Whispering-gallery states of antihydrogen near a curved surface

A. Yu. Voronin, V. V. Nesvizhevsky, and S. Reynaud
P. N. Lebedev Physical Institute, 53 Leninsky Prospect, RU-117924 Moscow, Russia,
Institut Laue-Langevin (ILL), 6 rue Jules Horowitz, F-38042, Grenoble, France, and
Laboratoire Kastler Brossel, Campus Jussieu, F-75252 Paris, France
(Received 15 October 2011; revised manuscript received 27 December 2011; published 18 January 2012)

We predict the existence of antihydrogen atom long-living quasistationary states, localized near a curved material surface due to quantum reflection from the van der Waals–Casimir potential. Such states are an atom-wave analog of the whispering-gallery (WG) modes, known in acoustics, optics, and neutron physics. We argue that the WG states of antihydrogen atoms could be regarded as a close analog of recently predicted gravitational states of antihydrogen where the centrifugal potential plays the role of the linear gravitational potential. We point out a method for the precision measurement of anti-atom-matter interactions, based on the study of interference of WG antihydrogen states.

DOI: 10.1103/PhysRevA.85.014902 PACS number(s): 34.35.+a, 36.10.Gv

I. INTRODUCTION

Since ancient times, the localization of waves near a curved material surface was known in acoustics; it was called the whispering-gallery (WG) effect. It was explained in detail by Lord Rayleigh in his *Theory of Sound* [1,2]. The WG effect in optics [3,4] has been the object of growing interest during the last decade due to their multiple applications [5,6]. The WG type of matter-wave localization was observed recently in neutron scattering on a cylinder [7,8]. WG states of atoms were studied in relation to the possibility of atom trapping in an *external* field near a curved surface [9–12] or to atom interaction with a quantized field in cavities (see Refs. [13–16] and references therein). Here, we study a different class of WG matter-wave localization phenomena. Namely, we predict that antihydrogen ($\bar{H}$) atoms could be settled in long-living WG states near a curved material surface in the absence of external fields.

$\bar{H}$ atoms, impinging a material surface, are partially reflected due to the phenomenon of overbarrier quantum reflection [17–20] from the van der Waals–Casimir (vdW–CP) atom-wall potential [21–26]. The smaller the normal incidence momentum k of $\bar{H}$, the larger the reflection coefficient. It tends to unity in the limit of zero k. Thus, a material wall appears to be an efficient reflector for slow enough antiatoms [27,28]. This is why an $\bar{H}$ atom may be localized in quantum quasistationary states near a material surface [29] in the gravitational field of the Earth. It bounces on a surface the same way neutrons bounce in gravitational quantum states, discovered recently [30–32]. The characteristic lifetime of $\bar{H}$ states above a conducting surface is long enough ($\tau \simeq 0.1$ s) and could be increased significantly by choosing a proper material. Gravitational states of $\bar{H}$ could be a useful tool for studying gravitational properties of antiatoms.

In a similar phenomenon of the localization of $\bar{H}$ atoms, moving in the vicinity of a curved surface, corresponding quasistationary quantum states are formed by a superposition of the effective centrifugal potential and the quantum reflection. We are interested in the states with high angular momentum such that the kinetic energy of tangential motion (parallel to the curved surface) is close to the total energy. This condition means that $\bar{H}$ radial motion normal to the material wall is slow, which provides a large probability of quantum reflection and, thus, a long lifetime of the quasistationary states. We show that, for a certain range of parameters (namely, the tangential velocities v of $\bar{H}$ and the surface curvature radius R), the problem of $\bar{H}$ transport near a curved surface is equivalent to the dynamics of $\bar{H}$ above a plane surface in the gravitational field with effective acceleration v^2/R. Indeed, the effective centrifugal potential can be approximated accurately by a linear potential in the vicinity of the curved surface. In the problem of interest, we deal with the effective centrifugal potential $(mv^2/R)x$ (where m is the inertial mass of $\bar{H}$ and x is the distance from the surface), instead of the gravitational potential Mgx (where M is the gravitational mass of $\bar{H}$ and g is the free-fall acceleration). Thus, mentioned WG states appear complementary to gravitational states in testing the weak equivalence principle (WEP). Additional benefits of exploiting the WG effect for the study of $\bar{H}$ interactions consist of long times in which $\bar{H}$ atoms spend in the vicinity of a material surface in WG states and the simplicity to tune the effective centrifugal acceleration by changing atom-beam velocity. They also could provide a promising tool for guiding and trapping antimatter with curved material walls.

