Generalized space-translated Dirac and Pauli equations for superintense laser-atom interactions

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We obtain a generalization of the nonrelativistic space-translation transformation to the Dirac equation in the case of a unidirectional laser pulse. This is achieved in a quantum-mechanical representation connected to the standard Dirac representation by a unitary operator T transforming the Foldy-Wouthuysen free-particle basis into the Volkov spinor basis. We show that a solution of the transformed Dirac equation containing initially low momenta p ($p/mc \ll 1$) will maintain this property at all times, no matter how intense the field or how rapidly it varies (within present experimental capabilities). As a consequence, the transformed four-component equation propagates independently electron and positron wave packets, and in fact the latter are propagated via two two-component Pauli equations, one for the electron, the other for the positron. These we shall denote as the Pauli low-momentum regime (LMR) equations, equivalent to the Dirac equation for the laser field. Successive levels of dynamical accuracy appear depending on how accurately the operator T is approximated. At the level of accuracy considered in this paper, the Pauli LMR equations contain no spin matrices and are in fact two-component Schrödinger equations containing generalized time-dependent potentials. The effects of spin are nevertheless included in the theory because, in the calculation of observables which are formulated in the laboratory frame, use is made of the spin-dependent transformation operator T. In addition, the nonrelativistic limit of our results reproduces known results for the laboratory frame with spin included. We show that in intense laser pulses the generalized potentials can undergo extreme distortion from their unperturbed form. The Pauli LMR equation for the electron is applicable to one-electron atoms of small nuclear charge ($\alpha Z \ll 1$) interacting with lasers of all intensities and frequencies $\omega \ll mc^2$.

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I. INTRODUCTION

The relativistic study of laser-atom interactions has become timely because electrons are driven with quasiluminal velocities by superintense lasers already in operation or envisaged [1]. This requires treatment of the electron motion using relativistic dynamics and that of the radiation including retardation in its propagation. (For a background on relativistic laser-atom interactions, see [2,3].) As a preliminary step toward this goal, substantial effort has been invested in exploring the retardation corrections to the nonrelativistic (NR) Schrödinger equation within the dipole approximation. The limitations of this approach are that the fields need to be weak and the electron velocities NR. The Dirac equation, on the other hand, does not have these limitations but is much more difficult to handle. In three dimensions (3D) it is also notoriously difficult to integrate numerically at high laser intensities. Moreover, in its standard laboratory-frame version, it is opaque to physical insight. An alternative point of view was desirable.

In the case of the Schrödinger equation, such an alternative point of view was offered by its space-translated version; see [4,5]. Via a time-dependent space translation, the laser-atom interaction is described in a reference frame in which the classical electron is at rest, while the nucleus oscillates driven by the field. A requirement is that one can apply the dipole approximation, i.e., that the wavelengths be sufficiently long, as in the optical case. This approach had rewarding consequences, such as a semianalytic treatment at high optical frequencies, the discovery of atomic dichotomy and stabilization, and the possibility of a simplified numerical treatment. When retardation is taken into account, the main obstacle is that an oscillating space-translated reference frame can no longer be defined. A space-translated version of the Schrödinger equation with retardation has nevertheless been found and applied to ionization [6–8].

The generalization of the NR space-translation method to the Dirac equation was also shown to be possible, albeit it proves to be considerably more intricate. It was done a long time ago, but has attracted little attention and was barely pursued. The work focused on the case of *monochromatic plane-wave* radiation. Thus, Kaminski formulated the operator transform mediating the generalization in connection with laser-assisted scattering [9]. The transformed Dirac equation was expressed in terms of an integral potential operator containing the original potential embedded in a double sum over generalized Bessel functions depending on the frequency and amplitude of the radiation; see [9], Eq. (24). Although quite complicated, this expression enabled the desired goal, namely, the relativistic generalization of the low-frequency Kroll-Watson formula.

Some years later, Krstic and Mittleman (KM) derived the space-translated Dirac equation in connection with atomic structure in a monochromatic plane wave, with emphasis on the high-frequency case [10]. Their version of the space-translated Dirac equation is similar to that of Kaminski except that their integral potential operator is expressed in a different form,

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convenient for the low-momentum approximation needed in the treatment of the bound-state atomic problem. In this case the integral potential operator could be reduced to a potential function $\mathcal{U}(\mathbf{r},t)$; see [10], Eq. (2.30). KM studied only the high-frequency form of $\mathcal{U}(\mathbf{r},t)$, when cycle averaging is meaningful and yields an average potential $\mathcal{U}_0(\mathbf{r})$ [see [10], Eqs. (2.33) and (2.34)]. $\mathcal{U}_0(\mathbf{r})$ generalizes the dressed potential of the NR case [4]. $\mathcal{U}_0(\mathbf{r})$ was found to display a figure-8 logarithmic singularity for linear polarization, and an estimate for its ground-state energy was given. In a subsequent paper [11], KM considered spin corrections to their equation. Several details about the differences between the KM approach and ours are given throughout this paper. The work of KM was continued by Ermolaev [12], who studied the properties of $\mathcal{U}_0(\mathbf{r})$ and computed the corresponding levels for a 1D model.

We consider that the space-translation approach to the Dirac equation could yield promising results, both in terms of physical insight and as a numerical avenue. In terms of the physical understanding, if carried through consistently, it could transcend the numerous results obtained by considering first-order retardation corrections to the Schrödinger equation as well as other approximations, for processes such as ionization and high-order harmonic generation. Moreover, by writing the Dirac equation in a different way, one might get answers to fundamental problems, such as the limitations of relativistic one-electron theory in superintense fields because of pair-production pheneomena, and information as to when one should resort to QED theory [13]. On the numerical side, any tractable algorithm derived from it would represent an advancement, as in 3D and with superintense fields the ordinary Dirac equation is still not tractable (e.g., see [14,15]; for the 1D and 2D cases, see [16] and references therein).

Therefore, we address in this paper the issue of finding the space-translated version of the Dirac equation for the realistic case of a *laser pulse* interacting with an atom. No limitations are imposed on the intensity of the field. We endeavor to give a transparent account of the rather tedious mathematics. We start with some remarks on Volkov spinors (Sec. II). Then the generalized space-translated transformation operator for laser pulses is formulated in terms of Volkov spinors. This operator is used to derive the exact form of the generalized space-translated Dirac equation in the quantum-mechanical representation, denoted the translated Dirac picture (Secs. III and IV). We indicate in Sec. V that there are reasons to assume that in the translated Dirac picture current laser-atom interactions can be described by spinors pertaining to the low-momentum regime (LMR). Indeed, we show that for LMR spinors the generalized translated Dirac equation simplifies substantially when we work with electron or positron wave packets; it is equivalent at all times to two independent Pauli equations, one for the electron case and the other for the positron case (Secs. VI–VIII). This shows that LMR spinors are appropriate to describe in the translated Dirac picture current laser-atom interactions. We designate the Pauli equations obtained as the LMR equations. The procedure to extract physical information from the formalism is discussed in Sec. IX. In Sec. X we analyze the properties of the generalized space-translated potential entering the Pauli LMR equations. The NR limit of these equations is considered in Sec. XI. In Sec. XII we comment on our results.

II. RELATIVISTIC-FIELD-DRIVEN QUANTUM MECHANICAL AND CLASSICAL ELECTRONS

We recall some basic facts, essential for what follows. The relativistic quantum-mechanical motion of a free electron (mass m, charge $e<0$) in a radiation field of vector potential $\mathbf{a}$ (we assume the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{a}=0$) is described by the Dirac equation

$$[c\boldsymbol{\alpha}\cdot(\mathbf{P}-mc\mathbf{a})+mc^2\beta]\Psi=i\frac{\partial\Psi}{\partial t}. \tag{1}$$

Here $\mathbf{P}$ is the momentum operator, $\boldsymbol{\alpha}$ and β are the standard Dirac matrices defined as in [17], and we have set $\hbar=1$ [18]. Throughout this paper the radiation is assumed to be a laser pulse of arbitrary shape and magnitude propagating in one direction $\mathbf{n}$, so that the vector potential is a function of $\tau\equiv t-\mathbf{n}\cdot\mathbf{r}/c$ only, i.e., $\mathbf{a}=\mathbf{a}(\tau)$. Transversality of the light oscillations in the Coulomb gauge requires that $\mathbf{a}(\tau)\cdot\mathbf{n}=0$. We shall assume that $\mathbf{a}(\tau)$ vanishes sufficiently rapidly for $\tau\to\pm\infty$, so that the electron can be considered initially and finally as free. We thereby exclude the case of a monochromatic plane wave with fixed amplitude in space-time.

Equation (1) can be solved exactly for an arbitrary field $\mathbf{a}(\tau)$ in terms of Volkov spinors [19]. These can be written as

$$\Psi_{\mathbf{p}\sigma}(\mathbf{r},t)=\Omega_{\mathbf{p}\sigma}(\tau)\,e^{i\,\Lambda_{\mathbf{p}\sigma}(\tau)}u_{\mathbf{p}\sigma}(\mathbf{r},t). \tag{2}$$

Here,

$$u_{\mathbf{p}\sigma}(\mathbf{r},t)=\frac{1}{(2\pi)^{3/2}}e^{i(\mathbf{p}\cdot\mathbf{r}-E_{\mathbf{p}\sigma}t)}\,\xi_{\mathbf{p}\sigma}\quad(\sigma=1,\ldots,4), \tag{3}$$

is the free-particle eigenspinor for momentum $\mathbf{p}$ and definite spin projection on the direction of $\mathbf{p}$ in the laboratory frame. The subscript values $\sigma=1,2$ correspond to positive-energy (electron) states, $E_{\mathbf{p}\sigma}=+|E_{\mathbf{p}}|\equiv\sqrt{m^2c^4+c^2\mathbf{p}^2}$, and $\sigma=3,4$ to negative-energy (positron) states, $E_{\mathbf{p}\sigma}=-|E_{\mathbf{p}}|$. We shall refer to the spinor basis $\{\xi_{\mathbf{p}\sigma}\}$ as the standard basis set. Further,

$$\Omega_{\mathbf{p}\sigma}(\tau)\equiv I+\frac{mc^2}{2(E_{\mathbf{p}\sigma}-c\mathbf{p}\cdot\mathbf{n})}\hat{v}\hat{a}(\tau) \tag{4}$$

and

$$\Lambda_{\mathbf{p}\sigma}(\tau)\equiv\frac{(mc^2)^2}{2(E_{\mathbf{p}\sigma}-c\mathbf{p}\cdot\mathbf{n})}\int_{-\infty}^{\tau}\left[\frac{2}{mc}\mathbf{p}\cdot\mathbf{a}(\chi)-\mathbf{a}^2(\chi)\right]d\chi. \tag{5}$$

Here, we have introduced the four-vectors $v(1,\mathbf{n})$ and $a(0,\mathbf{a})$, and, given the four-vector $q_\mu(q_0,\mathbf{q})$, we have denoted $\hat{q}\equiv q_0\gamma_0-\mathbf{q}\cdot\boldsymbol{\gamma}$, where $\gamma_0=\beta$, $\boldsymbol{\gamma}=\beta\boldsymbol{\alpha}$. Note that

$$\lim_{\tau\to-\infty}\Psi_{\mathbf{p}\sigma}(\mathbf{r},t)=u_{\mathbf{p}\sigma}(\mathbf{r},t).$$

The set $\{\Psi_{\mathbf{p}\sigma}(\mathbf{r},t)\}$ is orthonormal [20] and complete [21] at all times t.

For the same field conditions as above, the *relativistic classical motion* of an electron (charge $e<0$) or positron (charge $-e>0$), initially at rest at the origin ($\mathbf{r}_i=\mathbf{p}_i=\mathbf{0}$ for $t\to-\infty$), represented by $\boldsymbol{\gamma}_{e|p}(t)$, is given by the equations [e.g., see [2], Eq. (27), with $\mathbf{r}_0=\mathbf{0},\boldsymbol{\beta}_0=\mathbf{0}$, $\gamma_0=1$ at $t\to-\infty$]:

$$\boldsymbol{\gamma}_{e|p}(t)=\boldsymbol{\gamma}_{\perp}^{e|p}(t)+\boldsymbol{\gamma}_{\parallel}(t), \tag{6}$$

$$\boldsymbol{\gamma}_{\perp}^{e|p}(t) = \mp c \int_{-\infty}^{t-\boldsymbol{\gamma}_{\parallel}(t)/c} \mathbf{a}(\chi) d\chi,$$
$$\boldsymbol{\gamma}_{\parallel}(t) = \frac{c}{2}\mathbf{n} \int_{-\infty}^{t-\boldsymbol{\gamma}_{\parallel}(t)/c} a^2(\chi) d\chi. \tag{7}$$

$\boldsymbol{\gamma}_{\perp}^{e|p}(t)$ represents the oscillatory motion of the two particles in the direction of the electric field, which obviously is in opposite directions for the two particles [18]. $\boldsymbol{\gamma}_{\parallel}(t)$ represents the drift motion of the particles, parallel to the propagation direction, which is the same for both. The equation for $\boldsymbol{\gamma}_{\parallel}$ needs to be solved first and $\boldsymbol{\gamma}_{\perp}^{e|p}$ calculated thereafter. Equation (7) gives for the position at the end of the pulse [22]

$$\mathbf{r}_f \equiv \boldsymbol{\gamma}_{e|p}(\infty) = \mp c \int_{-\infty}^{+\infty} \mathbf{a}(\chi) d\chi + \frac{c}{2}\mathbf{n} \int_{-\infty}^{+\infty} a^2(\chi) d\chi. \tag{8}$$

In the NR limit and for the dipole approximation of a weak field [$\mathbf{a}(\tau) \simeq \mathbf{a}(t)$ and $O\left(a^2\right)$ neglected], Eq. (6) reduces to

$$\boldsymbol{\gamma}_{\mathrm{NR}}(t) = \mp c \int_{-\infty}^{t} \mathbf{a}(t') dt'. \tag{9}$$

III. GENERALIZED SPACE-TRANSLATED DIRAC EQUATION

We define the space-translation transformation as the transformation mapping a free-particle basis set in the absence of the field into a free-particle basis set in the presence of the field. In the NR case of the Schrödinger equation with the dipole approximation, this is equivalent to a change of physical reference frame, from the frame in which the classical electron is at rest to the laboratory frame in which the particle oscillates. With retardation included, the latter definition cannot be upheld, as a space-translated reference frame cannot be introduced if the field does not oscillate in phase throughout space. Moreover, in the relativistic case, because of the existence of spin, one needs to specify the spin basis sets considered.

