can be taken as $10^{11}\,{\rm s}^{-1}$ with the detuning of $10^{10}\,{\rm s}^{-1}$. Subjecting these parameters into Eq. (22) of Ref. [35], the strength of the Kerr effect is estimated as $U \approx 10^7\ {\rm s}^{-1}$. For a QD containing $N = 10^3$ localized excitons and a strong dipolar coupling rate $g = 10^{10}\ {\rm s}^{-1}$, the relative nonlinear coefficient $\tilde{U}/g = 1$ is satisfied.

In Fig. 2 it is manifested that when $\tilde{U}$ is much smaller than g a coherent photon field exists and its magnitude is greater than the exciton density. In this case the emergence of coherence is the photon-like polariton condensate. In fact the result we obtained for the small $\tilde{U}$ case is similar with those presented by the authors of Ref. [6]. For the case where $\tilde{U}$ and g are comparable, the coherent field density is smaller than the previous case and is accompanied by a larger magnitude of the exciton density. The tendency indicates that the repulsive interaction among photons tends to reduce the photonic component of the condensate, and in turn it will enhance the number of excitons for a given total excitation density. We further observe that when $\tilde{U} = 10g$ and $\rho_{\rm ex} > 0.5$, a total inversion arises in the exciton system and the order parameter shows a threshold behavior around $\rho_{\rm ex} = 0.5$. In such a case, the condensate is exciton-like rather than photon-like and we infer the underlying physics as follows. As the excitation density increases, the strong photon nonlinearity significantly prevents the external excitations entering the coherent photonic mode. As a consequence, the population of the exciton modes has been saturated before the photonic mode being populated. Such an exciton-like polariton condensate cannot be obtained with linear photons. For real experimental settings, polaritons

