Heisenberg uncertainty relation for position and momentum beyond central potentials

Łukasz Rudnicki*
Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, PL-02-668 Warsaw, Poland
(Received 20 December 2011; published 8 February 2012)

Recently, Sánchez-Moreno *et al.* [New J. Phys. **8**, 330 (2006)] have shown that the three-dimensional Heisenberg uncertainty relation $\sigma_r\sigma_p \geqslant 3\hbar/2$ might be significantly sharpened if the relevant quantum state describes the particle in a central potential. I extend that result to the case of states which are not the eigenstates of the square of the angular momentum operator. I derive a lower bound for $\sigma_r\sigma_p$ which involves the mean value and the variance of the square of the angular momentum operator.

DOI: 10.1103/PhysRevA.85.022112 PACS number(s): 03.65.Ta, 03.65.Ca

I. INTRODUCTION

The famous Heisenberg uncertainty relation [1,2] is one of the most fundamental results of whole quantum mechanics. Furthermore, testing various uncertainty relations provides a way to characterize many important quantum features. For a generic three-dimensional system described in terms of a wave function $\psi(\boldsymbol{r})$ the Heisenberg uncertainty relation reads

$$\sigma_r\sigma_p \geqslant \frac{3\hbar}{2}, \tag{1}$$

where $\sigma_r = \sqrt{\langle r^2\rangle - \langle \boldsymbol{r}\rangle\cdot\langle \boldsymbol{r}\rangle}$ and $\sigma_p = \sqrt{\langle p^2\rangle - \langle \boldsymbol{p}\rangle\cdot\langle \boldsymbol{p}\rangle}$ denote the standard deviations of position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$ variables, respectively.

In a special case where the wave function is an eigenstate of the square of the angular momentum operator $\hat{L}^2\psi(\boldsymbol{r}) = \hbar^2 l(l+1)\psi(\boldsymbol{r})$ the lower bound in Eq. (1) was significantly sharpened in [3] (this result has been recently rediscovered in [4]):

$$\sigma_r\sigma_p = \sqrt{\langle r^2\rangle\langle p^2\rangle} \geqslant \hbar\left(l + \tfrac{3}{2}\right). \tag{2}$$

The fact that $\langle \boldsymbol{r}\rangle = 0 = \langle \boldsymbol{p}\rangle$ for all eigenstates of $\hat{L}^2$ was taken into account. We shall also notice that all results presented in [3] were derived in a general case of D-dimensional systems.

The eigenstates of the $\hat{L}^2$ operator play a special role in atomic physics, where a quantum system is usually assumed to evolve in a central potential depending only on the distance from the origin $r^2 = \boldsymbol{r}\cdot\boldsymbol{r}$. For several central potentials the product $\sigma_r\sigma_p$ was analytically derived and checked numerically in [5]. Various uncertainty relations [6–13] are of special importance in numerical computations related to electronic structures, in particular, to that based on the density-functional theory [14], because they provide sensitive tools to verify physical adequateness of obtained electron densities [15]. However, since the results of numerical calculations and possible experiments cannot guarantee us that $\psi(\boldsymbol{r})$ has all expected properties, we shall ask about the validity of the uncertainty relation (2) when $\psi(\boldsymbol{r})$ is not exactly the eigenstate of $\hat{L}^2$. The aim of this paper is to provide an answer to that question.

Obviously, the standard deviations σ_r and σ_p are invariant under translations in both position and momentum (transformations $\boldsymbol{r} \mapsto \boldsymbol{r} + \boldsymbol{r}_0$ and $\boldsymbol{p} \mapsto \boldsymbol{p} + \boldsymbol{p}_0$), however, the average value of $\hat{L}^2$ is not. Therefore, let me introduce an invariant version of the angular momentum operator ($\boldsymbol{r}$ and $\boldsymbol{p}$ are operators):

$$\hat{\boldsymbol{L}}_{\text{inv}} = (\boldsymbol{r} - \langle \boldsymbol{r}\rangle) \times (\boldsymbol{p} - \langle \boldsymbol{p}\rangle). \tag{3}$$