There are various possibilities for selecting a complete set of Dirac free-particle spinors $\{v_{\mathbf{p}\sigma}(\mathbf{r},t)\}$, depending on the representation chosen for the Dirac matrices. In order to simplify the physical interpretation in terms of electrons and positrons (see Sec. VIII), we shall choose the set $\{v_{\mathbf{p}\sigma}(\mathbf{r},t)\}$ to be that of the Foldy-Wouthuysen (FW) representation i.e., satisfying the free-particle Dirac equation in the form ([23]; see also [17], Chap. XX, Secs. 32 and 33)

$$H'\Psi' = i\frac{\partial \Psi'}{\partial t}, \quad H' \equiv E\beta, \tag{10}$$

where E is the operator

$$E \equiv +(m^2c^4 + c^2\mathbf{P}^2)^{1/2}. \tag{11}$$

$v_{\mathbf{p}\sigma}(\mathbf{r},t)$ has the simple form

$$v_{\mathbf{p}\sigma}(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}} e^{i(\mathbf{p}\cdot\mathbf{r} - E_{\mathbf{p}\sigma}t)}\zeta_\sigma \quad (\sigma = 1,\ldots,4), \tag{12}$$

with

$$\zeta_\sigma = \begin{pmatrix} \chi_\sigma \\ 0 \end{pmatrix}, \quad \zeta_{\sigma+2} = \begin{pmatrix} 0 \\ \chi_\sigma \end{pmatrix}; \quad \sigma = 1,2,$$

and χ_σ are two-component spinors, which can be taken as (1,0), or (0,1). The FW free-particle spinor basis $\{\zeta_\sigma\}$ is connected to the standard basis $\{\xi_{\mathbf{p}\sigma}\}$ by

$$\xi_{\mathbf{p}\sigma} = S_{\mathrm{FW}}(\mathbf{p})\zeta_\sigma, \tag{13}$$

where $S_{\mathrm{FW}}(\mathbf{p})$ is the unitary transformation matrix

$$S_{\mathrm{FW}}(\mathbf{p}) = g[(|E_{\mathbf{p}}| + mc^2)\beta + c\boldsymbol{\alpha}\cdot\mathbf{p}]\beta;$$
$$g \equiv [2|E_{\mathbf{p}}|(|E_{\mathbf{p}}| + mc^2)]^{-1/2}; \tag{14}$$

see [23], Eq. (16).

We now introduce the generalized relativistic space-translation operator as the operator transforming the FW set of Dirac free-particle spinors $\{v_{\mathbf{p}\sigma}(\mathbf{r},t)\}$ into the Volkov spinors $\{\Psi_{\mathbf{p}\sigma}(\mathbf{r},t)\}$, Eq. (2). As both sets are orthonormal and complete at all t, this defines a unitary operator $T(t)$ on the Hilbert space $\mathcal{H}$ of quatum-mechanical spinors:

$$\Psi_{\mathbf{p}\sigma} = T(t)v_{\mathbf{p}\sigma}. \tag{15}$$

In coordinate representation the action of T on an arbitrary spinor $\Phi(\mathbf{r}) \in \mathcal{H}$ is defined by the integral operator

$$T(t)\Phi(\mathbf{r}) = \int \mathcal{T}(\mathbf{r},\mathbf{r}',t)\Phi(\mathbf{r}')d\mathbf{r}', \tag{16}$$

where [24]

$$\mathcal{T}(\mathbf{r},\mathbf{r}',t) \equiv \sum_{\sigma=1,\ldots,4} \int \Psi_{\mathbf{p}\sigma}(\mathbf{r},t)v^\dagger_{\mathbf{p}\sigma}(\mathbf{r}',t)d\mathbf{p}. \tag{17}$$

We recall that if the integral operator T is represented by the kernel $\mathcal{T}\left(\mathbf{r},\mathbf{r}',t\right)$, its adjoint $T^\dagger$ is represented by the kernel $\mathcal{T}^\dagger\left(\mathbf{r},\mathbf{r}',t\right) \equiv \left[\mathcal{T}\left(\mathbf{r}',\mathbf{r},t\right)\right]^{*tr}$, where the superscripts $*$ and tr stand for matrix complex conjugation and transposition. From Eq. (17),

$$\mathcal{T}^\dagger(\mathbf{r},\mathbf{r}',t) \equiv \sum_{\sigma=1,\ldots,4} \int v_{\mathbf{p}\sigma}(\mathbf{r},t)\,\Psi^\dagger_{\mathbf{p}\sigma}(\mathbf{r}',t)\,d\mathbf{p}\,. \tag{18}$$

As T is unitary, $TT^\dagger = T^\dagger T = I$.

It is not difficult to see that in the NR limit and with the dipole approximation, ignoring spin, T is expressible as

$$T_{\mathrm{NR}}(t) = \exp\left(\mp i\frac{mc^2}{2}\int_{-\infty}^{t}\mathbf{a}^2\left(t'\right)dt'\right)e^{-i\,\boldsymbol{\gamma}_{\mathrm{NR}}(t)\cdot\mathbf{P}}\,, \tag{19}$$

where $\boldsymbol{\gamma}_{\mathrm{NR}}(t)$ is the NR trajectory of the particle, Eq. (9). Apart from an irrelevant time-dependent phase factor, $T_{\mathrm{NR}}(t)$ stands for a *translation* of vector $\boldsymbol{\gamma}_{\mathrm{NR}}(t)$ representing the passage from the laboratory frame to an oscillating reference frame (e.g., see [4]). By contrast, the relativistic operator $T(t)$ does not represent a change of physical reference frame, but rather a change of quantum-mechanical representation. Nevertheless, because of its underlying NR limit, we shall designate $T(t)$ as the *generalized translation operator*.

We now use the unitary transformation $\Psi = T(t)\,\Psi'$ to transform the standard Dirac equation containing the field $\mathbf{A}(\tau)$ and the atomic potential $V(\mathbf{r})$:

$$[c\boldsymbol{\alpha}\cdot(\mathbf{P} - mc\mathbf{a}) + mc^2\beta + V(\mathbf{r})I]\Psi = i\frac{\partial\Psi}{\partial t} \tag{20}$$

to

$$H'\Psi' = i\frac{\partial\Psi'}{\partial t}, \quad H' \equiv E\beta + V'(t), \tag{21}$$

where E is defined by Eq. (11) and

$$V'(t) \equiv T^{\dagger}(t) V T(t) \tag{22}$$

is the generalized space-translated potential. The radiation field $\mathbf{a}(t)$ has been entirely absorbed in the potential V', which is a complicated integral operator. Its kernel is given by

$$\mathcal{V}'(\mathbf{r},\mathbf{r}',t) = \int \mathcal{T}^{\dagger}(\mathbf{r},\mathbf{r}'',t) V(\mathbf{r}'') \mathcal{T}(\mathbf{r}'',\mathbf{r}',t) d\mathbf{r}''. \tag{23}$$

In the following we shall have in mind the realistic case of the Coulomb potential $V(r) = -Z/r$, although our calculations may apply to more general model potentials.

Equation (21) will be designated as the *generalized translated Dirac equation*. Quantities pertaining to this equation and its interpretation will be referred to as belonging to the *translated Dirac picture*.

At this point it is useful to address an issue of terminology. The Dirac equation (20) for the negative electron in the absence of a radiation field has energy eigenstates, some of positive energy (discrete and continuous), associated with the threshold $+mc^2$. These will be termed electron states and any linear combination of them forms an electron wave packet. There are also the continuous eigenstates of negative energy, below the threshold $-mc^2$, termed positronic states due to their reinterpretation in QED. We refer to any wave packet made of these as a positron wave packet. At small Z ($\alpha Z \ll 1$) there is no problem in distinguishing the two kinds of states (there is no level "diving" into the negative-energy "sea").

IV. EXACT EXPRESSION FOR V'

For the calculation of V', Eq. (22), it is convenient to separate the contribution of the positive and negative energies in the exact $T(t)$, Eq. (17), according to

$$T = T_+ + T_-. \tag{24}$$

Hence, V' can be written as [see Eq. (23)]

$$V' \equiv T^{\dagger} V T = V'_{++} + V'_{--} + V'_{+-} + V'_{-+}, \tag{25}$$

where we have denoted $V'_{\pm\pm} \equiv T^{\dagger}_{\pm} V T_{\pm}$ and $V'_{\pm\mp} \equiv T^{\dagger}_{\pm} V T_{\mp}$. Here and in the following the upper subscripts represent one alternative and the lower subscripts the other.

According to Eqs. (17), (2), and (12), we have the exact expression

$$\mathcal{T}_{\pm}(\mathbf{r}'',\mathbf{r}',t) \equiv \sum_{\sigma} \frac{1}{(2\pi)^3} \int d\mathbf{p} e^{i[\mathbf{p}\cdot(\mathbf{r}''-\mathbf{r}')+\Lambda_{\mathbf{p}\sigma}(\tau'')]} \Omega_{\mathbf{p}\sigma}(\tau'') \xi_{\mathbf{p}\sigma}\, \zeta^{\dagger}_{\sigma}. \tag{26}$$

Here, for subscript $+$ the sum over σ extends over $\sigma = 1,2$, and for subscript $-$, over $\sigma = 3,4$. By introducing the notation $\Lambda^{\pm}_{\mathbf{p}} \equiv \Lambda_{\mathbf{p}\sigma}, \Omega^{\pm}_{\mathbf{p}} \equiv \Omega_{\mathbf{p}\sigma}$, where the superscript $+$ corresponds to $\sigma = 1,2$ and the superscript $-$ corresponds to $\sigma = 3,4$, we can write Eq. (26) as

$$\mathcal{T}_{\pm}(\mathbf{r}'',\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3} \int d\mathbf{p} e^{i[\mathbf{p}\cdot(\mathbf{r}''-\mathbf{r}')+\Lambda^{\pm}_{\mathbf{p}}(\tau'')]} \Omega^{\pm}_{\mathbf{p}}(\tau'') \sum_{\sigma} \xi_{\mathbf{p}\sigma}\, \zeta^{\dagger}_{\sigma}. \tag{27}$$

Using Eq. (13), we have

$$\sum_{\sigma} \xi_{\mathbf{p}\sigma}\, \zeta^{\dagger}_{\sigma} = S_{\mathrm{FW}}(\mathbf{p}) \frac{I \pm \beta}{2}. \tag{28}$$

Here again, the sum over σ extends over 1,2 or 3,4, and one should choose $+$ or $-$ on the right-hand side, respectively. The matrices $(I \pm \beta)/2$ are the projection operators for the electron and positron subspaces of the FW representation. By denoting

$$M_{\pm}(\mathbf{p};\tau) \equiv \Omega^{\pm}_{\mathbf{p}}(\tau) S_{\mathrm{FW}}(\mathbf{p}), \tag{29}$$

we have

$$\mathcal{T}_{\pm}(\mathbf{r}'',\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3} \int d\mathbf{p}\, M_{\pm}(\mathbf{p};\tau'') e^{i[\mathbf{p}\cdot(\mathbf{r}''-\mathbf{r}')+\Lambda^{\pm}_{\mathbf{p}}(\tau'')]} \frac{I \pm \beta}{2}. \tag{30}$$

Similarly, for the *exact* $\mathcal{T}^{\dagger}_{\pm}$, we have

$$\mathcal{T}^{\dagger}_{\pm}(\mathbf{r},\mathbf{r}'',t) = \frac{I \pm \beta}{2} \frac{1}{(2\pi)^3} \int M^{\dagger}_{\pm}(\mathbf{p}',\tau'')\, e^{-i[\mathbf{p}'\cdot(\mathbf{r}''-\mathbf{r})+\Lambda^{\pm}_{\mathbf{p}'}(\tau'')]} d\mathbf{p}', \tag{31}$$

where $\tau'' = t - \mathbf{n}\cdot\mathbf{r}''/c$, and $M^{\dagger}_{\pm}(\mathbf{p},\tau'')$ is the adjoint of the matrix Eq. (29).