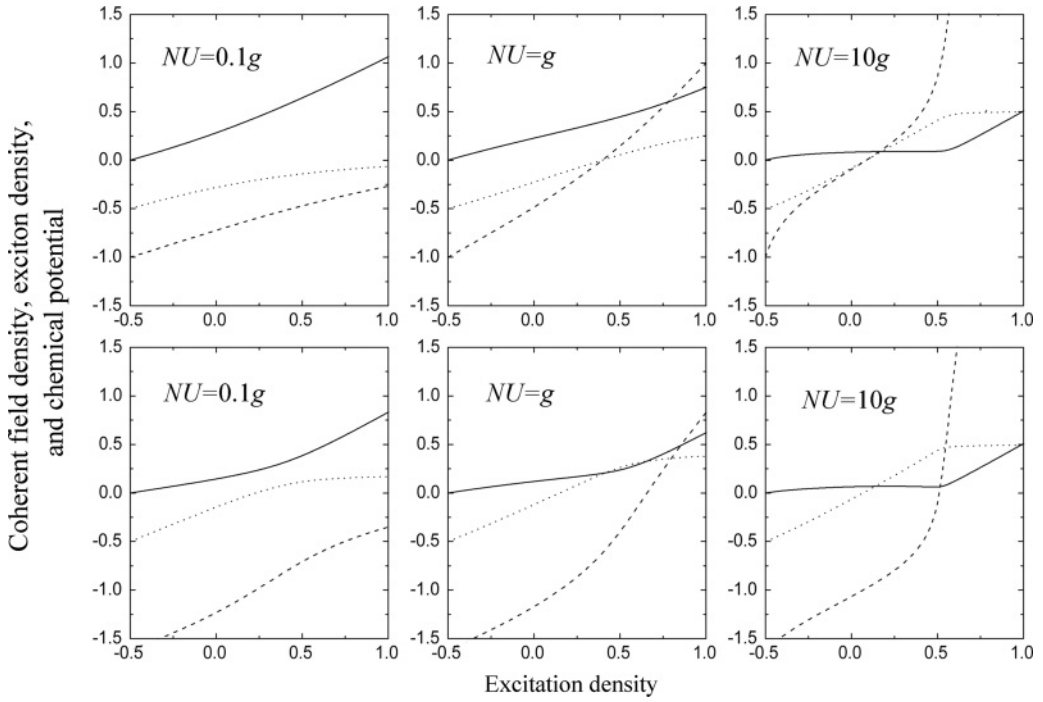

FIG. 2. The coherent field density $|\psi_0|^2$ (i.e., order parameter, solid line), the exciton density $\langle\sigma_z\rangle/N$ (dotted line), and the chemical potential $(\mu - \omega_c)/g$ (dashed line) as a function of the excitation density $\rho_{\rm ex}$ at zero temperature limit. The resonant case is presented in the top row for $\Delta = 0$, while the nonresonant case is in the bottom row for $\Delta = 1$. For each of the rows the nonlinear coefficient is set to be $NU = 0.1g$, g, and $10g$, increasing from left to right.

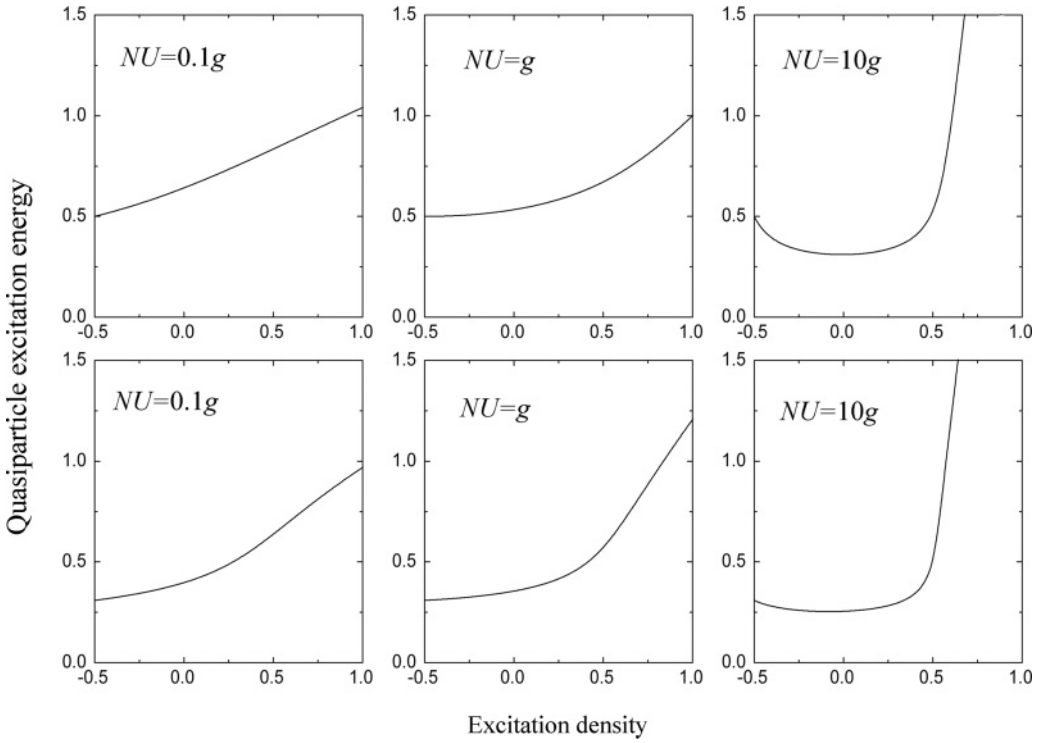

FIG. 3. The evolution of quasiparticle excitation energy $E = \sqrt{\tilde{\varepsilon}^2 + g^2|\psi_0|^2}$ (in units of g) with increasing excitation density $\rho_{\rm ex}$. Parameters are as same as those in Fig. 2.

have a very short lifetime due to fast photon decay [4]. Thus the lifetime of an exciton-like condensate is apparently longer than a photon-like condensate. With this respect the inclusion of photon nonlinearity has the experimental significance to reach equilibrium polariton condensate.

