Homodyne detection for atmosphere channels

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We give a systematic theoretical description of homodyne detection in the case where both the signal and the local oscillator pass through the turbulent atmosphere. Imperfect knowledge of the local-oscillator amplitude is effectively included in a noisy density operator, leading to postprocessing noise. Alternatively, we propose a technique with monitored transmission coefficient of the atmosphere, which is free of postprocessing noise.

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I. INTRODUCTION

Long-distance quantum communication [1,2] necessarily deals with strong unwanted effects of the environment. In this context, one usually compares two types of channels: optical fibers and free space. For purposes of quantum optics, it is important that in fibers [3] one usually deals with a stable intensity attenuation and with strong depolarization effects. In free-space channels [4,5], the situation is different: The attenuation randomly fluctuates and the depolarization effect is negligibly small.

An important method for measuring the quantum-light characteristics is the technique of balanced homodyne detection [6,7]. In this case, the signal field is combined through a 50:50 beam splitter with a strong coherent field, the local oscillator. The difference of photocounts in two outputs of the beam splitter is proportional to the field quadrature. By applying this procedure for different values of the local-oscillator phase, one could get complete information about the quantum state of the signal. Particularly, one can reconstruct the density operator in different representations [7–9].

The application of homodyne detection for long-distance quantum communications in free-space channels meets the problem of phase synchronization between the signal and the local oscillator. A possible way to overcome this difficulty could be based on the technique of the optical frequency comb [10]. In this case, the detected signal will be randomized by the atmosphere with respect to both the amplitude and the phase [11]. A more traditional way to provide such a synchronization is to derive the signal and the local oscillator from the same source. However, in this case the local oscillator will also be affected by the atmospheric turbulence. At least part of this problem can be resolved by sending the signal and the local oscillator from the same source in orthogonally polarized modes [12]. Since the atmospheric depolarization effects are negligible, the phase synchronization is not destroyed in such an experiment. On the other hand, in this case the local-oscillator amplitude randomly fluctuates due to the atmospheric noise. As a result, the problem is how to connect the photocount difference with the field quadrature.

In the present paper, we consider the situation when the signal and the local oscillator pass through the turbulent atmosphere in orthogonally polarized modes. First, we analyze the scheme proposed in Ref. [12]. In this case, the photocount difference can be connected with the field quadrature by using a certain reference value of the local-oscillator amplitude. This is equivalent to the use of a reference transmission coefficient of the atmosphere, for example, its mean value. This results in a kind of noise, which is related to the postprocessing procedure. Reconstructed with such a procedure, the density operator may even fail to satisfy the fundamental requirement of positive semidefiniteness, which is a serious disadvantage of this method. To resolve this deficiency, we propose a procedure with a permanently monitored transmission coefficient. This renders it possible to recover the true values of the field quadratures from the measured photocount differences. In this case, the postprocessing noise disappears.

The paper is organized as follows. In Sec. II, we derive an expression for the statistics of photocount differences for the scheme considered in Ref. [12]. This result is used in Sec. III, where the photocount difference is connected with the field quadrature by using a fixed reference transmission coefficient. A method based on monitoring the transmission coefficient is developed in Sec. IV. In Sec. V, we derive input-output relations for the normally ordered covariance matrix and consider the effect of quadrature squeezing of the light passing through the atmosphere. A summary and conclusions are given in Sec. VI.

II. STATISTICS OF PHOTOCOUNT DIFFERENCES

Let us consider the experimental scenario as implemented in Ref. [12] with the local oscillator copropagating with the signal field in the same spatial but different polarization modes; see Fig. 1. The half-wave plate HWP and the polarization beam splitter PBS at the receiver prepare 50:50 field superposition of the signal and the local oscillator as done in the standard homodyne detection. The detectors D_1 and D_2 are used for measuring the photocount difference Δn, which is applied for further analysis.