The kernel $\mathcal{V}'(\mathbf{r},\mathbf{r}',t)$ of V' is given by Eq. (23). We decompose $\mathcal{V}'(\mathbf{r},\mathbf{r}',t)$ as in Eq. (25) and write the Fourier transform of the (negative electron) potential energy as

$$V(\mathbf{r}) \equiv \int \mathcal{V}(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{q}. \tag{32}$$

This gives, for example,

$$\begin{aligned}\mathcal{V}'_{\pm\pm}&(\mathbf{r},\mathbf{r}',t) \\ &= \frac{1}{(2\pi)^6} \int d\mathbf{q} \mathcal{V}(\mathbf{q}) \int d\mathbf{r}'' \int d\mathbf{p} \int d\mathbf{p}' \frac{I \pm \beta}{2} \\ &\quad \times M^{\dagger}_{\pm}(\mathbf{p}';\tau'') M_{\pm}(\mathbf{p};\tau'') \frac{I \pm \beta}{2} \exp[i\{(\mathbf{p}-\mathbf{p}'+\mathbf{q})\cdot\mathbf{r}'' \\ &\quad + \mathbf{p}'\cdot\mathbf{r} - \mathbf{p}\cdot\mathbf{r}' + \Lambda^{\pm}_{\mathbf{p}}(\tau'') - \Lambda^{\pm}_{\mathbf{p}'}(\tau'')\}].\end{aligned} \tag{33}$$

As mentioned above, the upper signs represent one alternative and the lower signs the other. Four of the twelve integrals in the expression Eq. (33) can be carried out immediately. By decomposing $d\mathbf{r}'' = d\mathbf{r}''_{\perp} dz''$, where $\perp$ denotes the direction perpendicular to $\mathbf{n}$ and z is the direction parallel to $\mathbf{n}$, the integration over $\mathbf{r}''_{\perp}$ gives $\delta(\mathbf{p}_{\perp} - \mathbf{p}'_{\perp} + \mathbf{q}_{\perp})$, so that we have the exact formula

$$\begin{aligned}\mathcal{V}'_{\pm\pm}&(\mathbf{r},\mathbf{r}',t) \\ &= \frac{1}{(2\pi)^4} \int d\mathbf{q} \mathcal{V}(\mathbf{q}) \int dz'' \int d\mathbf{p} \int dp'_z \\ &\quad \times \frac{I \pm \beta}{2} M^{\dagger}_{\pm}(\mathbf{p}_{\perp} + \mathbf{q}_{\perp}, p'_z;\tau'') M_{\pm}(\mathbf{p}_{\perp}, p_z;\tau'') \frac{I \pm \beta}{2} \\ &\quad \times \exp[i\{(p_z - p'_z + q_z)z'' + (\mathbf{p}_{\perp} + \mathbf{q}_{\perp})\cdot\mathbf{r}_{\perp} \\ &\quad + p'_z z - \mathbf{p}_{\perp}\cdot\mathbf{r}'_{\perp} - p_z z' + \Lambda^{\pm}_{\mathbf{p}_{\perp}, p_z}(\tau'') - \Lambda^{\pm}_{\mathbf{p}_{\perp}+\mathbf{q}_{\perp}, p'_z}(\tau'')\}].\end{aligned} \tag{34}$$

Proceeding similarly, an exact expression can be obtained also for $\mathcal{V}'_{\pm\mp}(\mathbf{r},\mathbf{r}',t)$:

$$\begin{aligned}\mathcal{V}'_{\pm\mp}(\mathbf{r},\mathbf{r}',t) &= \frac{1}{(2\pi)^6}\int d\mathbf{q}\mathcal{V}(\mathbf{q})\int dz''\int d\mathbf{p}\int dp'_z \\ &\times\frac{I\pm\beta}{2}M^{\dagger}_{\pm}(\mathbf{p}_\perp+\mathbf{q}_\perp,p'_z;\tau'')M_{\mp}(\mathbf{p}_\perp,p_z;\tau'')\frac{I\mp\beta}{2} \\ &\times\exp[i\{(p_z-p'_z+q_z)z''+(\mathbf{p}_\perp+\mathbf{q}_\perp)\cdot\mathbf{r}_\perp \\ &+p'_z z-\mathbf{p}_\perp\cdot\mathbf{r}'_\perp-p_z z'+\Lambda^{\pm}_{\mathbf{p}_\perp,p_z}(\tau'')-\Lambda^{\mp}_{\mathbf{p}_\perp+\mathbf{q}_\perp,p'_z}(\tau'')\}]. \end{aligned} \tag{35}$$

V. ATOMIC LOW-MOMENTUM REGIME (LMR)

As Eqs. (20) and (21) are unitarily equivalent, use of one or the other is a matter of convenience, provided that the observables are properly transformed. Both equations can treat all physical phenomena that can be described within single-electron Dirac theory, involving one electron of any speed, photons of any energy, and nuclear potentials of any charge Z. Some of these are typical high-energy phenomena deriving from high-energy initial conditions, like fast-electron–heavy-nucleus collisions in the presence of radiation, leading to scattering with absorption or emission of high-energy photons ($\omega\ (\omega\gtrsim mc^2)$). Physically, pair-production phenomena (requiring photon energies larger than $2mc^2$) become possible, but these lie out of the range of single-electron Dirac theory. On the other hand, current *superintense laser phenomena* in the presence of small-Z atoms (e.g., laser-modified atomic structure, multiphoton ionization, free-free transitions) imply NR photon energies ($\omega\ll mc^2$) and low-momentum initial conditions ($p\ll mc$). These phenomena may need relativistic treatment because the electron can acquire velocities close to c due to the superintense fields acting on it. It is this kind of phenomenon that we want to address here.

For a relativistic description of such phenomena one could use the Dirac equation for the laboratory frame, Eq. (20), with an initial condition $\Psi_0(\mathbf{r})$ containing low NR momenta. During its evolution the wave spinor $\Psi(\mathbf{r},t)$ could acquire relativistic momenta $p\gtrsim mc$ in superintense fields. Alternatively, one could use the generalized space-translated Dirac equation Eq. (21), starting from an equivalent initial condition $\Psi'_0(\mathbf{r})$ given by $\Psi_0(\mathbf{r})=T(-\infty)\Psi'_0(\mathbf{r})$. As in this picture one has extracted the highly relativistic oscillatory motion of the free electron, it is conceivable that, for atomic-type processes with NR initial conditions, the solution $\Psi'(\mathbf{r},t)$ might contain only low momenta at all times. This would imply that, if $\Psi'(\mathbf{r},t)$ starts as an electron wave packet, it could not acquire positron components, and vice versa. Moreover, the single-electron Dirac equation would be adequate for treating the process considered and a QED description would not be needed. It might also allow the reduction of the generalized translated Dirac equation (21) to a simpler form. This is the problem we want to investigate in the following.

To fix the mathematical framework of the problem, let us consider the spinors of the translated Dirac picture which contain only low momenta $p<\varpi\ll mc$, where ϖ is a cutoff. We shall refer to them as pertaining to the atomic low-momentum regime (LMR). Such spinors Ψ' are characterized by the fact that their Fourier expansion

$$\Psi'(\mathbf{r})=\sum_{\sigma=1,\ldots,4}\int_{p\lesssim\varpi}c_{\mathbf{p}\sigma}v_{\mathbf{p}\sigma}(\mathbf{r},0)d\mathbf{p}, \tag{36}$$

with $v_{\mathbf{p}\sigma}(\mathbf{r},t)$ defined by Eq. (12), contains a cutoff $c_{\mathbf{p}\sigma}\simeq 0$, for $p>\varpi$ and all σ, i.e., the integration domain in Eq. (36) is limited to $p\lesssim\varpi$ [25]. Mathematically, the set of all LMR spinors Ψ' forms a subspace $\mathcal{D}$ of the quantum-mechanical Hilbert space $\mathcal{H}$ ($\mathcal{D}\subset\mathcal{H}$).

Because of the charge symmetry of the Dirac equation and the possibility of coupling between the positive-energy and the negative-energy components of the wave spinor Ψ', in the beginning we allow in the expansion Eq. (36) of LMR spinors electron as well as positron states $v_{\mathbf{p}\sigma}(\mathbf{r})$. The introduction of the cutoff at $\varpi\ll mc$ is the only restriction made on the dynamics in our calculation. The ratio ϖ/mc will serve to classify in the following the orders of magnitude encountered. Note that, for the bound states of a Coulomb potential, this limitation implies that the internal momenta, of order αZmc, should be small with respect to mc, i.e., $\alpha Z\ll 1$, and consequently Z should be sufficiently small. In this case there is a clear-cut distinction between the electron and positron states.

With the definitions above, we can rephrase the problem we shall be studying in the following (see Secs. V–X) as follows: if a solution of the generalized space-translated Dirac equation belongs initially to $\mathcal{D}$, will it remain in $\mathcal{D}$ during its time evolution according to the transformed equation (21)?

We outline our strategy for solving the problem. We shall apply the operator V', Eq. (22), to a LMR spinor $\Psi'(\mathbf{r})$ of the translated Dirac picture, and then the Hamiltonian H', Eq. (21). In this paper we shall present only the result for $H'\Psi'$ to lowest order in ϖ/mc, which we shall denote $O(1)$. The ordering in powers of ϖ/mc will occur naturally, by series expansion in p/mc of the Volkov spinors contained in the operator $\mathcal{T}(\mathbf{r},\mathbf{r}',t)$, Eq. (17). Having obtained $H'\Psi'$ we can find out if the solution of the generalized space-translated equation $H'\Psi'=i\partial\Psi'/\partial t$ remains within $\mathcal{D}$ at all times.

We emphasize that throughout the paper we endeavor to make no approximations regarding the magnitude of the field $a(\tau)$ or its retardation (dependence on τ). On the other hand, we anticipate here that in the course of the calculation there will appear the following condition on its space-time variation (see end of Sec. VI):

$$\frac{da(\tau)}{d\tau}\ll mc^2. \tag{37}$$

It immediately follows that the magnitude of the electric field should be limited by $E\ll E_S\equiv m^2c^3$, which is the so-called Schwinger field (see [1,2]). The latter has the very large value of $E_S\simeq 2.6\times 10^6$ a.u., corresponding to an intensity of $I_S=2.3\times 10^{29}$ W/cm^2, which shows that the condition is irrelevant in practice.

In addition to the field intensity, another important physical characteristic of the laser pulse is the spectral energy range $\Delta\omega$ of the photons contained in it, which determines the energy of the ionized electrons. In order that the LMR restriction can be maintained at all during the evolution of the state spinor Ψ', it is obvious that we have to limit $\Delta\omega$ such that no relevant

multiphoton absorptions can generate relativistic momentum components in the spinor Ψ'. This means that the range $\Delta\omega$ should be NR: $\Delta\omega \ll mc^2$.

Let us consider the effect of applying the operator $T(t)$ to a LMR spinor $\Psi' \in \mathcal{D}$. From Eqs. (16) and (17) we can write

$$T(t)\Psi'(\mathbf{r}) = \sum_{\sigma=1,\ldots,4} \int d\mathbf{p}\, \Psi_{\mathbf{p}\sigma}(\mathbf{r},t) \int v^{\dagger}_{\mathbf{p}\sigma}(\mathbf{r}',t)\Psi'(\mathbf{r}')d\mathbf{r}'. \tag{38}$$

Because of the low momentum content of Ψ', Eq. (36), only $\Psi_{\mathbf{p}\sigma}(\mathbf{r},t)$ with subscript $\mathbf{p}$ of small magnitude ($p \lesssim \varpi$) will appear in Eq. (38), for which only the low-$\mathbf{p}$ form of Eq. (2) is needed. Stated differently, LMR wave packets Ψ' in the Dirac picture correspond in the laboratory frame wave packets $T\Psi'$ containing Volkov spinors with small values of the subscript $\mathbf{p}$. An approximation of $T(t)$ obtained by cutting off the integration over $\mathbf{p}$ in Eq. (38) at some ϖ will be denoted by $T^{ap}(t)$. We shall need the explicit form of $T^{ap}(t)$ only later; see Sec. IX, Eq. (76).

When acting on the LMR subspace $\mathcal{D} \subset \mathcal{H}$, the operator $T(t)$ maps it into a subspace $\Delta(t) \subset \mathcal{H}$. As $T(t)$ depends on a, so does $\Delta(t)$; we write therefore $\Delta_a(t)$. The correspondence established between the elements of $\mathcal{D}$ and $\Delta_a(t)$ is one to one.

In view of Eq. (18), the action of $T^{\dagger}$ on a $\Phi \in \Delta_a(t)$ is

$$T^{\dagger}(t)\Phi(\mathbf{r}) = \sum_{\sigma=1,\ldots,4} \int d\mathbf{p}\, v_{\mathbf{p}\sigma}(\mathbf{r},t) \int \Psi^{\dagger}_{\mathbf{p}\sigma}(\mathbf{r}',t)\Phi(\mathbf{r}')d\mathbf{r}'. \tag{39}$$

Since $T^{\dagger}\Phi(\mathbf{r}) \in \mathcal{D}$, this means that its Fourier components are limited by the same cutoff ϖ which appears for $T\Psi'$ in Eq. (38), i.e., the integral over $\mathbf{p}$ in Eq. (39) extends over $p \lesssim \varpi$. Thus, for $\Psi^{\dagger}_{\mathbf{p}\sigma}(\mathbf{r}',t)$ we can use the low-$\mathbf{p}$ approximation we have used for $\Psi_{\mathbf{p}\sigma}(\mathbf{r},t)$ in $T^{ap}(t)$. $T^{\dagger}$ with the $p \lesssim \varpi$ limitation will be denoted by $(T^{\dagger})^{ap}$.We conclude that on $\Delta_a(t)$ we have, for $T^{\dagger}$, $(T^{\dagger})^{ap} \simeq (T^{ap})^{\dagger}$.

In the following we are interested in the action of V' on a LMR spinor Φ ($\Phi \in \mathcal{D}$). As discussed above, this means that the integration range over $\mathbf{p}$ in Eqs. (34) and (35) has a cutoff at ϖ, and we can use low-p approximations for the quantities $\Lambda^{\pm}_{\mathbf{p}}(\tau'')$ and $M_{\pm}(\mathbf{p};\tau'')$ involved, which are simpler than the exact expressions. These we obtain from series expansions in p/mc.