II. ANTIHYDROGEN IN A CYLINDER WAVEGUIDE

We are interested in the $\bar{H}$ atom transport through a cylindrical waveguide with radius R. Atom dynamics obeys the following Schrödinger equation in the cylindrical coordinates:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\right) + V(|R-\rho|)-\frac{p^2}{2m}\right]\Psi(z,\rho,\varphi)=0. \quad (1)$$

Here, $\Psi(z,\rho,\varphi)$ is the $\bar{H}$ wave function, z is the distance along the cylinder axis, ρ is the radial distance measured from the cylinder axis, φ is the angle, $V(|R-\rho|)$ is the $\bar{H}$-wall interaction potential [28,33], and p is the $\bar{H}$ momentum. The wave function depends on z in a trivial way, so we omit this dependence. Equation (1) is transformed into the following form by standard substitution $\Psi(\rho,\varphi) = \Phi(\rho,\varphi)/\sqrt{\rho}$:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2}\right) - \frac{\hbar^2}{2m\rho^2}\left(\frac{\partial^2}{\partial\varphi^2} + \frac{1}{4}\right) + V(|R-\rho|) - \frac{p^2}{2m}\right]\Phi(\rho,\varphi) = 0. \quad (2)$$

The above equation meets the following boundary conditions: regularity at $\rho = 0$, uniqueness under the substitution $\varphi \to \varphi + 2\pi$, and full absorption (annihilation) in the wall bulk at $\rho \geqslant R$ [28].

The wave function is decomposed into the angular momentum eigenfunctions basis,

$$\Phi(\rho,\varphi) = \sum_{\mu=-\infty}^{\mu=+\infty} \chi_\mu(\rho)\exp(i\mu\varphi), \quad (3)$$

with $\chi_\mu(\rho)$ as the radial motion wave functions, which obey the following one-dimensional radial equation:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\rho^2} + \frac{\hbar^2(\mu^2-1/4)}{2m\rho^2} + V(|R-\rho|) - \frac{p^2}{2m}\right]\chi_\mu(\rho) = 0. \quad (4)$$

The solutions of interest are regular at $\rho = 0$.

The $\bar{H}$-wall interaction potential has the asymptotic $C_4/|R-\rho|^4$ and the characteristic spatial scale $l_{CP} = \sqrt{2mC_4}/\hbar$, where C_4 is the Casimir-Polder coefficient [21], which depends on the wall material properties alone and stands for the retardation effects in the atom-wall interaction. The potential behaves like $C_3/|R-\rho|^3$ at short distances; here, C_3 is the van der Waals coefficient [21], determined by the wall material properties. The solution of Eq. (4) satisfies the condition of total absorption in the wall $\rho \to R$,

$$\chi_\mu(\rho) \sim \sqrt{|R-\rho|}H_1^{(1)}(2\sqrt{2mC_3/|R-\rho|}), \quad (5)$$

where H_1^1 is the Hankel function on the order of unity [34]. Thus, Eq. (5) can be used as a boundary condition at $\rho \to R$.

A solution of Eq. (4) for the given energy $p^2/(2m)$ and for all μ values provides a complete description of the $\bar{H}$ dynamic in a waveguide. We are interested in a special class of solutions χ_μ, which correspond to the $\bar{H}$ WG modes.

III. WG STATES

We are interested in the $\bar{H}$ states with large angular momentum μ such that $\hbar^2\mu^2/R^2 \approx p^2$ [8]. In this case, the radial motion is slow, which is necessary for efficient $\bar{H}$ quantum reflection from the surface. In order to solve Eq. (4), we expand the expression for the centrifugal potential in the vicinity of $\rho = R$, introducing the deviation from the cylinder surface $x = \rho - R$. In the first order of the small ratio x/R, we get the following equation:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(-x) + \hbar^2\frac{\mu^2-1/4}{2mR^2} \times \left(1 - \frac{2x}{R}\right) - \frac{p^2}{2m}\right]\chi_\mu(x) = 0. \quad (6)$$

The full absorption boundary condition Eq. (5) is now

$$\chi_\mu(x \to 0) \sim \sqrt{x}H_1^{(1)}(2\sqrt{2mC_3/x}). \quad (7)$$

Introducing a new variable,

$$\varepsilon_\mu = \frac{p^2}{2m} - \hbar^2\frac{\mu^2-1/4}{2mR^2} \simeq \frac{(pR)^2 - \hbar^2\mu^2}{2mR^2},$$

we get the following equation:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(-x) - \frac{mv_\mu^2}{R}x - \varepsilon_\mu\right]\chi_\mu(x) = 0. \quad (8)$$

Here, we introduce the tangential velocity v_μ such that $\mu = mv_\mu R/\hbar$. The value ε_μ could be understood as the radial motion energy. In the following, we omit the subscript μ, assuming that the following results are obtained for a fixed μ value.