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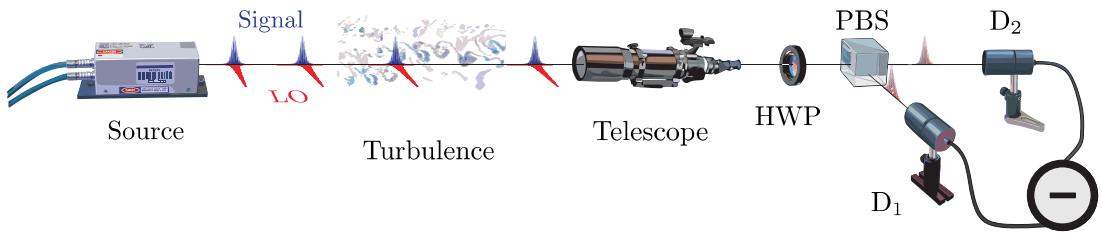

FIG. 1. (Color online) Scheme of homodyne detection for quantum light passing through the turbulent atmosphere as reported in Ref. [12]. The signal and the local oscillator are sent in two orthogonally polarized modes. After passing through the atmospheric channel and collection by a telescope, homodyne detection is realized with the half-wave plate HWP, the polarization beam splitter PBS, and the detectors D_1 and D_2.

The probability distribution of the photocount difference Δn in the Heisenberg picture is given by (cf. Ref. [13])

$$p_{\Delta n} = \mathrm{Tr}\left[\hat{\rho}\,\hat{K}_{\Delta n}^{\mathrm{noisy}}\right], \tag{1}$$

where $\hat{\rho}$ is the input-signal density operator and $\hat{K}_{\Delta n}^{\mathrm{noisy}}$ is the noisy positive operator-valued measure (POVM) of photocount differences. The latter,

$$\hat{K}_{\Delta n}^{\mathrm{noisy}} = \langle re^{i\varphi}|\langle 0|\sum_{n=\Delta n}^{+\infty}\hat{\Pi}_n \otimes \hat{\Pi}_{n-\Delta n}|0\rangle|re^{i\varphi}\rangle, \tag{2}$$

is expressed in terms of the POVM of photocounts,

$$\hat{\Pi}_{n_i} =: \frac{(\eta\,\hat{b}_i^\dagger\hat{b}_i + \bar{N}_{\mathrm{nc}})^{n_i}}{n_i!}\exp(-\eta\,\hat{b}_i^\dagger\hat{b}_i - \bar{N}_{\mathrm{nc}}) : , \tag{3}$$

where $:\cdots:$ denotes the normal ordering prescription. The coherent-state vector $|re^{i\varphi}\rangle$ represents the local oscillator of amplitude r and phase φ. The vacuum-state vector $|0\rangle$ includes all modes of the environment. Moreover, η is the detection efficiency, and $\bar{N}_{\mathrm{nc}}$ is the mean value of noise counts caused by stray light as well as dark counts; see Ref. [14]. The annihilation and creation operators $\hat{b}_i$ and $\hat{b}_i^\dagger$, respectively, represent the light modes at the ith output of the polarization beam splitter PBS. This beam splitter and the half-wave plate can be described by the input-output relations,

$$\hat{b}_1 = \frac{1}{\sqrt{2}}(\hat{b}_{\mathrm{s}} + \hat{b}_{\mathrm{lo}}), \tag{4}$$

$$\hat{b}_2 = \frac{1}{\sqrt{2}}(-\hat{b}_{\mathrm{s}} + \hat{b}_{\mathrm{lo}}), \tag{5}$$

where $\hat{b}_{\mathrm{s}}$ and $\hat{b}_{\mathrm{lo}}$ are annihilation operators of the signal and the local oscillator at the input of the beam splitter.

Next, we have to include in the consideration the effect of the atmosphere. It can be performed using the approach of fluctuating-loss channels [11,15–18]. Let $\hat{a}_{\mathrm{s}}$ and $\hat{a}_{\mathrm{lo}}$ be the annihilation operators of the signal and the local oscillator, respectively, at the sender. The corresponding input-output relation for light passing through the atmosphere reads as

$$\hat{b}_{\mathrm{s}} = T\hat{a}_{\mathrm{s}} + \sqrt{1-T^2}\hat{c}_{\mathrm{s}}, \tag{6}$$

$$\hat{b}_{\mathrm{lo}} = T\hat{a}_{\mathrm{lo}} + \sqrt{1-T^2}\hat{c}_{\mathrm{lo}}, \tag{7}$$

where $\hat{c}_{\mathrm{s}}$ and $\hat{c}_{\mathrm{lo}}$ are operators of the environment modes being in the vacuum state and T is the atmospheric transmission coefficient. The following properties of relations (6) and (7) are important. First, the transmission coefficient T is a random variable. Second, since the depolarization effect of the atmosphere is negligible, the transmission coefficients in both polarization modes are perfectly correlated and equal T. Third, in the considered case, the absence of the depolarization means the absence of dephasing, and hence T can be considered as a real random variable. Finally, the commutation rules require that $T \in [0,1]$.