For $\Omega^{\pm}_{\mathbf{p}}(\tau)$ and $\Lambda^{\pm}_{\mathbf{p}}(\tau)$ we find from Eqs. (4) and (5) the approximate expressions to first order in p/mc:

$$\Omega^{\pm}_{\mathbf{p}}(\tau) \simeq \left[I - \frac{1}{2}(I + \boldsymbol{\alpha}\cdot\mathbf{n})\left(\pm 1 + \frac{\mathbf{n}\cdot\mathbf{p}}{mc}\right)\right], \tag{40}$$

$$\Lambda^{\pm}_{\mathbf{p}}(\tau) \simeq \mathbf{p}\cdot\mathbf{R}^{\pm}(\tau) \mp \frac{mc^2}{2}\int_{-\infty}^{\tau} a^2(\chi)d\chi. \tag{41}$$

Here,

$$\mathbf{R}^{\pm}(\tau) = \mathbf{R}^{\pm}_{\perp}(\tau) + Z(\tau)\mathbf{n}, \tag{42}$$

$$\mathbf{R}^{\pm}_{\perp}(\tau) = \pm c\int_{-\infty}^{\tau} \mathbf{a}(\chi)d\chi; Z(\tau) = -\frac{c}{2}\int_{-\infty}^{\tau} a^2(\chi)d\chi. \tag{43}$$

By also approximating consistently $S_{\mathrm{FW}}(\mathbf{p})$:

$$S^{ap}_{\mathrm{FW}}(\mathbf{p}) \simeq I - \beta\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2mc}, \tag{44}$$

we find from Eq. (29) the expression for $M_{\pm}(\mathbf{p};\tau)$ to first order in p/mc:

$$M_{\pm}(\mathbf{p};\tau) \simeq I + \frac{\hat{\mathbf{v}}\hat{\mathbf{a}}(\tau)}{2}\left(\pm 1 + \frac{\mathbf{n}\cdot\mathbf{p}}{mc}\right) - \left[I \pm \frac{\hat{\mathbf{v}}\hat{\mathbf{a}}(\tau)}{2}\right]\beta\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2mc}. \tag{45}$$

Note that we have introduced above expressions up to first order in p/mc although we aim at a result for H' which is of *order* $O(1)$ .This is due to a peculiarity of the problem which will become evident in Sec. VII [26].

Whereas the expression of $\mathcal{T}_{\pm}(\mathbf{r}'',\mathbf{r}',t)$ in Eq. (23) could be approximated by invoking LMR conditions, this is not warranted *a priori* for the expression of $\mathcal{T}^{\dagger}_{\pm}(\mathbf{r},\mathbf{r}'',t)$. In principle, one should use for the latter its exact expression Eq. (31), with the integration over $\mathbf{p}'$ extended to infinity. However, we now show that this is not necessary and that we can limit the $\mathbf{p}'$ integration in Eqs. (33)–(35) by the same cutoff ϖ used for the $\mathbf{p}$ integration ($p' \lesssim \varpi$) and, consequently, use the same approximations Eqs. (40)–(45) for the $\mathbf{p}'$ integration in Eq. (33). Indeed, note first that the range of large momenta $p' \gtrsim mc$ in Eq. (33) corresponds to the range of large momenta $q_{\perp}, p'_z \gtrsim mc$ in Eqs. (34) and (35). Next, we compare the orders of magnitude of $\mathcal{V}(\mathbf{q})$ at LMR momenta $q = O(1)$ and at large momenta $q_{\perp} \simeq q = O(mc)$. For the Coulomb case, for which $\mathcal{V}(\mathbf{q}) \sim 1/q^2$, $\mathcal{V}(\mathbf{q})$ is $c^2 \simeq 18\,700$ smaller in the large-momentum case and hence negligible. [The presence of $M^{\dagger}_{\pm}(\mathbf{p}_{\perp}+\mathbf{q}_{\perp}, p'_z;\tau'')$—see Eq. (29)—does not alter this conclusion.] Thus, the range of large momenta $q_{\perp}, p'_z \gtrsim mc \gg \varpi$ in Eqs. (34) and (35) or $p' \gg \varpi$ in Eq. (33) gives indeed a negligible contribution, which proves our statement [27].

With the approximations above, V' will be denoted as V'^{ap}. In Secs. VI and VII we proceed to evaluate the terms of V'^{ap} according to the decomposition in Eq. (25).

VI. CALCULATION OF $\mathcal{V}'^{ap}_{\pm\pm}(\mathbf{r},\mathbf{r}',t)$

For $\mathcal{V}'^{ap}_{\pm\pm}(\mathbf{r},\mathbf{r}',t)$, Eq. (34), we calculate the approximate LMR expressions

$$\Lambda^{\pm}_{\mathbf{p}_{\perp},p_z}(\tau'') - \Lambda^{\pm}_{\mathbf{p}_{\perp}+\mathbf{q}_{\perp},p'_z}(\tau'') \simeq (p_z - p'_z)Z(\tau'') - \mathbf{q}_{\perp}\cdot\mathbf{R}^{\pm}_{\perp}(\tau''), \tag{46}$$

$$\begin{aligned}\frac{I\pm\beta}{2}M^{\dagger}_{\pm}(\mathbf{p}_{\perp}+\mathbf{q}_{\perp},p'_z;\tau'')M_{\pm}(\mathbf{p}_{\perp},p_z;\tau'')\frac{I\pm\beta}{2} &\simeq \left\{\left[1+\frac{a^2}{2}-\frac{1}{2mc}\mathbf{a}\cdot(2\mathbf{p}_{\perp}+\mathbf{q}_{\perp})\right]I\right.\\ &+\frac{1}{4mc}(p_z+p'_z)a^2\beta+\frac{i}{2mc}\left[\mathbf{a}\times\mathbf{q}_{\perp}\right.\\ &\left.\left.-\frac{1}{2}(\mathbf{q}_{\perp}\times\mathbf{n})a^2\beta\right]\cdot\boldsymbol{\Sigma}\right\}\frac{I\pm\beta}{2},\end{aligned} \tag{47}$$

where we have neglected terms in $(p/mc)^2$; all $\mathbf{a}$ depend on τ''. Upon inserting these expressions in Eq. (34), we can replace

the integration variable p_z' appearing on the right-hand side of Eq. (47) with the operator P_z acting on the variable z, over which one does not integrate. This allows the p_z' integration to be carried out, which leads to $\delta(z''-z+Z(\tau''))$. Hence,

$$\begin{aligned}\mathcal{V}'^{ap}_{\pm\pm}(\mathbf{r},\mathbf{r}',t) = \frac{1}{(2\pi)^3}\int d\mathbf{q}\mathcal{V}(\mathbf{q})\int dz''\int d\mathbf{p}\frac{I\pm\beta}{2}\\ \times M^{\dagger}_{\pm}(\mathbf{p}_\perp+\mathbf{q}_\perp,P_z;\tau'')M_{\pm}(\mathbf{p}_\perp,p_z;\tau'')\frac{I\pm\beta}{2}\\ \times\delta(z''-z+Z(\tau''))\exp[i\{(p_z+q_z)z''\\ +(\mathbf{p}_\perp+\mathbf{q}_\perp)\cdot\mathbf{r}_\perp-\mathbf{p}_\perp\cdot\mathbf{r}'_\perp-p_zz'+p_zZ(\tau'')\\ -\mathbf{q}_\perp\cdot\mathbf{R}^{\pm}_\perp(\tau'')\}].\end{aligned}$$

We would like to integrate next over z'', but the δ function does not have a suitable form. However, by using known formulas [28] we can write

$$\begin{aligned}\delta(z''-z+Z(\tau'')) &= \frac{1}{c}\delta\left(\tau''-\tau-\frac{1}{c}Z(\tau'')\right)\\ &= \frac{1}{c}\frac{1}{1+a^2(\overline{\tau})/2}\,\delta(\tau''-\overline{\tau}(\tau)),\end{aligned}\tag{48}$$

where $\overline{\tau}(\tau)$ is the solution of the equation

$$\overline{\tau}=\tau+\frac{1}{c}Z(\overline{\tau}).\tag{49}$$

With the definition of $Z(\tau)$, Eqs. (42) and (43), we can write Eq. (49) as

$$\overline{\tau}=\tau-\frac{1}{2}\int_{-\infty}^{\overline{\tau}}a^2(\chi)d\chi.\tag{50}$$

Setting $\overline{\tau}\equiv\tau+\zeta$ here results in an equation for $\zeta(\tau)$ which is identical to the equation for $\boldsymbol{\gamma}_\parallel(t)$ in Eq. (7) with t replaced by τ, if one sets $\boldsymbol{\zeta}(\tau)=-\boldsymbol{\gamma}_\parallel(\tau)/c$. We recall that we are denoting by $\boldsymbol{\gamma}_{e|p}(t)$ the solution Eq. (6) of the relativistic equation of motion. Hence, we have found that the solution we seek is

$$\overline{\tau}(\tau)=\tau-\frac{1}{c}\gamma_\parallel(\tau).\tag{51}$$

Besides, due to Eq. (49), $Z(\overline{\tau})=-\gamma_\parallel(\tau)$. Taking into account also Eqs. (42), (6), and (7), we have more generally

$$\mathbf{R}^{\pm}(\overline{\tau})=-\boldsymbol{\gamma}_{e|p}(\tau).\tag{52}$$

Returning to Eq. (48), by applying Eq. (49), we can also write this as

$$\delta(z''-z+Z(\tau''))=\frac{1}{1+a^2(\overline{\tau})/2}\delta(z''-z+Z(\overline{\tau})).\tag{53}$$

Performing the z'' integration then gives

$$\begin{aligned}&\mathcal{V}'^{ap}_{\pm\pm}(\mathbf{r},\mathbf{r}',t)\\ &=\frac{1}{(2\pi)^3}\int d\mathbf{q}\mathcal{V}(\mathbf{q})\int d\mathbf{p}\frac{I\pm\beta}{2}M^{\dagger}_{\pm}(\mathbf{p}_\perp+\mathbf{q}_\perp,P_z;\overline{\tau})\\ &\times M_{\pm}(\mathbf{p}_\perp,p_z;\overline{\tau})\frac{I\pm\beta}{2}\frac{1}{1+a^2(\overline{\tau})/2}\\ &\times\exp\left(i\{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')-\mathbf{q}\cdot[\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau)]\}\right).\end{aligned}\tag{54}$$

Further, we can extract the matrices in Eq. (54) out of the integrals if we replace the vectors $\mathbf{p},\mathbf{q}_\perp$ in Eq. (47) by the momentum operators $\mathbf{P}',\mathbf{P}_\perp$, acting on the variables $\mathbf{r}'_\perp,\mathbf{r}_\perp$, respectively (over which one does not integrate), as follows: $\mathbf{p}\to-\mathbf{P}'$, $\mathbf{q}_\perp\to\mathbf{P}_\perp$. Equation (54) becomes

$$\begin{aligned}\mathcal{V}'^{ap}_{\pm\pm}(\mathbf{r},\mathbf{r}',t) = \frac{I\pm\beta}{2}M^{\dagger}_{\pm}(\mathbf{P}_\perp-\mathbf{P}'_\perp,P_z;\overline{\tau})M_{\pm}(-\mathbf{P}'_\perp,-P_z;\overline{\tau})\\ \times\frac{I\pm\beta}{2}\int d\mathbf{q}\mathcal{V}(\mathbf{q})\int d\mathbf{p}\frac{1}{1+a^2(\overline{\tau})/2}\\ \times\exp\left(i\{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')-\mathbf{q}\cdot[\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau)]\}\right).\end{aligned}\tag{55}$$

As, from Eq. (32),

$$\int d\mathbf{q}\mathcal{V}(\mathbf{q})\exp\left(i\{\mathbf{q}\cdot[\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau)]\}\right)=V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau)),\tag{56}$$

taking into account the expression of the matrix product Eq. (47), Eq. (55) becomes

$$\begin{aligned}\mathcal{V}'^{ap}_{\pm\pm}(\mathbf{r},\mathbf{r}',t) = [AI+\mathbf{B}\cdot\boldsymbol{\Sigma}]\frac{1}{1+a^2(\overline{\tau})/2}\delta(\mathbf{r}-\mathbf{r}')\\ \times V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))\frac{I\pm\beta}{2},\end{aligned}\tag{57}$$

where

$$A\equiv\left[1+\frac{a^2}{2}-\frac{1}{2mc}\mathbf{a}\cdot(\mathbf{P}_\perp-2\mathbf{P}'_\perp)\right]I+\frac{1}{4mc}(P_z-P'_z)a^2\beta,\tag{58}$$

$$\mathbf{B}\equiv\frac{i}{4mc}\{[\mathbf{n}\times\mathbf{P}_\perp]a^2\beta+2\mathbf{a}\times\mathbf{P}_\perp\}.\tag{59}$$

Here all $\mathbf{a}$ depend on $\overline{\tau}$ and the momentum operators $\mathbf{P}$ and $\mathbf{P}'$ act on all functions depending on $\mathbf{r}$ or $\mathbf{r}'$, respectively, to their right in Eq. (57). With the definition of an integral operator [see, for example, Eq. (16)] and using Eqs. (57)–(59), we find that $V'^{ap}_{\pm\pm}$ acting in the subspace $\mathcal{D}$ can be expressed up to $O(p/mc)$ as

$$V'^{ap}_{\pm\pm}=\left[V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))I+\frac{1}{mc}(\widetilde{A}\,I+\widetilde{\boldsymbol{B}}\cdot\boldsymbol{\Sigma})\right]\frac{I\pm\beta}{2},\tag{60}$$

where

$$\begin{aligned}\widetilde{A}\equiv&-\frac{i}{2(1+a^2/2)^2}\mathbf{a}\cdot\frac{d\mathbf{a}}{dz}V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))\beta\\ &+\frac{1}{(1+a^2/2)}\left\{-\frac{1}{2}\langle\mathbf{a}\cdot[\mathbf{P}_\perp V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))]\rangle\right.\\ &+\frac{1}{4}a^2[P_zV(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))]\beta+V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))\\ &\left.\times\left\langle-\mathbf{a}\cdot\mathbf{P}_\perp+\frac{1}{4}a^2P_z\beta\right\rangle\right\},\end{aligned}\tag{61}$$

$$\begin{aligned}\widetilde{\boldsymbol{B}}\equiv&\frac{i}{4(1+a^2/2)}\{\mathbf{n}\times[\mathbf{P}_\perp V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))]a^2\beta\\ &+2\mathbf{a}\times[\mathbf{P}_\perp V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))]\}.\end{aligned}\tag{62}$$

In the equations above the only place where the action of the momentum operators $\mathbf{P}_\perp,P_z$ extends beyond the brackets they are enclosed in is the last bracket $\langle\cdots\rangle$ of Eq. (61).