This is the angular momentum in the position and momentum reference frames centered at $\langle \boldsymbol{r}\rangle$ and $\langle \boldsymbol{p}\rangle$, respectively. Let me then assume that we have at our disposal the average value of the square of this angular momentum operator described by a dimensionless parameter L_{inv}

$$\hbar^2 L_{\text{inv}}^2 = \langle \hat{L}_{\text{inv}}^2\rangle = \int d^3r\, \psi^*(\boldsymbol{r})\hat{L}_{\text{inv}}^2\psi(\boldsymbol{r}). \tag{4}$$

In Sec. II, I present a simple example which shows that when one does not know the state (one knows only the parameter L_{inv}) one cannot refine the general Heisenberg bound (1). This happens because even if $L_{\text{inv}}^2 \approx l(l+1)$ for some $l \in \mathbb{N}$, one can still construct states *laying far away* from the eigenstate of $\hat{L}_{\text{inv}}^2$ labeled by the quantum number l. To overcome this issue I shall employ the variance of $\hat{L}_{\text{inv}}^2$ and, in addition to L_{inv}, use the dimensionless parameter

$$\mathcal{R}_{\text{inv}} = \hbar^{-4}\left\langle\left(\hat{L}_{\text{inv}}^2 - \langle \hat{L}_{\text{inv}}^2\rangle\right)^2\right\rangle. \tag{5}$$

Of course $\mathcal{R}_{\text{inv}} = 0$ only for the eigenstates of $\hat{L}_{\text{inv}}^2$. Thus, relatively small values of $\mathcal{R}_{\text{inv}}$ shall justify the approximation that $\psi(\boldsymbol{r})$ is some eigenstate of $\hat{L}_{\text{inv}}^2$.

The main result of this paper is the sharpened version of the Heisenberg uncertainty relation (1):

$$\sigma_r\sigma_p \geqslant \frac{3\hbar}{2} + \frac{\hbar L_{\text{inv}}^3}{2}\,\frac{\sqrt{L_{\text{inv}}^2 + 4L_{\text{inv}}^4 + 4\mathcal{R}_{\text{inv}}} - L_{\text{inv}}}{\mathcal{R}_{\text{inv}} + L_{\text{inv}}^4}, \tag{6}$$

derived in Sec. III. In the discussion section (Sec. IV) I show that this uncertainty relation links in a continuous manner the previous result (2) recovered from Eq. (6) for $L_{\text{inv}} = \sqrt{l(l+1)}$ and $\mathcal{R}_{\text{inv}} = 0$, with the Heisenberg uncertainty relation (1) achieved when $L_{\text{inv}} = 0$ or $\mathcal{R}_{\text{inv}} \to \infty$. We shall notice that for $\langle \boldsymbol{r}\rangle = 0 = \langle \boldsymbol{p}\rangle$ we obtain $\hat{L}_{\text{inv}}^2 = \hat{L}^2$, which provides a full correspondence with the case (2).

II. THE LOWER BOUND (2) BEYOND EIGENSTATES OF $\hat{L}_{\text{inv}}^2$

In this section I show that the constraint on the average value L_{inv}^2 (from now on we put $\hbar = 1$) of the $\hat{L}_{\text{inv}}^2$ operator (or simply $\hat{L}^2$ operator in the reference frame where $\langle \boldsymbol{r}\rangle = 0 = \langle \boldsymbol{p}\rangle$) does

*rudnicki@cft.edu.pl

not lead by itself to an inequality stronger than the Heisenberg uncertainty relation (1). To this end let me use the function

$$f_l(r,\theta,\varphi) = \frac{2^{1+l}}{\pi^{1/4}}\sqrt{\frac{l!}{(2l+1)!}}\,r^l e^{-r^2/2} Y_l^0(\theta,\varphi), \qquad (7)$$

which is the normalized eigenstate of some isotropic harmonic oscillator with the angular momentum quantum number l, the magnetic number $m = 0$, and the principal quantum number $n = 0$. Now we shall take the superposition of the ground state f_0 (angular momentum number equal to 0) with the state f_{l_0}, for $l_0(l_0+1) > L_{\rm inv}^2$ and $l_0 > 1$,

$$\Psi_{l_0} = \sqrt{\frac{l_0(l_0+1) - L_{\rm inv}^2}{l_0(l_0+1)}}\, f_0 + \frac{L_{\rm inv}}{\sqrt{l_0(l_0+1)}} f_{l_0}. \qquad (8)$$