Equation (8) describes the $\bar{H}$ motion in a constant effective field $a = -v^2/R$ superposed with the $\bar{H}$-wall CP–vdW potential $V(x)$. Equation (8) gives the spatial l_0 and energy ε_0 scales, characteristic for the effective linear potential $m(v^2/R)x$,

$$l_0 = \sqrt[3]{\frac{\hbar^2 R}{2m^2v^2}}, \quad (9)$$

$$\varepsilon_0 = \sqrt[3]{\frac{\hbar^2 m v^4}{2R^2}}. \quad (10)$$

We are interested in such a range of the parameters of the problem (R and v) that

$$l_0 \gg l_{CP}. \quad (11)$$

This condition means, in particular, that the effect of the vdW–CP potential on the $\bar{H}$ motion in a linear potential can be described using a modified boundary condition. Under the condition, Eq. (11), the problem of interest is equivalent to the problem of $\bar{H}$ motion in a superposition of the linear *gravitational* potential and the CP–vdW antiatom-wall interaction potential, studied in Ref. [29]. In particular, for the parameters $R = 0.1$ m and $v = 0.99$ m/s,

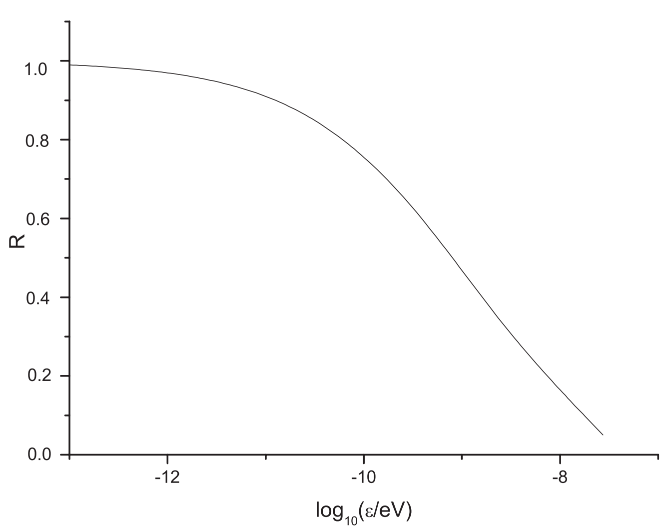

FIG. 1. The reflection coefficient from the CP–vdW interaction potential as a function of the incident (radial) energy.

the effective centrifugal acceleration coincides with the free-fall acceleration $v^2/R = g$. The corresponding WG spatial scale is $l_0 = 5.87\ \mu$m, while the energy scale is $\varepsilon_0 = 0.60$ peV.

An important feature of the problem of interest is high reflectivity of slow $\bar{H}$ from the vdW–CP interaction potential $V(-x)$, which takes place at distances, small compared to the characteristic spatial scale l_0. In Fig. 1, we show the reflection coefficient as a function of the radial energy ε_μ.

Using the analogy with gravitational quantum states of $\bar{H}$ atoms [29], we find that the lifetime $\tau = \hbar/\Gamma$ of the quasistationary states is 0.1 s for $v = 0.99$ m/s and $R = 0.10$ m for an ideally conducting surface, and it goes up to 0.2 s for the silica surface. The $\bar{H}$ atom angular position changes by

$$\Delta\varphi = \frac{\tau v}{R} = \frac{\hbar}{2mv|\mathrm{Im}\, a_{CP}|}, \quad (12)$$

during this lifetime. Here, a_{CP} is the complex scattering length on the vdW–CP potential. $a_{CP} = -0.0027 - i0.0287\ \mu$m for an ideally conducting surface, $a_{CPs} = -0.0035 - i0.0144\ \mu$m for the silica surface, and $\Delta\varphi = 1.136$ rad for an ideally conducting surface and twice larger for the silica surface.

Although the properties of WG states of $\bar{H}$ near a cylindrical surface are equivalent to the gravitational states of $\bar{H}$ near a plane surface, only the inertial mass of $\bar{H}$ is involved in the case of WG states, and here, the effective acceleration v^2/R is a tunable parameter.