Equation (61) contains the derivative da/dz, appearing for the first time in the calculation. We want to confine the

corresponding term in Eq. (60) to being a correction term to $V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))$, while assuming, as was done before, that a can be large [$a = O(1)$]. This requires that $(1/mc)(da/dz) \ll 1$, and Eq. (37) follows. The significance of this condition was discussed in Sec. V.

VII. CALCULATION OF $\mathcal{V}'^{ap}_{\pm\mp}(\mathbf{r},\mathbf{r}',t)$

For $V'^{ap}_{\pm\mp}$, Eq. (35), we calculate the approximate LMR expressions:

$$\Lambda^{\pm}_{\mathbf{p}_\perp,\,p_z}(\tau'') - \Lambda^{\mp}_{\mathbf{p}_\perp+\mathbf{q}_\perp,\,p'_z}(\tau'') \simeq \mp 2mcZ(\tau'') + (p_z - p'_z)Z(\tau'') - (2\mathbf{p}_\perp + \mathbf{q}_\perp)\cdot\mathbf{R}^{\pm}_\perp(\tau''), \tag{63}$$

where we have used $\mathbf{R}^-_\perp = -\mathbf{R}^+_\perp$ [see Eq. (42)] and

$$\begin{aligned} Q_{\pm\mp} &\equiv \frac{I\pm\beta}{2}\, M^{\dagger}_{\pm}(\mathbf{p}_\perp+\mathbf{q}_\perp,p'_z;\tau'')M_{\mp}(\mathbf{p}_\perp,p_z;\tau'')\frac{I\mp\beta}{2} \\ &\simeq \left\{\frac{i}{2mc}(\mathbf{n}\times\mathbf{a})\cdot\mathbf{q}_\perp\rho + \left(1-\frac{a^2}{2}\right)\beta\boldsymbol{\alpha}\cdot\frac{\mathbf{q}_\perp}{2mc}\right. \\ &\quad + \boldsymbol{\alpha}\cdot\mathbf{n}\left[\frac{a^2}{2} - \frac{\mathbf{a}\cdot(2\mathbf{p}_\perp+\mathbf{q}_\perp)}{2mc}\right] \\ &\quad \left. + \beta\,\boldsymbol{\alpha}\cdot\mathbf{n}\,\frac{(1+a^2/2)}{2mc}(p'_z - p_z)\right\}\frac{I\mp\beta}{2}, \end{aligned} \tag{64}$$

where we have introduced the odd matrix $\rho = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ [29]. We have neglected here second-order terms in $\mathbf{p}$ and $\mathbf{p}'$; all $\mathbf{a}$ depend on τ''.

In the following we want to determine $V'_{\pm\mp}$, Eq. (35), only to $O(1)$. To this end we can discard in Eq. (64) all linear terms in the momenta as giving contributions of $O(p/mc)$, except for the term containing p'_z/mc on the last line of Eq. (64). By a peculiarity of the problem, this gives a contribution of $O(1)$ instead of $O(p/mc)$; see below. Consequently, we shall be working with the matrix

$$\overline{Q}_{\pm\mp}(p'_z,\tau'') \equiv \frac{I\pm\beta}{2}\left[\frac{a^2}{2}\,I + \left(1+\frac{a^2}{2}\right)\frac{p'_z}{2mc}\beta\right] \boldsymbol{\alpha}\cdot\mathbf{n}\left(\frac{I\mp\beta}{2}\right). \tag{65}$$

We perform first the p'_z integration. To this end we can replace the variable p'_z contained in Eq. (64) by the operator P_z acting on the variable z, over which one does not integrate in Eq. (35). This allows the integral over p'_z to be carried out, yielding $\delta(z - z'' - Z(\tau''))$. The integration over z'' is done using Eq. (53), which gives

$$\begin{aligned} \mathcal{V}'^{ap}_{\pm\mp}(\mathbf{r},\mathbf{r}',t) \simeq \frac{1}{(2\pi)^3}\int d\mathbf{q}\mathcal{V}(\mathbf{q})\int d\mathbf{p}\exp[i\{\mathbf{p}_\perp\cdot(\mathbf{r}_\perp-\mathbf{r}'_\perp) \\ -p_z z' - \mathbf{q}_\perp\cdot\mathbf{r}_\perp\}\overline{Q}_{\pm\mp}(P_z;\overline{\tau})] \\ \times\frac{1}{1+a^2(\overline{\tau})/2}\exp\left(i\{\mp 2mcZ(\overline{\tau}) + (p_z+q_z)\right. \\ \left.\times[z - Z(\overline{\tau})] - (2\mathbf{p}_\perp+\mathbf{q}_\perp)\cdot\mathbf{R}^{\pm}_\perp(\overline{\tau})\}\right). \end{aligned} \tag{66}$$

where the operator P_z acts on all functions placed to its right in Eq. (66). Recall that the δ function changes τ'' into $\overline{\tau}$, which is z dependent; see Eq. (51).

When P_z is applied in Eq. (66), several terms appear. One of them contains the derivative

$$\begin{aligned} &[P_z\exp\{\mp 2imcZ(\overline{\tau})\}] \\ &= \mp 2mca^2(\overline{\tau})\frac{d\overline{\tau}}{d\tau}\left(\frac{\partial\tau}{\partial z}\right)\exp[\mp 2imcZ(\overline{\tau})] \\ &= \mp 2mca^2(\overline{\tau})\left(1-\frac{1}{c}\frac{d\gamma_\parallel(\tau)}{d\tau}\right)\exp[\mp 2imcZ(\overline{\tau})], \end{aligned} \tag{67}$$

where we have used Eq. (51). By considering the equation of motion for $\gamma_\parallel(\tau)$ from Eq. (7) with t replaced by τ, we find

$$\frac{1}{c}\frac{d\gamma_\parallel(\tau)}{d\tau} = \frac{1}{2}\left(1-\frac{1}{c}\frac{d\gamma_\parallel(\tau)}{d\tau}\right)a^2(\overline{\tau}).$$

This can be solved for $d\gamma_\parallel(\tau)/d\tau$ to give

$$\frac{1}{c}\frac{d\gamma_\parallel(\tau)}{d\tau} = \frac{a^2(\overline{\tau})/2}{1+a^2(\overline{\tau})/2}. \tag{68}$$

Thus, Eq. (67) becomes

$$\frac{P_z}{mc}\exp[\mp 2imcZ(\overline{\tau})] = \mp\frac{a^2(\overline{\tau})}{1+a^2(\overline{\tau})/2}\exp[\mp 2imcZ(\overline{\tau})]. \tag{69}$$

Here, the exponential on the left-hand side being of order $O(1)$, one might have expected to get a result of order $O(p/mc)$, but the result turns out to be of order $O(1)$. The change in the order of magnitude occurs because the argument of $\exp[\mp 2imcZ(\overline{\tau})]$ contains the large factor mc. This explains why we have kept the term p'_z/mc in Eq. (64). The other terms resulting from the application of the operator P_z in Eq. (66) are of $O(p/mc)$.

By making use of Eqs. (65) and (69) in Eq. (66), we have to $O(1)$

$$\begin{aligned} &\overline{Q}_{\pm\mp}(P_z;\overline{\tau})\frac{1}{1+a^2(\overline{\tau})/2}\exp i\{\cdots\} \\ &\simeq \frac{I\pm\beta}{2}(I\mp\beta)\frac{a^2}{2}\boldsymbol{\alpha}\cdot\mathbf{n}\left(\frac{I\mp\beta}{2}\right)\frac{1}{1+a^2(\overline{\tau})/2}\exp i\{\cdots\}. \end{aligned}$$

As $(I\pm\beta)(I\mp\beta) = 0$,

$$\mathcal{V}'^{ap}_{\pm\mp}(\mathbf{r},\mathbf{r}',t) = 0. \tag{70}$$

VIII. GENERALIZED SPACE-TRANSLATED PAULI EQUATIONS

In the following we shall concentrate on our results to $O(1)$. Taking into account Eqs. (25), (60), and (70), the expression of the operator V'^{ap} takes to this order the remarkably simple form of a multiplicative operator:

$$V'^{ap}(t) = V(\mathbf{r}+\boldsymbol{\gamma}_e(\tau))\frac{I+\beta}{2} + V(\mathbf{r}+\boldsymbol{\gamma}_p(\tau))\frac{I-\beta}{2}. \tag{71}$$

We are now in a position to write the action of the Hamiltonian $H'(t)$, Eq. (21), on a LMR spinor $\Phi(\mathbf{r}) \in \mathcal{D}$. We note that

the kinetic energy term $E\beta$ of H' was obtained using the exact form of the operator T. This represents a higher level of accuracy than we are interested in, and we need to take the $O(1)$ approximation of $E\beta$ with respect to p/mc, which gives $E\beta\Phi \simeq (mc^2 + \mathbf{P}^2/2m)\beta\Phi$. We can express the result for $H'\Phi \simeq H'^{ap}\Phi$ to $O(1)$ as

$$H'^{ap}(t)\Phi \simeq [mc^2 + \mathbf{P}^2/2m + V(\mathbf{r} + \boldsymbol{\gamma}_e(\tau))]\frac{I+\beta}{2}\Phi + [-mc^2 - \mathbf{P}^2/2m + V(\mathbf{r} + \boldsymbol{\gamma}_p(\tau))]\frac{I-\beta}{2}\Phi. \tag{72}$$

In Sec. V we showed that $V'^{ap}\Phi \in \mathcal{D}$ (see also [27]), and hence also $H'^{ap}(t)\Phi \in \mathcal{D}$, whatever t.

We now consider what happens to a spinor Ψ' starting from a LMR initial condition at some time t_0 and evolving according to the translated Dirac equation (21). We recall that a differential equation like Eq. (21), containing only the first time derivative $\partial/\partial t$, implies the existence of a unitary evolution operator $U(\mathbf{r};t,t_0)$. In general, $U(\mathbf{r};t,t_0)$ can be represented by Neumann's series [30]:

$$U(\mathbf{r};t,t_0) = I + \sum_{n=1}^{\infty}(-i)^n \int_{t_0}^{t} dt_1 H(\mathbf{r};t_1) \times \int_{t_0}^{t_1} dt_2 H(\mathbf{r};t_2)\cdots\int_{t_0}^{t_{n-1}} dt_n H(\mathbf{r};t_n),$$

where $t_0 < t_{n-1} < \cdots < t_1 < t$. The existence of the solution $\Psi'(\mathbf{r},t)$ implies the convergence of the series. When applying $U(\mathbf{r};t,t_0)$ to $\Psi'_0(\mathbf{r}) \in \mathcal{D}$, one first encounters the integral

$$\chi(\mathbf{r},t_{n-1}) = \int_{t_0}^{t_{n-1}} dt_n H(\mathbf{r};t_n)\Psi'_0(\mathbf{r}).$$

As $H(\mathbf{r};t_n)\Psi'_0 \cong H^{ap}(\mathbf{r};t_n)\Psi'_0 \in \mathcal{D}$ at all times t_n, $\chi(\mathbf{r},t_{n-1})$, which is an integral over elements of $\mathcal{D}$ (i.e., a linear combination of them), will also belong to $\mathcal{D}$, $\chi(\mathbf{r},t_{n-1}) \in \mathcal{D}$. This means that we can repeat the argument stepwise to conclude that $\Psi'(\mathbf{r},t) = U(\mathbf{r};t,t_0)\Psi'_0(\mathbf{r}) \in \mathcal{D}$, and hence that the solutionremains in the LMR at all times t and the approximation for the Hamiltonian $H' \simeq H'^{ap}$ remains valid. The space-translated Dirac equation (21) is therefore equivalent to $O(1)$ to the equation

$$\left\{[mc^2 + \mathbf{P}^2/2m + V(\mathbf{r} + \boldsymbol{\gamma}_e(\tau))] - i\frac{\partial}{\partial t}\right\}\frac{I+\beta}{2}\Psi' + \left\{[-mc^2 - \mathbf{P}^2/2m + V(\mathbf{r} + \boldsymbol{\gamma}_p(\tau))] - i\frac{\partial}{\partial t}\right\}\frac{I-\beta}{2}\Psi' = 0. \tag{73}$$

If initially Ψ' is an electron (positron) wave packet, i.e., only its first two (last two) components are nonvanishing, because of the projection operators, Eq. (73) can be split into two independent equations. By introducing the four-component spinors φ and χ containing, respectively, the upper and lower components of a solution Ψ' according to

$$\Psi' \equiv e^{-imc^2t}\frac{I+\beta}{2}\varphi + e^{+imc^2t}\frac{I-\beta}{2}\chi^*,$$

the two equations can be written as

$$[\mathbf{P}^2/2m + V(\mathbf{r} + \boldsymbol{\gamma}_e(\tau))]\varphi - i\frac{\partial\varphi}{\partial t} = 0, \tag{74}$$

$$[\mathbf{P}^2/2m - V(\mathbf{r} + \boldsymbol{\gamma}_p(\tau))]\chi - i\frac{\partial\chi}{\partial t} = 0. \tag{75}$$

Although Eqs. (74) and (75) are 4×4 equations, because of the structure of φ and χ they reduce for practical purposes to two-component Pauli equations [31]. These we shall denote as Pauli LMR equations. In fact, due to our restriction to $O(1)$ in p/mc, they are two-component Schrödinger equations because they contain no spin matrices. The fact that these are not present is deceptive, as will be shown in Secs. IX and XI.