The state (8) satisfies $\langle \boldsymbol{r}\rangle = 0$, $\langle \boldsymbol{p}\rangle = 0$, and $\langle \hat{L}_{\rm inv}^2\rangle = \langle \hat{L}^2\rangle = L_{\rm inv}^2$, but

$$\sigma_r\sigma_p = \sqrt{\langle r^2\rangle\langle p^2\rangle} = \frac{3}{2} + \frac{L_{\rm inv}^2}{l_0+1} \xrightarrow[l_0\to\infty]{} \frac{3}{2}. \qquad (9)$$

This observation means that keeping $\langle \hat{L}_{\rm inv}^2\rangle$ constant we are able to go arbitrarily close to the ground state f_0 taking arbitrarily large l_0. However, for the example state (8) we can check that

$$\mathcal{R}_{\rm inv} = L_{\rm inv}^2\left[l_0(l_0+1) - L_{\rm inv}^2\right] \xrightarrow[l_0\to\infty]{} \infty, \qquad (10)$$

which explains why the Heisenberg lower bound 3/2 can be asymptotically reached in Eq. (9).

III. PROOF OF THE UNCERTAINTY RELATION (6) BASED ON CALCULUS OF VARIATIONS

A main ingredient of my derivation shall be the variational approach used recently in [16] to prove a new Heisenberg-like uncertainty relation for photons. The most important advantage of this method is that one does not need to rely on commutation relations between the conjugate variables. In the first step I define the following functionals:

$$X^2[\psi^*,\psi] = \int d^3r\, r^2 \psi^*(\boldsymbol{r})\psi(\boldsymbol{r}), \qquad (11)$$

$$P^2[\psi^*,\psi] = \int d^3r\, \psi^*(\boldsymbol{r})(-\Delta)\psi(\boldsymbol{r}), \qquad (12)$$

where Δ is a three-dimensional Laplacian and we set $\hbar = 1$.

A. Ordinary Heisenberg uncertainty relation

Following the idea presented in [16] I shall briefly describe the variational method using the example of the Heisenberg uncertainty relation (1). However, I modify this derivation and use the Lagrange multiplier to include the normalization condition. To prove Eq. (1) one can start with solving the following variational equation:

$$\frac{\delta}{\delta\psi^*}\left[X^2[\psi^*,\psi]P^2[\psi^*,\psi] - \lambda\left(\int d^3r\, \psi^*(\boldsymbol{r})\psi(\boldsymbol{r}) - 1\right)\right] = 0, \qquad (13)$$

where λ is the Lagrange multiplier related to the normalization constraint of the wave function. Equation (13) gives

$$\left(\frac{\omega^2}{X^2} r^2 - X^2\Delta - \lambda\right)\psi(\boldsymbol{r}) = 0, \qquad (14)$$

where we denote $\omega = XP$. In the next step we impose the normalization condition [we multiply Eq. (14) by ψ^* and integrate over d^3r] and find that $\lambda = 2\omega^2$. Finally, we introduce a dimensionless variable $\boldsymbol{\xi} = \boldsymbol{r}/X$ and obtain the Schrödinger equation for the three-dimensional isotropic harmonic oscillator

$$\left(-\frac{1}{2}\Delta + \frac{\omega^2\xi^2}{2}\right)\psi(\boldsymbol{\xi}) = \omega^2\psi(\boldsymbol{\xi}). \qquad (15)$$

The eigenvalues of the *left-hand side* of Eq. (15) are $\omega(n + 3/2)$, $n \in \mathbb{N}$, thus, the eigenvalue equation for Eq. (15) gives

$$\omega\left(n + \tfrac{3}{2}\right) = \omega^2 \quad \Longrightarrow \quad \omega = n + \tfrac{3}{2}. \qquad (16)$$