The above treatment can be easily used in Eq. (1) with the Glauber-Sudarshan P representation [19] for the signal density operator,

$$\hat{\rho} = \int_{-\infty}^{+\infty} d^2\alpha|\alpha\rangle P(\alpha)\langle\alpha|, \tag{8}$$

where $P(\alpha)$ is the Glauber-Sudarshan P function of the input signal field (at the sender) and $|\alpha\rangle$ is a coherent-state vector. Substituting Eqs. (2)–(8) into Eq. (1) and taking into account that T is a random variable, one gets

$$p_{\Delta n} = \int_{-\infty}^{+\infty} d^2\alpha\, P(\alpha) K_{\Delta n}^{\mathrm{noisy}}(\alpha), \tag{9}$$

where

$$\begin{aligned} K_{\Delta n}^{\mathrm{noisy}}(\alpha) = \int_0^1 dT\mathcal{P}(T)&\left(\frac{\eta\theta_1 + 2\bar{N}_{\mathrm{nc}}}{\eta\theta_2 + 2\bar{N}_{\mathrm{nc}}}\right)^{\Delta n/2} \\ &\times \mathrm{I}_{\Delta n}[\sqrt{(\eta\theta_1 + 2\bar{N}_{\mathrm{nc}})(\eta\theta_2 + 2\bar{N}_{\mathrm{nc}})}] \\ &\times \exp\left[-\eta(T^2r^2 + T^2|\alpha|^2) - 2\bar{N}_{\mathrm{nc}}\right] \end{aligned} \tag{10}$$

is the Husimi-Kano Q symbol [20] of the POVM of photocount differences, where

$$\theta_{1,2} = T^2[r^2 + |\alpha|^2 \pm 2r\,\mathrm{Re}(\alpha e^{-i\varphi})], \tag{11}$$

$\mathcal{P}(T)$ is the probability distribution of the transmission coefficient (PDTC) of the atmospheric channel, $\mathrm{I}_{\Delta n}$ is the modified Bessel function. For simplicity, further we refer to the Q symbol of the POVM as the POVM.

For the purposes of balanced homodyne detection, one usually uses a local oscillator, which is strong compared to the signal. After transmission through the atmosphere, the intensity of the signal is still small compared to the local oscillator, $T^2|\alpha|^2 \ll T^2r^2$. The contribution of the latter can be comparable with the noise counts, $T^2r^2 \sim \bar{N}_{\mathrm{nc}}$. Following

the argumentation of Ref. [13], one can approximate Eq. (10) by

$$K_{\Delta n}^{\text{noisy}}(\alpha) = \int_0^1 dT\mathcal{P}(T)\frac{1}{\sqrt{2\pi(\eta T^2 r^2 + 2\bar{N}_{\text{nc}})}} \times \exp\left[-\frac{(\Delta n - 2\eta T^2 r\,\text{Re}[\alpha e^{-i\varphi}])^2}{2(\eta T^2 r^2 + 2\bar{N}_{\text{nc}})}\right]. \quad (12)$$

Equations (9) and (12) can be directly used for evaluating the statistics of photocount differences when both the signal and the local oscillator pass through the turbulent atmosphere.

III. POSTPROCESSING NOISE

The next problem is to connect the photocount differences $\Delta\hat{n}$ with the field quadrature,

$$\hat{x}(\varphi) = \frac{1}{\sqrt{2}}[\hat{a}^\dagger e^{i\varphi} + \hat{a}e^{-i\varphi}]. \quad (13)$$

The corresponding relation is given by (see, e.g., Ref. [7]),

$$\hat{x}(\varphi) = \frac{\Delta\hat{n}}{r_{\text{out}}\sqrt{2}}, \quad (14)$$

where r_{out} is the local-oscillator amplitude scaled by atmosphere losses and detection efficiency. Its real value is

$$r_{\text{out}} = T\sqrt{\eta}r. \quad (15)$$

However, since in the considered experiment we do not have any information about the current value of the fluctuating transmission coefficient T, we can use a certain reference value T_{ref}, and set

$$r_{\text{out}} = T_{\text{ref}}\sqrt{\eta}r. \quad (16)$$

Because the real value of r_{out} given by Eq. (15) randomly changes in the atmosphere and deviates from its reference value (16), the obtained quadrature value suffers from a kind of noise, which we refer to as postprocessing noise. For this reason, the reconstructed density operator and any characteristics obtained from the approximate quadrature values are, in fact, contaminated by the corresponding noise effects.