We can finally answer the problem posed in Sec. IV: with LMR initial conditions the solutions of the generalized translated Dirac equation Eq. (21) remain LMR at all times, and the electron and positron components of the solutions evolve according to two independent Pauli equations. For LMR initial conditions all information contained in the original Dirac equation is present also in Eqs. (74) and (75), if terms of order $O(\varpi/mc)$ in the translated Dirac picture are neglected. As a consequence, under these conditions the electron cannot generate electron-positron pairs no matter how intense the field and how rapidly it varies (recall however, the limitation mentioned at the end of Sec. VI, irrelevant in practice) [13,32].

IX. CALCULATION OF OBSERVABLES

Physical observables are defined in the laboratory frame and are calculated in terms of the laboratory-frame wave spinors $\Psi(\mathbf{r},t)$, using the appropriate form of the scalar product. We show in the following how to express these quantities in terms of the corresponding spinors $\Psi'(\mathbf{r},t)$ of the translated Dirac picture. The exact transformation operator T being unitary ensures the conservation of the scalar product when passing from one representation to the other: $\langle\Psi \mid \Phi\rangle = \langle T\Psi' \mid T\Phi'\rangle = \langle\Psi' \mid \Phi'\rangle$, as $T^\dagger T = I$ exactly. It is easy to show that the approximate $T^{ap}_{\pm}$ also has this property. Indeed, if we replace V by I in Eq. (71), we find $T^{\dagger ap}T^{ap} = I$ to $O(1)$, which means that also T^{ap} conserves the scalar product to $O(1)$ [33].

We now calculate the appropriate forms of T^{ap} and $T^{\dagger ap}$. Starting from the exact $\mathcal{T}_{\pm}(\mathbf{r},\mathbf{r}',t)$, Eq. (30), by applying the approximations Eqs. (40)–(45), it is easy to show that

$$\mathcal{T}^{ap}_{\pm}(\mathbf{r},\mathbf{r}',t) = \exp\left(\mp i\frac{mc^2}{2}\int_{-\infty}^{\tau} a^2(\chi)d\chi\right)M_{\pm}(-\mathbf{P}';\tau) \times \frac{I\pm\beta}{2}\delta(\mathbf{r}-\mathbf{r}'+\mathbf{R}^{\pm}(\tau)), \tag{76}$$

where $M_{\pm}(\mathbf{P};\tau)$ is defined by Eq. (45), with $\mathbf{p}$ replaced by the momentum operator $\mathbf{P}$. In Eq. (76) $\mathbf{P}'$ acts on the coordinate $\mathbf{r}'$ of the δ function to its right. Consequently, $T^{ap}_{\pm}$ acts on a spinor $\Psi'(\mathbf{r})$ as

$$T^{ap}_{\pm}(t)\Psi'(\mathbf{r}) = \exp\left(\mp i\frac{mc^2}{2}\int_{-\infty}^{\tau} a^2(\chi)d\chi\right) \times \left\{M_{\pm}(+\mathbf{P}';\tau)\frac{I\pm\beta}{2}\Psi'(\mathbf{r}')\right\}_{\mathbf{r}'=\mathbf{r}+\mathbf{R}^{\pm}(\tau)} \tag{77}$$

Recall that $\mathbf{R}^{\pm}(\tau)$ can be expressed with Eq. (52). The complete transform of Ψ' is given by $T^{ap}\Psi'(\mathbf{r}) \equiv T_{+}^{ap}\Psi'(\mathbf{r}) + T_{-}^{ap}\Psi'(\mathbf{r})$.

Let us consider the special cases $t \to \pm\infty$, when the field vanishes in the vicinity of the atom. From Eqs. (77) and (29), we find for $t \to -\infty$

$$[T_{\pm}^{ap}]_{-\infty} = S_{\mathrm{FW}}^{ap}(\mathbf{P})\frac{I \pm \beta}{2}. \tag{78}$$

Here, $S_{\mathrm{FW}}^{ap}(\mathbf{P})$ is given by Eq. (44) with $\mathbf{p}$ replaced by $\mathbf{P}$. The fact that $[T_{\pm}^{ap}]_{-\infty}$ differs from $(I \pm \beta)/2$ and $[T^{ap}]_{-\infty} \neq I$ is due to the initial spinor change of basis, from the standard basis set to the FW basis set; see Eq. (13). Thus, the connection between the initial conditions of the laboratory frame and the translated Dirac picture is

$$\Psi_i'(\mathbf{r}) = \frac{I \pm \beta}{2} S_{\mathrm{FW}}^{ap\dagger}(\mathbf{P})\Psi_i(\mathbf{r}). \tag{79}$$

For $t \to +\infty$, the distances $\mathbf{r}$ of interest being finite, $\tau, \overline{\tau} \to \infty$. We are then dealing with Eqs. (42) with $\boldsymbol{\gamma}_{e|p}(\infty)$; see Eq. (8). Equation (77) gives

$$[\Psi_{\pm}(\mathbf{r})]_f = \exp\left(\mp i\frac{mc^2}{2}\int_{-\infty}^{+\infty} a^2(\chi)d\chi\right) S_{\mathrm{FW}}^{ap}(\mathbf{P}) \times \frac{I \pm \beta}{2}\Psi_f'(\mathbf{r} - \mathbf{r}_f), \tag{80}$$

where $\mathbf{r}_f$ is the forward drift of the classical electron, Eq. (8). The complete transform of $\Psi_f'(\mathbf{r})$ is

$$\Psi_f(\mathbf{r}) = \left\{\exp\left(-i\frac{mc^2}{2}\int_{-\infty}^{+\infty} a^2(\chi)\,d\chi\right) S_{\mathrm{FW}}^{ap}(\mathbf{P})\frac{I+\beta}{2} + \exp\left(+i\frac{mc^2}{2}\int_{-\infty}^{+\infty} a^2(\chi)\,d\chi\right) S_{\mathrm{FW}}^{ap}(\mathbf{P})\frac{I-\beta}{2}\right\} \times \Psi_f'(\mathbf{r} - \mathbf{r}_f). \tag{81}$$

We shall also need in the following the operator $T_{\pm}^{\dagger ap}$, for whose kernel we find the two alternative forms

$$\mathcal{T}_{\pm}^{\dagger ap}(\mathbf{r},\mathbf{r}',t) = \exp\left(\pm i\frac{mc^2}{2}\int_{-\infty}^{\tau'} a^2(\chi)d\chi\right)\frac{I \pm \beta}{2} \times M_{\pm}^{\dagger}(\mathbf{P};\tau')\delta(\mathbf{r}' - \mathbf{r} + \mathbf{R}_{\pm}(\tau')) \tag{82}$$

$$= \exp\left(\pm i\frac{mc^2}{2}\int_{-\infty}^{\overline{\tau}} a^2(\chi)d\chi\right)\frac{I \pm \beta}{2} \times M_{\pm}^{\dagger}(\mathbf{P};\overline{\tau})\frac{\delta(\mathbf{r}' - \mathbf{r} + \mathbf{R}_{\pm}(\overline{\tau}))}{1 + a^2(\overline{\tau})/2} \tag{83}$$

to be used depending on the integration variable $\mathbf{r}$ or $\mathbf{r}'$. The second form was obtained from the first one by using Eq. (53) [28]. Here $M_{\pm}^{\dagger}(\mathbf{P};\tau)$ is the adjoint of the operator in Eq. (77), taking duly into account the order of noncommuting operators it contains. Note that Eqs. (76) and (82) contain $\mathbf{R}^{\pm}(\tau)$, whereas Eq. (83) contains $\mathbf{R}^{\pm}(\overline{\tau})$; the latter can be expressed in terms of τ via Eq. (52). The complete transform $T^{\dagger ap}$ is $T^{\dagger ap} \equiv T_{+}^{\dagger ap} + T_{-}^{\dagger ap}$.

In most problems the quantity of interest is the final wave spinor for the laboratory frame, $\Psi_f(\mathbf{r})$. The standard procedure would be to obtain it by integrating the Dirac equation for the laboratory frame starting from a certain atomic initial condition, $\Psi_i(\mathbf{r})$. With our procedure, however, the propagation of the wave spinor in time is carried out using the Pauli LMR equations (74) and (75), starting from an initial condition $\Psi_i'(\mathbf{r})$, connected to $\Psi_i(\mathbf{r})$ by Eq. (79). Once $\Psi_f'(\mathbf{r})$ is obtained, $\Psi_f(\mathbf{r})$ results from Eq. (81). Concerning the initial condition $\Psi_i(\mathbf{r})$ for the laboratory frame, this should be either an electron or a positron wave packet, which is an eigenspinor of some observables, such as energy, spin, etc. Clearly, $\Psi_i(\mathbf{r})$ should be expressed at the level of approximation of our theory, i.e., reflecting the light-atom condition $\alpha Z \ll 1$.

Although there are no spin matrices in Eqs. (74) and (75), the fact that the solution is an electron or positron wave spinor on which $S_{\mathrm{FW}}^{ap}(\mathbf{P})$ needs to be applied to get the laboratory spinor $\Psi_f(\mathbf{r})$ indicates that spin is taken into account. We illustrate this as follows. If the initial state is a relativistic Coulomb-energy bound state with fine-structure quantum numbers n,l,j,m, and $\Psi_i(\mathbf{r}) = u_{nljm}(\mathbf{r})$, the $\alpha Z \ll 1$ approximation that should be used for $u_{nljm}(\mathbf{r})$ is, in split notation,

$$u_{nljm}(\mathbf{r}) \simeq \left\{R_{nl}(r)\mathcal{Y}_{lj}^{m}(\hat{\mathbf{r}}); (\boldsymbol{\sigma}\cdot\mathbf{P}/2mc)R_{nl}(r)\mathcal{Y}_{lj}^{m}(\hat{\mathbf{r}})\right\}, \tag{84}$$

where $R_{nl}(r)$ are the NR radial functions and $\mathcal{Y}_{lj}^{m}(\hat{\mathbf{r}})$ are the Pauli eigenspinors common to $\mathbf{J}^2$, $\mathbf{L}^2$, and J_z. The initial condition in the translated Dirac picture deriving from Eq. (84) is, according to Eq. (79), in split notation

$$\Psi_i' = \left\{(1 + P^2/4m^2c^2)R_{nl}(r)\mathcal{Y}_{lj}^{m}(\hat{\mathbf{r}}); 0\right\} \simeq \left\{R_{nl}(r)\mathcal{Y}_{lj}^{m}(\hat{\mathbf{r}}); 0\right\}, \tag{85}$$

where we have neglected the term with P^2 as being $O(\alpha^2Z^2)$. Referring to Eq. (74), the initial condition Eq. (85) for φ has two different nonvanishing components. Thus, we need (in general) to integrate the Schrödinger equation in (74) twice, for two different initial conditions in order to obtain the $\Psi_f'(\mathbf{r})$.

As an example for the calculation of an observable quantity we take the *survival probabilitiy* of the atom in the ground state after the passage of a radiation pulse. Since there are two degenerate ground states $n = 1$, $l = 0$, $j = 1/2$, $m = \pm 1/2$, we have for the survival probability of the atom initially in the m spin state the laboratory-frame formula (electron case)

$$P_m^s \equiv \sum_{n'l'j'm'} |\langle u_{n'l'j'm'}(\mathbf{r}) \mid \Psi_{fm}(\mathbf{r})\rangle|^2, \tag{86}$$

where $\Psi_{fm}(\mathbf{r})$ is the final wave spinor evolving from the initial state m. Insertion of Eq. (80) here leads to

$$P_m^s = \sum_{n'l'j'm'}\left|\left\langle u_{n'l'j'm'}(\mathbf{r})\middle|\frac{I+\beta}{2}S_{\mathrm{FW}}^{ap}(\mathbf{P})\Psi_{f\mu}'(\mathbf{r} - \mathbf{r}_f)\right\rangle\right|^2. \tag{87}$$

As Ψ_{fm}' has two nonvanishing components that are different for $m = \pm 1/2$, the corresponding P_m^s will in general differ.

X. GENERALIZED COULOMB POTENTIAL

The generalized potential $V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau))$ is a quantity which requires attention. For the Coulomb electron case we have (in a.u.):

$$V^e_{\mathrm{LMR}}(\mathbf{r}) \equiv V_C(\mathbf{r}+\boldsymbol{\gamma}_e(\tau)) = -\frac{Z}{|\mathbf{r}+\boldsymbol{\gamma}_e(t-(\mathbf{n}\cdot\mathbf{r})/c)|}\,. \tag{88}$$

This has a moving singularity and also can manifest a large time-dependent distortion compared to the pure Coulomb potential.

Let us consider first the motion of its singularity, $\mathbf{r}_s(t)$. This is given by

$$\mathbf{r}_s(t)+\boldsymbol{\gamma}_e(\tau_s)=0,$$

where $\boldsymbol{\gamma}_e(\tau_s)$ is obained from Eq. (7) with t replaced by $\tau_s = t-\mathbf{n}\cdot\mathbf{r}_s(t)/c$. In this case the upper limit of the integrals in Eq. (7) is $\tau_s-\gamma_{\parallel}(\tau_s)/c=t$. We find for $\mathbf{r}_s(t)$

$$\mathbf{r}_s(t) = +c\int_{-\infty}^{t}\mathbf{a}(\chi)d\chi - \frac{c}{2}\mathbf{n}\int_{-\infty}^{t}a^2(\chi)d\chi\,. \tag{89}$$

By comparing Eqs. (7) and (89), it is apparent that $\mathbf{r}_s(t)$ has mathematically the same trajectory as $-\boldsymbol{\gamma}_e(t)$, except that the two motions are delayed in time. However, whereas $\boldsymbol{\gamma}_e(t)$ represents a trajectory in the laboratory frame, $\mathbf{r}_s(t)$ is a trajectory in a coordinate space that is not associated with a physical reference frame. At times $\mathbf{r}_s(t)$ moves with superluminal velocities, but this is of no concern.