The smallest possible value $\omega_{\rm min}$ of ω in Eq. (16) is equal to 3/2 for $n = 0$ and when $\psi(\boldsymbol{\xi})$ is the ground state. This result provides the inequality

$$XP \geqslant \omega_{\rm min} = \tfrac{3}{2}. \qquad (17)$$

Now we choose the coordinate and momentum reference frames such that $\langle \boldsymbol{r}\rangle = 0 = \langle \boldsymbol{p}\rangle$. In this specific frame we have $X = \sigma_r$ and $P = \sigma_p$, and our result (17) reads

$$\sigma_r\sigma_p \geqslant \tfrac{3}{2}. \qquad (18)$$

The observation that Eq. (18) is invariant under translations (is valid in all reference frames) completes the proof of Eq. (1). In other words, in order to prove the Heisenberg uncertainty relation (1) one needs to show that $\sqrt{\langle r^2\rangle\langle p^2\rangle} \geqslant 3/2$, when $\langle \boldsymbol{r}\rangle = 0 = \langle \boldsymbol{p}\rangle$.

B. Derivation including constraints related to angular momentum

In order to prove the main result of this paper – Eq. (6) I shall use the method described in Sec. III A together with two additional constraints: $\langle \hat{L}^2\rangle \equiv L^2 = \text{const}$ and $\langle(\hat{L}^2 - \langle \hat{L}^2\rangle)^2\rangle \equiv \mathcal{R} = \text{const}$. Note that in this part of the derivation I use the ordinary angular momentum operator $\hat{\boldsymbol{L}}$. The variational equation in that case reads ($F = \mathcal{R} + L^4$)

$$\begin{aligned}\frac{\delta}{\delta\psi^*}\Bigg[&X^2[\psi^*,\psi]P^2[\psi^*,\psi] - \lambda\left(\int d^3r\, \psi^*(\boldsymbol{r})\psi(\boldsymbol{r}) - 1\right)\\ &+ 2\eta\left(\int d^3r\, \psi^*(\boldsymbol{r})\hat{L}^2\psi(\boldsymbol{r}) - L^2\right)\\ &+ 2\Lambda\left(\int d^3r\, \psi^*(\boldsymbol{r})\hat{L}^4\psi(\boldsymbol{r}) - F\right)\Bigg] = 0,\end{aligned} \qquad (19)$$

and leads to the equation

$$\left(\frac{\omega^2}{X^2} r^2 - X^2\Delta - \lambda + 2\eta\hat{L}^2 + 2\Lambda\hat{L}^4\right)\psi(\boldsymbol{r}) = 0. \qquad (20)$$

The parameters λ, η, and Λ in Eq. (19) play the role of Lagrange multipliers and the factor -2 before η and Λ was introduced for further convenience. Imposing the normalization constraint we find that $\lambda = 2\omega^2 + 2\eta L^2 + 2\Lambda F$. Thus, we have to solve the following counterpart of the Schrödinger equation (15):

$$\left(-\frac{1}{2}\Delta + \frac{\omega^2\xi^2}{2} + \eta\hat{L}^2 + \Lambda\hat{L}^4\right)\psi(\boldsymbol{\xi}) = \mathcal{E}\psi(\boldsymbol{\xi}), \quad (21)$$

where $\mathcal{E} = \omega^2 + \eta L^2 + \Lambda F$.

1. Solutions to the eigenproblem

In the first step we notice that all eigenstates $\Phi(\boldsymbol{\xi})$ of the isotropic harmonic oscillator [17]

$$\left(-\frac{1}{2}\Delta + \frac{\omega^2\xi^2}{2}\right)\Phi(\boldsymbol{\xi}) = \omega\left(\frac{3}{2} + 2n + l\right)\Phi(\boldsymbol{\xi}) \quad (22)$$

are also solutions of Eq. (21). However, taking $\psi(\boldsymbol{\xi})$ to be one of these eigenstates, we will not be able to fulfill the constraints on angular momentum [in particular, all eigenstates $\Phi(\boldsymbol{\xi})$ have $\mathcal{R} = 0$]. Thus, in order to solve Eq. (21) together with the constraints we shall look for the solution of Eq. (21) in the form of a superposition of the eigenstates of the harmonic oscillator

$$\psi(\boldsymbol{\xi}) = \sum_{i=1}^{N} C_i\Phi_i(\boldsymbol{\xi}), \qquad \sum_{i=1}^{N}|C_i|^2 = 1. \quad (23)$$