IV. CONCLUSION

We have predicted the existence of long-living quantum states of $\bar{H}$ atoms, moving near a cylindrical material surface. Such states are a matter-wave analog of the WG wave. The localization of $\bar{H}$ atoms near a surface of a curved mirror is due to the combined effect of quantum reflection and centrifugal potential. We have shown that, under a certain choice of the curvature radius and the $\bar{H}$ velocity, the problem of $\bar{H}$ transport near a curved surface is equivalent to $\bar{H}$ behavior near a plane material surface in the presence of a linear potential. Interference of long-living WG states of $\bar{H}$ could provide a promising tool in precision measurements of $\bar{H}$-surface interactions, gravitational properties and WEP tests, $\bar{H}$ guiding, and trapping.

[1] Baron J. W. Strutt Rayleigh, *The Theory of Sound* (Macmillan, London, 1878), Vol. 2.
[2] L. Rayleigh, Philos. Mag., **27**, 100 (1914).
[3] G. Mie, Ann. Phys. **25**, 371 (1908).
[4] P. Debye, Ann. Phys. **30**, 57 (1909).
[5] A. N. Oraevsky, Quant. Electron. **32**, 377 (2002).
[6] K. J. Vahala, Nature (London) **424**, 839 (2003).
[7] V. V. Nesvizhevsky, A. Y. Voronin, R. Cubitt, and K. V. Protasov, Nat. Phys. **6**, 114 (2010).
[8] R. Cubitt, V. V. Nesvizhevsky, K. V. Protasov, and A. Y. Voronin, New J. Phys. **12**, 113050 (2010).
[9] D. W. Vernooy and H. J. Kimble, Phys. Rev. A **55**, 1239 (1997).
[10] H. Mabichi and H. J. Kimble, Opt. Lett. **19**, 749 (1994).
[11] R. P. Bertram *et al.*, Phys. Rev. A **63**, 053405 (2001).
[12] C. G. Aminoff *et al.*, Phys. Rev. Lett. **71**, 3083 (1993).
[13] D. Meschede, H. Walther, and G. Muller, Phys. Rev. Lett. **54**, 551 (1985).
[14] H. Wallis, J. Dalibard, and C. Cohen-Tannoudji, Appl. Phys. B **54**, 407 (1992).
[15] M. P. Klembovsky, M. L. Gorodetsky, and T. Becker, Phys. Lett. A **372**, 5246 (2008).
[16] D. J. Alton, N. P. Stern, T. Aoki, E. Ostby, H. Lee, K. J. Vahala, and H. J. Kimble, Nat. Phys. **7**, 159 (2011).
[17] J. E. Lennard-Jones, Trans. Faraday Soc. **28**, 333 (1932).
[18] H. Friedrich, G. Jacoby, and C. G. Meister, Phys. Rev. A **65**, 032902 (2002).
[19] G. Feinverg and J. Sucher, Phys. Rev. A **2**, 2395 (1970).
[20] E. A. Power and T. Thirunamachandran, Phys. Rev. A **50**, 3929 (1994).
[21] H. B. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).
[22] I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Adv. Phys. **10**, 165 (1960).
[23] T. A. Pasquini, Y. Shin, C. Sanner, M. Saba, A. Schirotzek, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. **93**, 223201 (2004).
[24] T. A. Pasquini *et al.*, Phys. Rev. Lett. **97**, 093201 (2006).
[25] F. Shimizu, Phys. Rev. Lett. **86**, 987 (2001).
[26] J. M. Obrecht, R. J. Wild, M. Antezza, L. P. Pitaevskii, S. Stringari, and E. A. Cornell, Phys. Rev. Lett. **98**, 063201 (2007).
[27] A. Y. Voronin, P. Froelich, and B. Zygelman, Phys. Rev. A **72**, 062903 (2005).
[28] A. Y. Voronin and P. Froelich, J. Phys. B **38**, L301 (2005).
[29] A. Y. Voronin, P. Froelich, and V. V. Nesvizhevsky, Phys. Rev. A **83**, 032903 (2011).
[30] V. V. Nesvizhevsky, H. G. Börner, A. K. Petukhov, H. Abele, S. Baeßler, F. J. Rueß, T. Stöferle, A. Westphal, A. M. Gagarski, G. A. Petrov, and A. V. Strelkov, Nature (London) **415**, 297 (2002).
[31] V. V. Nesvizhevsky *et al.*, Phys. Rev. D **67**, 102002 (2003).
[32] V. V. Nesvizhevsky *et al.*, Eur. Phys. J. C **40**, 479 (2005).
[33] M. Marinescu, A. Dalgarno, and J. F. Babb, Phys. Rev. A **55**, 1530 (1997).
[34] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, 10th ed. (Dover, New York, 1972).