To illustrate the distortion of $V^e_{\mathrm{LMR}}(\mathbf{r})$, let us consider its behavior in the vicinity of the singularity where the interaction with the radiation takes place. We assume that the pulse is linearly polarized and propagates along Oz, with the vector potential and electric field oscillating along Ox, i.e., of the form $\mathbf{a}\{a(\tau),0,0\}$. By introducing the polar coordinates $\boldsymbol{\rho}=\mathbf{r}-\mathbf{r}_s(t)$, the expansion of the denominator in Eq. (88) for $\rho<1$ leads to

$$V^e_{\mathrm{LMR}}(\mathbf{r}) \simeq -\frac{Z}{\left[\rho^2-\lambda(t)\rho_z^2+\mu(t)\rho_x\rho_z+O(\rho^3)\right]^{1/2}}, \tag{90}$$

where

$$\lambda(t)=\left[\frac{a^2/2}{1+a^2/2}\right]^2_{\overline{\tau}_s},\quad \mu(t)=\left[\frac{2a}{1+a^2/2}\right]_{\overline{\tau}_s}. \tag{91}$$

We have calculated the expressions for $d\boldsymbol{\gamma}_{\perp}(\tau)/d\tau$ and $d\boldsymbol{\gamma}_{\parallel}(\tau)/d\tau$ required for Eq. (90) using Eq. (7). The result for $d\boldsymbol{\gamma}_{\perp}(\tau)/d\tau$ was given in Eq. (68), and we obtain similarly

$$\frac{1}{c}\frac{d\gamma_{\perp}(\tau)}{d\tau} = -\frac{a^2(\overline{\tau})}{1+a^2(\overline{\tau})/2}.$$

The coefficients $\lambda(t)$ and $\mu(t)$ in Eq. (90) need to be calculated at the point $\mathbf{r}=\mathbf{r}_s(t)$, which makes them depend on the value of $\overline{\tau}$ at this point, namely, $\overline{\tau}_s \equiv \tau_s - \boldsymbol{\gamma}_{e|p}(\tau_s)\cdot\mathbf{n}/c$.

When expressing the quadratic form contained in Eq. (90) with respect to the principal axes, we get

$$V^e_{\mathrm{LMR}}(\mathbf{r}) = -\frac{Z}{\left[(\rho'_x/\alpha)^2+\rho'^2_y+(\rho'_z/\beta)^2\right]^{1/2}}. \tag{92}$$

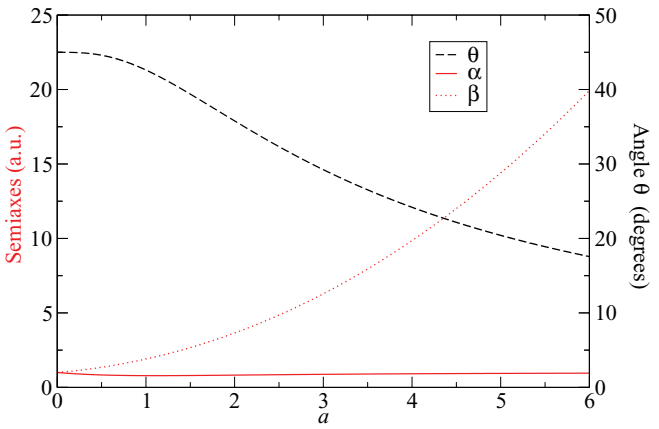

FIG. 1. (Color online) Dependence of α (continuous line) and β (dotted line), the lengths of the semiaxes of the equipotential ellipses of V_{LMR} at $\rho'_y=0$ and $Z=1$ [Eq. (92)], on the dimensionless field strength parameter a in Eq. (1). The values in a.u. are given on the ordinate at left. Dashed line: a dependence of the rotation angle θ of the axes of the ellipse with respect to their initial position; values given in degrees on the ordinate at right.

The coordinates ρ'_x,ρ'_z are obtained from the original ρ_x,ρ_z by a rotation by angle θ in the positive sense, defined by

$$\tan\theta = -(\lambda/\mu)+\sqrt{1+(\lambda/\mu)^2} \tag{93}$$

and $\rho'_y=\rho_y$. The dependence of θ on a is given in Fig. 1. At given ρ'_y, the level lines of $V_{\mathrm{LMR}}(\mathbf{r})$ are ellipses with the semiminor axis proportional to α and the semimajor axis proportional to β. α and β are functions of a that can be readily calculated analytically. Their graphical representation is also shown in Fig. 1, for $a>0$. As seen, $\alpha(a)$ stays rather close to 1 (to within 20%) for all a, whereas β grows with a (parabolically at large a). As $\alpha\approx 1$, Eq. (92) shows that the 3D equipotential surfaces of $V_{\mathrm{LMR}}(\mathbf{r})$ are rather close to being revolution ellipsoids with the axis along $O\rho'_z$.

For a laser pulse, $a(\tau)$ oscillates, taking positive and negative values, depending on its specific form. Let us follow the rotation and distortion of a typical equipotential ellipse obtained by setting $V_{\mathrm{LMR}}=-Z$ and taking $\rho'_y=0$ in Eq. (92). Figure 1 shows that in the limit $a=0$ we have $\alpha=\beta=1$, i.e., the ellipse reduces to a circle of radius 1, and moreover $\theta=\pi/4$. As a increases the axes turn in the clockwise sense. This goes on until a attains its maximum value and starts decreasing. Then α, β, and θ retrace their paths until the circle at $a=0$ is regained. Figure 2 shows the rotation and distortion of the ellipse at several values of $a>0$. The distortion can be substantial for high fields and low frequencies. As an illustration we take the case of a pulse $a(\tau)=a_0(\tau)\sin\omega\tau$, the envelope $a_0(\tau)$ having a maximum $a_0(0)$ at $\tau=0$. It can be shown [5] that in this case the maximum value of the electric field is given by $E_{\max}=c\omega a_0(0)$ a.u. Taking here $I_{\max}=1000$ a.u. $=3.5\times10^{19}$ W/cm^2, $E_{\max}=31.6$ a.u., and $\omega=0.057$ ($\lambda=800$ nm), we get $a_0(0)=4.06$. Figure 1 shows that the ratio of the semimajor and semiminor axes is then quite large, $\beta/\alpha\simeq\beta\simeq10.8$. The corresponding rotation angle is $\theta=24.2^\circ$.

Note that the elongation of the major axis of the ellipse from its unperturbed value 1 represents a deepening of the potential

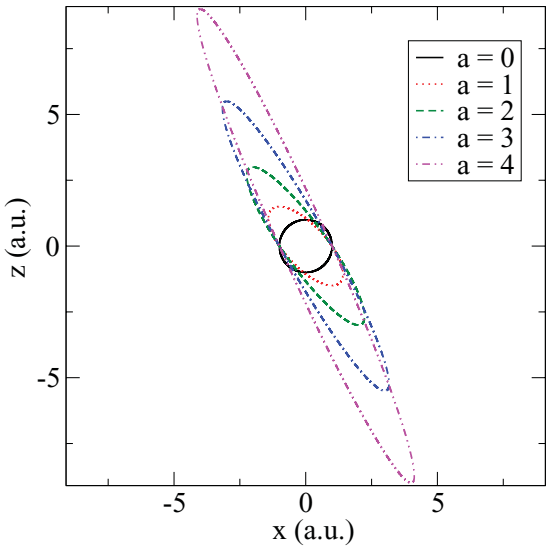

FIG. 2. (Color online) Rotation and elongation of a typical equipotential ellipse of $V_{\rm LMR}(\mathbf{r})$, Eq. (92) ($V_{\rm LMR} = -Z$ with $\rho_y' = 0$), at various values of the dimensionless field strength a. As a increases, the ellipse starts from a circle of radius 1 at $a = 0$ (unperturbed Coulomb case), then elongates in the direction of the major axis and rotates in the positive sense, while the semiminor axis stays fairly close to 1. The units on the axes are a.u.

in the direction of this axis with respect to the original Coulomb potential. Equation (92) gives $V^e_{\rm LMR}(0,0,\rho_z) = \beta V_C(0,0,\rho_z)$, which means a 9.8 times deepening in the preceding example. For a short and intense laser pulse this is particularly marked during the time interval close to the peak and could lead to interesting physical consequences.

XI. NONRELATIVISTIC LIMIT

In the translated Dirac picture the NR limit is achieved by imposing the condition that the field $\mathbf{a}(\tau)$ be weak such that $mca = O(1)$ in a.u. (otherwise, $|a| \lesssim 1/137$). In this case the momenta in the laboratory frame will automatically be NR [of order $O(1)$ a.u.], if they were so in the absence of the radiation. In the following we establish the equivalence of the NR limit of our Dirac picture Hamiltonian for the electron [see Eq. (74)] with the NR Hamiltonian of the Dirac equation for the laboratory frame, $H_{\rm NR}$ [see Eq. (101) below], to order $O(a)$.

We start from the LMR Dirac Hamiltonian H'^{ap}, Eq. (72), and endeavor to show that there exists an operator $\widetilde{T}$ which is an approximation of the exact T, such that the transformed Hamiltonian H^{ap},

$$H^{ap} \equiv i\frac{\partial \widetilde{T}}{\partial t}\widetilde{T}^\dagger + \widetilde{T}H'^{ap}\widetilde{T}^\dagger \tag{94}$$

coincides with $H_{\rm NR}$, and

$$\widetilde{T}\widetilde{T}^\dagger \simeq I + O(a^2). \tag{95}$$

As a candidate for $\widetilde{T}$ we choose a simplified form of T^{ap}, Eqs. (76) and (77), with the operator matrix $M_\pm(+\mathbf{P}';\tau)$ replaced by [34]

$$\widetilde{M}_\pm(\tau) \equiv I \pm \frac{i}{2}\boldsymbol{\Sigma}\cdot(\mathbf{a}\times\mathbf{n}). \tag{96}$$

The total $\widetilde{T}$ is given by $\widetilde{T} = \widetilde{T}_+ + \widetilde{T}_-$. Similarly, $\widetilde{T}^\dagger$ is defined by Eq. (82) with $M_\pm(+\mathbf{P}';\tau)$ replaced by Eq. (96). We shall now check that $\widetilde{T}$ is an appropriate choice.

First, let us consider the different terms of Eq. (94). We begin by calculating the transform V^{ap} of the potential term V'^{ap}, Eq. (71), contained in H'^{ap}, Eq. (72). To obtain the kernel of V^{ap}, we use a formula similar to Eq. (23):

$$\mathcal{V}^{ap}(\mathbf{r},\mathbf{r}',t) = \int \widetilde{\mathcal{T}}(\mathbf{r},\mathbf{r}'',t)V'^{ap}(\mathbf{r}'')\widetilde{\mathcal{T}}^\dagger(\mathbf{r}'',\mathbf{r}',t)d\mathbf{r}''.$$

When we insert Eq. (71) here and the appropriate forms for $\widetilde{\mathcal{T}},\widetilde{\mathcal{T}}^\dagger$ resulting from Eqs. (76) and (82), we get

$$\mathcal{V}^{ap}(\mathbf{r},\mathbf{r}',t) = \left[\widetilde{M}_+(\tau)\frac{I+\beta}{2}\widetilde{M}^\dagger_+(\tau')f_+J_+ + \widetilde{M}_-(\tau)\frac{I-\beta}{2}\widetilde{M}^\dagger_-(\tau')f_-J_-\right], \tag{97}$$

where

$$f_\pm = \exp\left(\mp i\frac{mc^2}{2}\int_{-\infty}^{\tau} a^2(\chi)\,d\chi\right) \times \exp\left(\pm i\frac{mc^2}{2}\int_{-\infty}^{\tau'} a^2(\chi)\,d\chi\right),$$

and for $J_\pm$ we find

$$J_\pm = \delta(\mathbf{r}+\mathbf{R}^{(\pm)}(\tau)-\mathbf{r}'-\mathbf{R}^{(\pm)}(\tau')) \times V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau)+\mathbf{R}^{(\pm)}(\tau)).$$

The δ function appearing here can be handled similarly to that in Eq. (53); see [28]. This gives

$$J_\pm = \frac{1}{[1+a^2(\tau)/2]}\delta(\mathbf{r}-\mathbf{r}')V(\mathbf{r}+\boldsymbol{\gamma}_{e|p}(\tau)+\mathbf{R}^{(\pm)}(\tau)).$$

Neglecting $O(a^2)$ and inserting this in Eq. (97), we get

$$\mathcal{V}^{ap}(\mathbf{r},\mathbf{r}',t) = \delta(\mathbf{r}-\mathbf{r}')V(\mathbf{r}), \quad V^{ap}(\mathbf{r}) \equiv \widetilde{T}V'^{ap}\widetilde{T}^\dagger = V(\mathbf{r})I, \tag{98}$$

as

$$\widetilde{M}_+(\tau)\frac{I+\beta}{2}\widetilde{M}^\dagger_+(\tau) + \widetilde{M}_-(\tau)\frac{I-\beta}{2}\widetilde{M}^\dagger_-(\tau) = I + O(a^2). \tag{99}$$

Equation (98) allows us to verify that $\widetilde{T}$ satisfies Eq. (95). This is obtained by setting $V \to 1$ in Eqs. (71) and (98).