We assume that each of the two states Φ_i and Φ_j (for $i \neq j$) in Eq. (23) differ by at least one quantum number, i.e., $n_i \neq n_j$ or $l_i \neq l_j$. Thus, all states $\Phi_i(\boldsymbol{\xi})$ in the superposition (23) are independently the eigenstates of Eq. (21) with the energies

$$\mathcal{E}_i = \omega\left(\tfrac{3}{2} + 2n_i + l_i\right) + \eta l_i(l_i+1) + \Lambda l_i^2(l_i+1)^2. \quad (24)$$

In that way we obtain a set of N independent equations that must be simultaneously satisfied:

$$\mathcal{E}_i = \mathcal{E}, \quad i = 1,2,\ldots,N. \quad (25)$$

But, in our problem we have only three constants to specify: $\mathcal{E}$, η, Λ [it is important to notice that amplitudes C_i do not appear in Eq. (25)], thus, allowed solutions (23) of Eq. (21) can be the superposition of at most three eigenstates of the harmonic oscillator. In other words, only the value $N = 3$ allows us to find the solutions $\eta(\omega,n_i,l_i)$, $\Lambda(\omega,n_i,l_i)$, and $\mathcal{E}(\omega,n_i,l_i)$. I will not write them down explicitly, but I restrict myself to give in Sec. III C a simple derivation of ω consistent with these solutions.

It might appear unusual that we have not specified the Lagrange multipliers using the related constraints, but during the process of solving the variational equation (19). However, we still have at our disposal three amplitudes C_1, C_2, and C_3 and we shall use these coefficients to fulfill the remaining constraints.

2. Solutions to the constraints

For the sake of simplicity let me introduce the following notation: $\alpha = l_1(l_1+1)$, $\beta = l_2(l_2+1)$, and $\gamma = l_3(l_3+1)$. The three constraints we have imposed lead to three equations for the probabilities $|C_i|^2$:

$$|C_1|^2 + |C_2|^2 + |C_3|^2 = 1, \quad (26a)$$

$$\alpha|C_1|^2 + \beta|C_2|^2 + \gamma|C_3|^2 = L^2, \quad (26b)$$

$$\alpha^2|C_1|^2 + \beta^2|C_2|^2 + \gamma^2|C_3|^2 = F. \quad (26c)$$

The solutions of Eqs. (26a)–(26c) are

$$|C_1|^2 = \frac{\beta\gamma - L^2(\beta+\gamma) + F}{(\alpha-\beta)(\alpha-\gamma)}, \quad (27a)$$

$$|C_2|^2 = \frac{\alpha\gamma - L^2(\alpha+\gamma) + F}{(\beta-\alpha)(\beta-\gamma)}, \quad (27b)$$

$$|C_3|^2 = \frac{\alpha\beta - L^2(\alpha+\beta) + F}{(\gamma-\alpha)(\gamma-\beta)}. \quad (27c)$$

In that way we have specified the moduli of C_1, C_2, and C_3 coefficients, but their phases might be arbitrary. The solutions (27a)–(27c) possess a permutational symmetry (a permutation of l_i produces the permutation of $|C_i|$); however, at this point we shall decide about some hierarchy between the l_i numbers. Let me choose $l_1 \geqslant l_2 \geqslant l_3$, which means that $\alpha \geqslant \beta \geqslant \gamma$. Since the solutions (27a)–(27c) must be positive we obtain the following conditions for α, β, and γ:

$$\beta\gamma - L^2(\beta+\gamma) + F \geqslant 0, \quad (28a)$$

$$\alpha\gamma - L^2(\alpha+\gamma) + F \leqslant 0, \quad (28b)$$

$$\alpha\beta - L^2(\alpha+\beta) + F \geqslant 0. \quad (28c)$$

The conditions (28a)–(28c) might be also presented in terms of two, mutually exclusive cases:

$$0 \leqslant \gamma \leqslant L^2 \leqslant \beta \leqslant \frac{F - \gamma L^2}{L^2 - \gamma} \leqslant \alpha \quad (29)$$

or

$$0 \leqslant \gamma \leqslant \beta \leqslant L^2 \leqslant \frac{F - \gamma L^2}{L^2 - \gamma} \leqslant \alpha \leqslant \frac{F - \beta L^2}{L^2 - \beta}. \quad (30)$$

In fact, from the beginning we could expect that either one or two parameters among (α,β,γ) shall be greater than L^2.