We also note that when the inversion of two-level oscillators comes up, the dimensionless chemical potential $(\mu - \omega_c)/g$ grows fast and changes its sign from negative to positive. As we already mentioned this is a new feature brought by the photon nonlinearity. For a large $\tilde{U}$, the term $2\tilde{U}|\psi_0|^2$ in the left-hand side of Eq. (10) grows fast with the increase of $|\psi_0|$. Thus it needs a negative value of $\tilde{\omega}_c$ to balance the equation. As a result, the chemical potential will be evolved into values greater than the photon energy. Comparing the upper part of Fig. 2 to the lower part, we learn that the photonic component in the detuning case is less than the resonance case. In a coupled matter-light system, a nonvanishing detuning amounts to weaken the dipolar interaction [38]. This will enhance the relative strength of the photon nonlinearity versus the photon-exciton coupling. As a result, increasing the detuning has the same effect as increasing the nonlinear coefficient $\tilde{U}$.

The physics read from Fig. 2 is in accordance with the qualitative energetics analysis toward the free energy (9). Figure 2 clearly demonstrates that at zero temperature the polariton condensation can occur with the presence of the photon nonlinearity, which plays a role of determining the components of the condensate. To explore the consequences of photon nonlinearity on quasiparticle excitations, we calculate the quasiparticle excitation energy $E = \sqrt{\tilde{\varepsilon}^2 + g^2|\psi_0|^2}$ as a function of the excitation density $\rho_{\rm ex}$ and illustrate the results in Fig. 3 (in units of g). Here the parameters are as same as those in Fig. 2. In Fig. 3 a fully gapped energy spectrum shows up. The quasiparticle excitation energy is gapless at the parameter regime where the condition $|\psi_0| = 0$ and $\mu = E_g$ are both satisfied. To meet with the second condition a positive chemical potential is required. Although this is allowed in our model, from Fig. 2 we note that the two conditions cannot satisfy simultaneously. A positive $(\mu - \omega_c)/g$ is always accompanied by a nonzero coherent field $|\psi_0| \neq 0$. Therefore the quasiparticle energy is still fully gapped. For the nonlinear coefficient smaller than or comparable with the dipolar coupling, the quasiparticle energy is a monotonous increasing function of the excitation density. However, in the strong nonlinear region, the quasiparticle energy first decreases and then increases with the density. A minimal width of the energy gap can be found at round $\rho_{\rm ex} = 0$. When $\rho_{\rm ex} > 0.5$, the width grows fast. This tells us that in a certain parameter regime exciting a quasiparticle needs a lot of energy. Although increasing the detuning has an effect to make the gap narrower, the general characters of curves still remain. In the next section we make use of the mean-field description as a platform on which to construct a picture of low-energy collective excitations.

IV. COLLECTIVE EXCITATIONS

In this section we proceed to consider the picture of low-energy collective modes. To appreciate this point, we include a small incoherent fluctuation about the saddle point $\psi = \psi_0 + \delta\psi$ and expand the logarithm in Eq. (6) to quadratic

order with respect to $\delta\psi$. The contribution from logarithm term $-\frac{1}{2}\mathrm{Tr}(M_0^{-1}\delta M M_0^{-1}\delta M)$, together with the bare photon action yields an effective action for fluctuations

$$S_{\mathrm{fluct}} = \frac{\beta}{2}\sum_{\omega}(\delta\bar{\psi}(\omega),\ \delta\psi(-\omega))G^{-1}(\omega)\begin{pmatrix}\delta\psi(\omega)\\ \delta\bar{\psi}(-\omega)\end{pmatrix}. \quad (13)$$

Here

$$G^{-1}(\omega) = \begin{pmatrix}K_1^* & K_2^*\\ K_2 & K_1\end{pmatrix} \quad (14)$$

is the inverse thermal Green's function for fluctuations, where

$$\begin{aligned}K_1 &= i\omega + \tilde{\omega}_c + 2\tilde{U}|\psi_0|^2 + \frac{g^2}{E}\tanh\left(\frac{1}{2}\beta E\right)\\ &\quad\times\frac{i\tilde{\varepsilon}\omega - 2\tilde{\varepsilon}^2 - g^2|\psi_0|^2}{\omega^2+4E^2} + \delta(\omega)\alpha g^2|\psi_0|^2,\\ K_2 &= 2\tilde{U}\bar{\psi}_0^2 + \frac{g^4|\psi_0|^2}{E(\omega^2+4E^2)}\tanh\left(\frac{1}{2}\beta E\right) + \delta(\omega)\alpha g^4|\psi_0|^2,\end{aligned} \quad (15)$$