Next we consider the contribution of the kinetic energy terms of Eq. (72) to Eq. (94). This is

$$\widetilde{T}\left[(mc^2+\mathbf{P}^2/2m)\frac{I+\beta}{2} + (-mc^2-\mathbf{P}^2/2m)\frac{I-\beta}{2}\right]\widetilde{T}^\dagger = \left(mc^2+\frac{\mathbf{P}^2}{2m}\right)\beta,$$

as $\mathbf{P}^2$ commutes with $\widetilde{T}$ if terms of order $O(a^2)$ and crossed order $O(a) \times O(p/mc)$ are neglected; we have also used Eq. (95).

Finally, the contribution of the first term in Eq. (94) can be written to order $O(a)$ as

$$i\frac{\partial \widetilde{T}}{\partial t}\widetilde{T}^{\dagger} = i\frac{\partial \widetilde{T}_{+}}{\partial t}\widetilde{T}_{+}^{\dagger} + i\frac{\partial \widetilde{T}_{-}}{\partial t}\widetilde{T}_{-}^{\dagger} = \left\{\left[\frac{1}{2}mc^2a^2(\tau) - c\mathbf{a}(\tau)\cdot\mathbf{P}\right]I - \frac{e}{2mc}\boldsymbol{\sigma}\cdot\mathbf{H}\right\}\beta. \tag{100}$$

To derive this equation we have used the form of $\widetilde{\mathcal{T}}_{\pm}(\mathbf{r},\mathbf{r}',t)$ resulting from Eq. (76) and calculated $\partial\widetilde{\mathcal{T}}_{\pm}(\mathbf{r},\mathbf{r}',t)/\partial t$, noting that $\mathbf{n}\times\partial\mathbf{a}/\partial t = -c\,\mathrm{rot}\,\mathbf{a} = -(e/mc)\mathbf{H}$, where $\mathbf{H}$ is the magnetic field. Neglecting terms of $O(a^2)$, the matrix $(I\pm\beta)/2$ can be factored out to the right of the derivative $\partial\widetilde{\mathcal{T}}_{\pm}(\mathbf{r},\mathbf{r}',t)/\partial t$. The same matrix can be factored out to the left of $[\widetilde{\mathcal{T}}_{\pm}(\mathbf{r},\mathbf{r}',t)]^{\dagger}$ when calculating Eq. (100). One can then apply Eq. (95) which leads to the result Eq. (100). Note that the term $mc^2a^2(\tau)$, although proportional to $a^2(\tau)$, cannot be neglected in Eq. (100) because of its large coefficient mc^2.

Collecting the contributions to H^{ap}, we find that it coincides with

$$H_{\mathrm{NR}} = \left[mc^2 + \frac{[\mathbf{P}-mc\mathbf{a}(\tau)]^2}{2m} - \frac{e}{2mc}\boldsymbol{\sigma}\cdot\mathbf{H}(\tau)\right]\beta + V(\mathbf{r})\,I, \tag{101}$$

which is the textbook result for the NR limit of the Dirac equation; see, e.g., [17], Chap. XX, Eq. (188). We have thus proven that the unitary transformation $\widetilde{T}$ establishes the equivalence, in the NR limit and to $O(a)$, of our LMR Eq. (74) and the Dirac equation for the laboratory frame, Eq. (101) [35].

Since H_{NR}, Eq. (101), contains the Hamiltonian for the Schrödinger equation with full retardation, this means that our result Eq. (74) encompasses any approximate form derived from H_{NR}, such as retaining only the first-order retardation correction in $\mathbf{A}(\tau)$ beyond the dipole approximation $\mathbf{A}(t)$, a case considered in [36], Eq. (4); [37], Eqs. (2) and (3); [38], Eq. (2); and [39], Eq. (14). Our result also encompasses the space-transformed Schrödinger equation obtained in [6], Eq. (5), since this too derives from the retarded Schrödinger equation. A more direct contact cannot be established with [6], as the operator transform U used there [see [6], Eqs. (2) and (3)] does not represent the NR nondipole limit of our T^{ap}.

XII. FINAL COMMENTS AND CONCLUSION

In this final section we endeavor to give more insight into our calculations. We start by noting that in the NR case the space-translation transformation for the Schrödinger equation with a (dipole approximation) radiation field is achieved by passing to an oscillating reference frame. In the relativistic case, because of retardation, this oscillating frame cannot be defined, as the field does not oscillate in phase throughout space. A generalization of the space translation is defined in the form of a change of quantum-mechanical representation, defined with the help of the operator $T(t)$, Eq. (17). The resulting generalized translated Dirac equation (21) of the new representation (the Dirac picture) contains a potential term which is a complicated integral operator $V'(\mathbf{r},t)$, Eq. (22), that makes the equation intractable in general. Its advantage, however, lies in the fact that the field dependence is entirely contained in $V'(\mathbf{r},t)$.

We have focused here on the special case of the interaction of an atomic electron with an optical laser pulse of arbitrary intensity and have considered the realistic situation of laser physics in which the initial condition for the atom in the laboratory frame, $\Psi_i(\mathbf{r})$, is represented by a spinor containing only low momenta $p/mc \leqslant \varpi/mc \ll 1$, where ϖ is a cutoff. Such spinors we have denoted as being in the low-momentum regime (LMR). This implicitly limits our study to small-Z atoms ($\alpha Z \ll 1$). When evolving from a LMR initial condition $\Psi_i(\mathbf{r})$ according to the Dirac equation for the laboratory frame, a spinor $\Psi(\mathbf{r},t)$ will nevertheless develop in superintense laser fields high relativistic momenta ($p/mc \gtrsim 1$). In the translated Dirac picture the corresponding initial condition $\Psi_i'(\mathbf{r})$ is also a LMR spinor, as $\Psi_i(\mathbf{r}) = T(-\infty)\Psi_i'(\mathbf{r})$ and $T(-\infty)$ represents only a change of spinor basis. We have shown that the spinor $\Psi_i'(\mathbf{r},t)$, the solution of the generalized Dirac equation, Eq. (21), remains a LMR spinor during its evolution, no matter how intense the field (see Secs. VI–VIII). A limitation was needed on the space-time variation of the field, Eq. (37), but it turns out to be irrelevant in practice.

On the other hand, when acting on a LMR spinor of the transformed Dirac picture, the operator $V'(\mathbf{r},t)$ reduces to a remarkably simple form: it becomes a multiplicative function. Moreover, for initial conditions chosen to be electron or positron wave packets, the transformed Dirac equation reduces in fact to two independent Pauli equations (denoted the Pauli LMR equations), one for the electron, the other for the positron. They contain all the information the Dirac equation can offer for laser interactions with small-Z atoms.This implies that in the conditions we have considered no pair production can occur, no matter how intense the field $\mathbf{a}(\tau)$, and that the nonquantized Dirac equation is capable of handling our situation, even though the laser field is allowed to be arbitrarily intense.

The LMR equations can be obtained with various degrees of accuracy with respect to the LMR cutoff ϖ/mc. In this paper we have neglected ϖ/mc corrections to the dominant-order results, denoted as $O(1)$. The LMR equations then contain no Pauli spin matrices, and are proportional to the 2×2 matrix I. This means that they reduce to two-component Schrödinger equations (74) and (75), one for the electron, the other for the positron. They contain the time-dependent generalized potentials $V_{\mathrm{LMR}}^{e|p}(\mathbf{r},t)$ [see Eq. (88)], manifesting rather unusual properties, discussed in Sec. X.

It may appear puzzling that we started from a four-component Dirac equation which contains the physics of spin and have ended up with Schrödinger equations with no spin in them. The explanation lies in the way physical observables are calculated. For the Dirac equation these are basically formulated in the laboratory frame in terms of scalar products containing the laboratory-frame wave spinors. When expressed in terms of the Dirac-picture wave spinors, use

is made of the transformation operator $T^{ap}(t)$ which does contain the spin (see Sec. IX). Moreover, we have shown in Sec. XI that when transformed back to the laboratory frame, Eqs. (74) and (75) yield in the NR limit (i.e., the case of small momenta $p/mc \ll 1$ in the laboratory frame and weak fields) the well-known textbook result with spin included.

We note that it would be in principle possible to apply our procedure to obtain a higher-order approximation with respect to the LMR cutoff ϖ/mc. The analog of Eqs. (74) and (75) would now contain the Pauli spin matrices.

The simple form of the electronic equation (74) makes it potentially attractive for numerical integration. As noted in the Introduction, the 3D Dirac equation for superintense laser fields is notoriously difficult to integrate numerically. In our case, instead of dealing with the Dirac system of four coupled equations, one would be dealing with a Schrödinger equation which, although of a more complicated sort, is tractable with existing programs. The solution of Eq. (74) would be started with an initial condition transformed from that given for the laboratory frame, propagated according to the equation, and finally transformed back to the laboratory frame where observables are calculated (e.g., ionization survival probability and angular distributions, high-order harmonic emission spectrum, etc.).

To conclude, in this paper we have derived the Dirac analog of the space-translated Schrödinger equation for laser pulses. When applied to typical laser-atom interactions the equation simplifies dramatically, reducing to two-component Schrödinger equations (for the electron and positron, respectively) with modified time-dependent Coulomb potentials. This opens the way to relativistic theoretical treatments similar to those applied in the NR case (like the high-frequency, high-intensity approximation) and their application to physical phenomena (ionization, high-order harmonic generation, etc). In addition, the application in the relativistic case of well-tested NR numerical methods could be of great use.

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and their transformed Dirac equation is written in a different representation than our Eqs. (21) and (22).

[25] In practice, Eq. (36) should be understood in the sense that for $p > \varpi$, $|c_{\mathbf{p}\sigma}|$ decreases rapidly with p.

[26] Krstic and Mittleman [24] retained in their transformation operator U_{KM} linear terms in p in the exponent of $\Psi_{\mathbf{p}\sigma}$ [corresponding to our $\Lambda_{\mathbf{p}\sigma}$, Eq. (41)], but in the matrix prefactor [corresponding to our $\Omega_{\mathbf{p}\sigma}$, Eq. (40)] they took only the p-independent approximation. This inconsistency has serious consequences; see [33].

[27] An alternative way of phrasing the problem encountered here is as follows. Considering Eq. (22), the operator $T(t)$ acting on an element $\Phi \in \mathcal{D}$ transforms it by definition into an element of Δ_a. Multiplication by V might well give rise to an element $VT\Phi$ lying outside Δ_a, which $T^{\dagger}(t)$ could no longer return to $\mathcal{D}$, and therefore one could not use the approximation $T^{\dagger ap}(t)$ [see Eq. (39) and the remark thereafter]. Our proof shows that this is not the case due to the analytic form of the Coulomb potential V_c [i.e., $V'^{ap}\Phi = \left(T^{\dagger}\right) V T^{ap}\Phi \in \mathcal{D}$] and hence that the use of the operator $\left(T^{\dagger}\right)^{ap} \simeq (T^{ap})^{\dagger}$ is justified.

[28] For example, $\delta(f(x)) = \sum_n \delta(x - x_n) / |f'(x_n)|$, where x_n are the zeros of $f(x)$ and $f'(x_n) \neq 0$.

[29] A Dirac matrix t is said to be even or odd if, in split notation, its submatrices t_{ij} have the properties $t_{12} = t_{21} = 0$ or $t_{11} = t_{22} = 0$, respectively.

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[31] The possibility of attaining the split form Eqs. (73)–(75) for the translated Dirac equation and hence the manifest decoupling of the electron and positron motions is a consequence of having chosen the operator $T(t)$ to connect the Volkov spinors of the laboratory frame to the FW basis set, rather than the standard spinor basis set, Eq. (3).

[32] The nonoccurrence of pair production in extremely high fields can be explained on physical grounds by the fact that in the laboratory frame the electron oscillates almost freely, with little influence from the nucleus (the so-called strong-field approximation is applicable). As for a free electron the momentum distribution is the same before and after the incidence of the pulse, no pair production can occur.

[33] Had we adopted the Krstic and Mittleman approximation [26], this would have led to the unfortunate consequence that $\mathcal{V}'^{ap}_{\pm\mp}(\mathbf{r},\mathbf{r}',t) \neq 0$ instead of Eq. (70), which in turn would have implied a fictitious coupling of the electron and positron states in $H'\Phi$ of Eq. (72), and the impossibility of separating the electronic and positronic equations (74) and (75). At the same time, $T^{\dagger ap}T^{ap} \neq I$ to $O(1)$. These difficulties appear also for the operator U_{KM} and the corresponding H'_{KM} used in [10], but they were not mentioned.

[34] $\widetilde{M}_{\pm}(\tau)$ is obtained from $M_{\pm}(+\mathbf{P};\tau)$ by neglecting the $\mathbf{p}/mc$ terms in Eq. (45) and keeping the even part of the matrix $\hat{v}\hat{\mathbf{a}}(\tau)$.

[35] The procedure we have adopted in the text is that of checking that the operator $\widetilde{T}$ is the right one connecting to the NR limit Eq. (101). This procedure is a shortcut compared to the more "natural" one, which consists in returning from the Dirac picture back to the laboratory-frame representation using the same operator T^{ap}, Eq. (77), as was used in going. This will not retrieve the original Dirac Hamiltonian for the laboratory frame (due to the LMR approximations), but rather a simplified form of it, $\overline{H}^{ap}$. However, this $\overline{H}^{ap}$ is not yet an adequate form of Hamiltonian for the NR limit as it contains odd-matrix contributions. According to the FW procedure such odd contributions should be eliminated by a succession of unitary transformations $S_1, S_2, \ldots$ of increasing order in $1/mc^2$, which will eventually lead to the NR Hamiltonian having the desired accuracy. It turns out that in our case it is sufficient to perform only one transformation S_1 in order to obtain Eq. (101). The overall transformation is given then by the product $S_1 T^{ap}$. One can show that the choice made in the text for $\widetilde{T}$ is precisely equal to this product: $\widetilde{T} \simeq S_1 T^{ap}$.

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