C. Final uncertainty relation

Since all constraints are fulfilled we are able to derive in a simple way the coefficient ω consistent with the Eq. (25). To this end I write down these equations explicitly (for $i \in \{1,2,3\}$):

$$\omega\left(\tfrac{3}{2} + 2n_i + l_i\right) + \eta l_i(l_i+1) + \Lambda l_i^2(l_i+1)^2 = \omega^2 + \eta L^2 + \Lambda F, \quad (31)$$

multiply ith equation by $|C_i|^2/\omega$ and sum up over i. Due to the conditions (26a)–(26c) all the terms with the Lagrange multipliers η and Λ present on both sides of Eq. (31) cancel, and we immediately obtain the solution

$$\omega(n_i,l_i) = \frac{3}{2} + \sum_{i=1}^{3}|C_i|^2(2n_i + l_i). \quad (32)$$

The final step is to find values of the quantum numbers n_i and l_i that minimize ω. Since the probabilities $|C_i|^2$ do not depend on n_i we shall take the lowest levels $n_i = 0$, as in the case of the ordinary Heisenberg uncertainty relation. An answer to the question which values of the parameters l_i minimize the function $\omega(l_i) \equiv \omega(0,l_i)$ is more subtle, because we have assumed that $l_i \in \mathbb{N}$. However, for a moment we shall treat l_i as continuous parameters (we assume that $l_i \in \mathbb{R}$) and calculate the following derivatives:

$$\frac{\partial\omega(l_i)}{\partial l_1} = \frac{(l_1-l_2)(l_1-l_3)(S+l_1)}{Q_{12}Q_{13}Q_{23}}|C_1|^2 \geqslant 0, \tag{33a}$$

$$\frac{\partial\omega(l_i)}{\partial l_2} = \frac{(l_2-l_1)(l_2-l_3)(S+l_2)}{Q_{12}Q_{13}Q_{23}}|C_2|^2 \leqslant 0, \tag{33b}$$

$$\frac{\partial\omega(l_i)}{\partial l_3} = \frac{(l_3-l_1)(l_3-l_2)(S+l_3)}{Q_{12}Q_{13}Q_{23}}|C_3|^2 \geqslant 0, \tag{33c}$$

where $S = 2 + l_1 + l_2 + l_3$ and $Q_{jk} = 1 + l_j + l_k$.

We have defined $\omega_{\min}$ as a minimal value of $\omega(l_i)$ with the assumption that $l_i \in \mathbb{N}$. Thus, since $\omega(l_i)$ increases with l_1 and l_3 and decreases with l_2 we can obtain a lower bound $\Omega \leqslant \omega_{\min}$ taking the smallest possible values of l_1 and l_3 and the largest value of l_2, according to the ranges (29) and (30). This means that the optimal values of $l_i \in \mathbb{R}$ are

$$l_1 = l_2 = \frac{\sqrt{4F+L^2}}{2L} - \frac{1}{2}, \quad l_3 = 0, \tag{34}$$

and lead to the result

$$\Omega(L,\mathcal{R}) = \frac{3}{2} + \frac{L^4}{2F(L,\mathcal{R})}\left(\sqrt{1+\frac{4F(L,\mathcal{R})}{L^2}} - 1\right). \tag{35}$$

When we substitute in Eq. (35) the value $F(L,\mathcal{R}) = \mathcal{R} + L^4$ we obtain the inequality

$$XP \geqslant \omega_{\min} \geqslant \Omega = \frac{3}{2} + \frac{L^3}{2}\frac{\sqrt{L^2+4L^4+4\mathcal{R}} - L}{\mathcal{R}+L^4}. \tag{36}$$