with $\alpha = -(\beta/4E^2)\mathrm{sech}^2(\beta E/2)$. By action (13) the stability of our mean-field theory can be examined. It is provided that the mean-field condensed solution is stable against thermal fluctuations, and the unstable solution shows up when the phase transition takes place. The derivation of the above statements is similar to the one in Ref. [6] since only coherent photons participate in the condensation and the order parameter vanishes at the transition. In Eq. (15) the terms proportional to the Kronecker δ function stem from the summation over Fermionic Matsubara frequencies when we take the trace. These terms do not survive the analytic continuation, and so they do not appear in the retarded Green's function as well as in the collective excitation spectrum [9]. The calculation of the retarded Green's function $G^R(\omega)$ is straightforward by an analytic continuation $G^R(\omega) = G(i\omega_n = -\omega + \mu - i0^+)$. The poles of the retarded Green's function determine the collective modes which are experimentally appreciable [4,39]. In the normal state the thermal Green's function is exactly the same as in Ref. [6], and we do not intend to reproduce it here. However, in the condensed state we deduce that

$$G_{11}(i\omega) = \frac{(\tilde{\omega}_c + 2\tilde{U}|\psi_0|^2)(\omega^2 + 2g^2|\psi_0|^2) - i\omega[\omega^2 + 4E^2 + 2\tilde{\varepsilon}(\tilde{\omega}_c + 2\tilde{U}|\psi_0|^2)]}{(\omega^2+4E^2)^2 + A(\omega^2+4E^2) + B}, \quad (16)$$

where

$$\begin{aligned}A &= 4[\tilde{\varepsilon}(\tilde{\omega}_c + 2\tilde{U}|\psi_0|^2) - E^2] + \tilde{\omega}_c(\tilde{\omega}_c + 4\tilde{U}|\psi_0|^2),\\ B &= -4(\tilde{\omega}_c + 2\tilde{U}|\psi_0|^2)[(4\tilde{\varepsilon} + \tilde{\omega}_c\\ &\quad + 2\tilde{U}|\psi_0|^2)E^2 + 2\tilde{U}g^2|\psi_0|^4].\end{aligned} \quad (17)$$

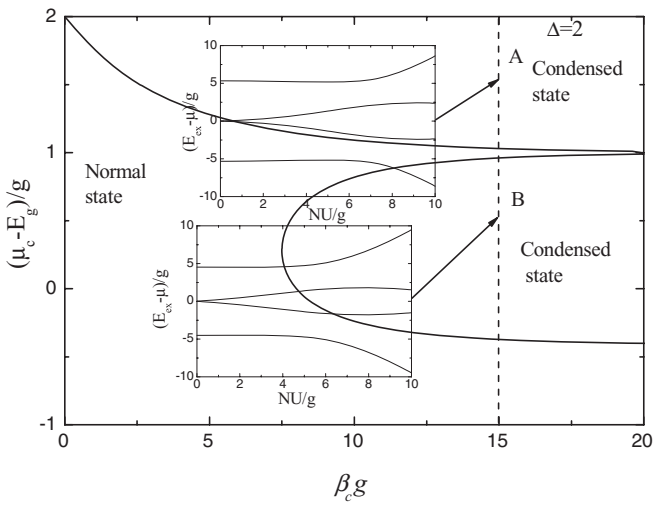

FIG. 4. The frequencies of collective modes as a function of the nonlinear coefficient (in units of g) superimposed on the phase boundary of critical chemical potential μ_c versus the transition temperature β_c for $\Delta = 2$, $(\mu - E_g)/g = 1.5$ (point A), $(\mu - E_g)/g = 0.5$ (point B), and along a constant temperature $g\beta = 15$, represented by a dashed line.