Similarly to the result (17), the uncertainty relation (36) in the reference frame defined by $\langle \boldsymbol{r}\rangle = 0 = \langle \boldsymbol{p}\rangle$ reads

$$\sigma_r\sigma_p \geqslant \frac{3}{2} + \frac{L^3}{2}\frac{\sqrt{L^2+4L^4+4\mathcal{R}} - L}{\mathcal{R}+L^4}, \tag{37}$$

where now, since $\hat{\boldsymbol{L}} = \hat{\boldsymbol{L}}_{\text{inv}}$, we can replace $\hat{\boldsymbol{L}}$ by $\hat{\boldsymbol{L}}_{\text{inv}}$ and the couple of parameters $(L,\mathcal{R})$ by $(L_{\text{inv}},\mathcal{R}_{\text{inv}})$. Finally we shall use the fact that $\hat{\boldsymbol{L}}_{\text{inv}}$ is invariant under translations both in positions and momenta, thus the uncertainty relation (37) is also invariant and the main result (6) of this paper is proven.

In fact, we are able to improve immediately the inequality (6) because we can take an exact, but more complicated value $\omega_{\min}$ instead of its lower bound Ω. If $l_i \in \mathbb{N}$, then the optimal values of l_i are $l_3 = 0$, and

$$l_1 = \left\lceil \frac{\sqrt{4F+L^2}}{2L} - \frac{1}{2}\right\rceil, \quad l_2 = \left\lfloor \frac{\sqrt{4F+L^2}}{2L} - \frac{1}{2}\right\rfloor, \tag{38}$$

where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ denote the integer-valued ceiling and floor functions, respectively [18]. Finally we have

$$\sigma_r\sigma_p \geqslant \omega_{\min} = W\left(\frac{\sqrt{4\mathcal{R}_{\text{inv}}+4L_{\text{inv}}^4+L_{\text{inv}}^2}}{2L_{\text{inv}}} - \frac{1}{2}\right), \tag{39}$$

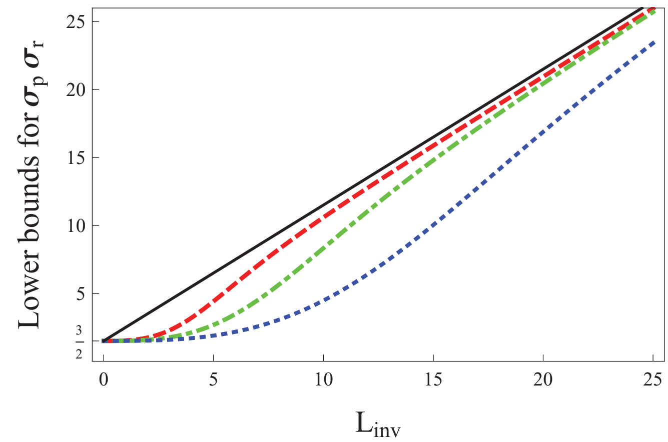

FIG. 1. (Color online) The dependence on L_{inv} of the new bound (6) for some fixed values of $\mathcal{R}_{\text{inv}}$: $\mathcal{R}_{\text{inv}} = 10^3$, red dashed; $\mathcal{R}_{\text{inv}} = 10^4$, green dashed-dotted; $\mathcal{R}_{\text{inv}} = 10^5$, blue dotted. The black line represents the reference bound (2). All bounds are above the Heisenberg bound 3/2.

where the function $W(x)$ reads ($F_{\text{inv}} = \mathcal{R}_{\text{inv}} + L_{\text{inv}}^4$)

$$W(x) = \frac{L_{\text{inv}}^2[(1+\lceil x\rceil)^2 + (1+\lfloor x\rfloor)^2 + \lceil x\rceil\lfloor x\rfloor - 1] - F_{\text{inv}}}{(1+\lceil x\rceil)(1+\lfloor x\rfloor)}. \tag{40}$$

One can check that this result is actually a minor improvement of Eq. (6); moreover, the function $W(x)$ is not even continuous.

IV. DISCUSSION

In this Sec. I would like to present the connection of the uncertainty relation (6) with the previous results (1) and (2). First of all we shall note, looking at the expression (36), that

$$\Omega(0,\mathcal{R}) = \tfrac{3}{2} \quad \text{and} \quad \lim_{\mathcal{R}\to\infty} \Omega(L,\mathcal{R}) = \tfrac{3}{2}. \tag{41}$$

These results mean that in both cases, when the angular momentum is $L = 0$ or when the variance $\mathcal{R}$ is very large, we have no improvement of the ordinary Heisenberg uncertainty relation (1). This conclusion is in full agreement with logical expectations. Furthermore, it is easy to check that

$$\Omega(\sqrt{l(l+1)},0) = \tfrac{3}{2} + l, \tag{42}$$

which coincides with the uncertainty relation (2). The value $\mathcal{R} = 0$ means that the state is a true eigenstate of $\hat{L}^2$ (or $\hat{L}_{\text{inv}}^2$ in a general reference frame) and the average value L^2 must be equal to $l(l+1)$, where l is the related quantum number. Figure 1 summarizes these observations.

V. CONCLUSIONS

In this paper I have discussed the Heisenberg uncertainty relation for position and momentum with additional information about angular momentum. I derived a lower bound for the product of standard deviations $\sigma_r\sigma_p$ which depends on the average value and the variance of the $\hat{L}_{\text{inv}}^2$ operator. This operator is a square of the angular momentum operator defined in the reference frame where $\langle \boldsymbol{r}\rangle = 0 = \langle \boldsymbol{p}\rangle$. I showed that relation (6) links the ordinary Heisenberg uncertainty relation (1) with stronger relation (2) valid for the eigenstates of $\hat{L}^2_{\rm inv}$.

ACKNOWLEDGMENTS

I am especially grateful to Iwo Białynicki-Birula for long discussions and several important comments. Furthermore, I would like to thank all members of the Atomic Physics Group from University of Granada for their kind hospitality and fruitful discussions. Finally, the referee deserves thanks for his/her accurate remarks, which helped me clarify the main message of this paper. Financial support by the Grant of the Polish Ministry of Science and Higher Education for years 2010–2012 is gratefully acknowledged.

[1] W. Heisenberg, Z. Phys. **43**, 172 (1927).
[2] E. Kennard, Z. Phys. **44**, 326 (1927).
[3] P. Sánchez-Moreno, R. González-Férez, and J. S. Dehesa, New J. Phys. **8**, 330 (2006).
[4] C. Bracher, Am. J. Phys. **79**, 313 (2011).
[5] M. E. Grypeos *et al.*, J. Phys. A **37**, 7895 (2004).
[6] I. Bialynicki-Birula and J. Mycielski, Commun. Math. Phys. **44**, 129 (1975).
[7] I. Bialynicki-Birula, Phys. Rev. A **74**, 052101 (2006).
[8] S. Zozor, M. Portesi, P. Sánchez-Moreno, and J. S. Dehesa, Phys. Rev. A **83**, 052107 (2011).
[9] J. C. Angulo, Phys. Rev. A **83**, 062102 (2011).
[10] P. Sánchez-Moreno, A. R. Plastino, and J. S. Dehesa, J. Phys. A **44**, 065301 (2011).
[11] P. Sánchez-Moreno, S. Zozor, and J. S. Dehesa, J. Math. Phys. **52**, 022105 (2011).
[12] E. Romera, P. Sánchez-Moreno, and J. S. Dehesa, J. Math. Phys. **47**, 103504 (2006).
[13] Á. Guerrero, P. Sánchez-Moreno, and J. S. Dehesa, Phys. Rev. A **84**, 042105 (2011).
[14] I. A. Howard, N. H. March, and L. M. Nieto, Phys. Rev. A **66**, 054501 (2002).
[15] J. S. Dehesa, R. González-Férez, P. Sánchez-Moreno, and R. J. Yáñez, New J. Phys. **9**, 131 (2007).
[16] I. Bialynicki-Birula and Z. Bialynicka-Birula, e-print arXiv:1110.2415v1.
[17] I. Bialynicki-Birula, M. Cieplak, and J. Kamiński, *Theory of Quanta* (Oxford University Press, New York, 1992).
[18] These functions may be defined as $\lceil x \rceil = \min(m \in \mathbb{Z} : x \leqslant m)$ and $\lfloor x \rfloor = \max(n \in \mathbb{Z} : x \geqslant n)$.