The poles of Eq. (16) are given as

$$(E_{\mathrm{ex}} - \mu)^2 = 4E^2 + \frac{A}{2} \pm \frac{1}{2}\sqrt{A^2 - 4B}, \quad (18)$$

from which a symmetric excitation spectrum around the chemical potential is predicted. These modes are formed by the collective excitations of quasiparticles. Notice that the quasiparticle energies are always gapped from weak to strong nonlinearity, and thus the collective modes is not damped at the low temperature regime.

For $\tilde{U} = 0$, the denominator of Eq. (16) reduces to $(i\omega)^2(i\omega + \xi)(i\omega - \xi)$ with $\xi = \sqrt{(\tilde{\omega}_c + 2\tilde{\varepsilon})^2 + 4g^2|\psi_0|^2}$. In this case, the $(i\omega)^2$ is associated with phase modes (Goldstone modes) and two poles at $i\omega = \pm\xi$ determine two collective modes, $E_\pm - \mu = \pm\xi$. For $\tilde{U} \neq 0$, the structure of the collective modes changes dramatically. There are four poles indicating four non-zero-frequency modes. In Fig. 4 we plot the excitation energies of these modes as a function of the nonlinear coefficient superimposed on the mean-field phase boundary [i.e., the curve of critical chemical potential μ_c versus the transition temperature β_c, which is obtained by requiring $\psi_0 = 0$ in Eq. (10)]. For the detuning case $\Delta = 2$, the corresponding chemical potential relates to the two markers labeled as A at $(\mu - E_g)/g = 1.5$ and B at $(\mu - E_g)/g = 0.5$, along a constant temperature $g\beta = 15$, represented by a dashed line. It is demonstrated that the inclusion of photon nonlinearity leads to mixed excitations of amplitude and phase modes. The zero-frequency modes no longer exist and it is bifurcated into a pair of non-zero-frequency modes. This means that the phase fluctuations of the condensate are now related to the amplitude

fluctuations and they will occur simultaneously. For point A, the excitation energies are a monotonous function with increasing nonlinear coefficient. When the chemical potential takes a smaller value at point B, the frequency of the collective modes bifurcated from phase modes first increase and then decrease with the nonlinear coefficient. The experimental observation of the excitation spectrum may be done as one may probe the response to inserting a real photon, and detecting its emission at a later time.

V. CONCLUSION

In summary, we have addressed the Bose condensation of polaritons with the Kerr-type nonlinear photons. We have introduced a generalized Dicke Hamiltonian with the pairwise repulsion of photons to model our system. By the path integral representation of the partition function, the thermodynamics of our model has been investigated and the consequences of the additional photon nonlinearity are examined. On the mean-field level, it is shown that the polariton condensation can occur and the strength of the Kerr nonlinearity affects the character of the condensate. When the nonlinearity is weaker than or comparable with the dipolar coupling, the emergence of spontaneous coherence is the condensate of photon-like polaritons. The photonic component is decreased with the increase of the nonlinear coefficient. However, when the nonlinearity is much stronger than the dipolar coupling, the coherent photon field exhibits a threshold behavior which is accompanied by a large excitons density. In this case the condensate is more exciton-like. Thus we point out that the Kerr nonlinearity has an effect of suppressing the photonic component in the condensate. Moreover, in our system the quasiparticle excitation energy is fully gapped, even though the strict condition that the chemical potential has to be less than the photon energy no longer exists. By going beyond the mean-field description, an effective action for thermal fluctuations is derived. We extract properties of the condensate collective modes from the thermal Green's function. With the presence of photon nonlinearity, a mixture of the phase modes with the amplitude modes forms two pairs of non-zero-frequency modes. The phase and amplitude fluctuations of the condensate are related by the photon nonlinearity.

The present paper is an extension of previous work in Ref. [6]. It would be of considerable interest to further extend our model including pumping and dissipation. Actually the dissipations will lead to nonequilibrium condensation. This could be done by the real-time Keldysh field theory [40], and it will be studied in a forthcoming publication.

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