Let us consider in more detail the effects of the postprocessing noise. Based on Eqs. (9), (12), (14), and (16), the quadrature distribution in the considered case is given by

$$p(x;\varphi) = \int_{-\infty}^{+\infty} d^2\alpha\, P(\alpha)K_{\text{noisy}}[x(\varphi);\alpha], \quad (17)$$

where

$$K_{\text{noisy}}[x(\varphi);\alpha] = \int_0^1 dT\mathcal{P}(T)\frac{1}{\sqrt{\pi\left(\frac{T^2}{T_{\text{ref}}^2} + \frac{2\bar{N}_{\text{nc}}}{\eta r^2 T_{\text{ref}}^2}\right)}} \times \exp\left[-\frac{\left(x - \sqrt{2\eta}\,\frac{T^2}{T_{\text{ref}}}\,\text{Re}[\alpha e^{-i\varphi}]\right)^2}{\frac{T^2}{T_{\text{ref}}^2} + \frac{2\bar{N}_{\text{nc}}}{\eta r^2 T_{\text{ref}}^2}}\right] \quad (18)$$

is the resulting noisy quadrature POVM. Alternatively, Eq. (17) can be rewritten in the Schrödinger picture as

$$p(x;\varphi) = \int_{-\infty}^{+\infty} d^2\alpha\, P_{\text{noisy}}(\alpha)K[x(\varphi);\alpha], \quad (19)$$

where

$$K[x(\varphi);\alpha] = \frac{1}{\sqrt{\pi}}\exp[(x - \sqrt{2}\,\text{Re}[\alpha e^{-i\varphi}])^2] \quad (20)$$

is the noiseless quadrature POVM,

$$P_{\text{noisy}}(\alpha) = \int_0^1 dT\mathcal{P}(T)\frac{T_{\text{ref}}^2}{T^4\eta}\exp\left[\left(\frac{T^2 - T_{\text{ref}}^2}{8T_{\text{ref}}^2} + \frac{\bar{N}_{\text{nc}}}{4r^2T_{\text{ref}}^2\eta}\right)\Delta_\alpha\right] \times P\left(\frac{T_{\text{ref}}}{T^2\sqrt{\eta}}\alpha\right) \quad (21)$$

is the noisy P function of the detected signal, and $\Delta_\alpha = \frac{\partial^2}{\partial^2\text{Re}\alpha} + \frac{\partial^2}{\partial^2\text{Im}\alpha}$ is the Laplace operator in phase space. Equation (21) can be considered as the quantum-state input-output relation, where the noisy density operator, represented by the P function, is affected by (i) fluctuating losses due to the signal transmission through the atmosphere, (ii) detection losses and noise counts, (iii) postprocessing noise caused by imperfect knowledge of the transmission coefficient. Any reconstruction of the density operator using homodyne-detection data, obtained from Eq. (14) together with the approximation (16), will yield the noisy quantum state (21). Similarly, any characteristics obtained from such quadratures also correspond to the noisy density operator.

As follows from Eq. (21), the contribution of noise counts in the considered scheme can always be made negligible by choosing a sufficiently strong local oscillator. In this case, the quantum-state input-output relation reduces to

$$P_{\text{noisy}}(\alpha) = \int_0^1 dT\,\mathcal{P}(T)\frac{T_{\text{ref}}^2}{T^4\eta}\exp\left[\frac{T^2 - T_{\text{ref}}^2}{8T_{\text{ref}}^2}\Delta_\alpha\right] \times P\left(\frac{T_{\text{ref}}}{T^2\sqrt{\eta}}\alpha\right). \quad (22)$$

This equation can be interpreted as the input-output relation of a fluctuating-loss channel (cf. Ref. [11]) with the effective transmission coefficient

$$T_{\text{eff}} = \frac{T^2}{T_{\text{ref}}}. \quad (23)$$

Additionally, the measurement procedure suffers from a kind of effective noise counts whose mean value is

$$\bar{N}_{\text{eff}} \sim \frac{T^2 - T_{\text{ref}}^2}{8T_{\text{ref}}^2}. \quad (24)$$

However, the upper bound of T_{eff} is not restricted anymore by the value 1. Similarly, $\bar{N}_{\text{eff}}$ may take negative values. As a result, the noisy density operator obtained by using $P_{\text{noisy}}(\alpha)$ in Eq. (8) may fail to obey the requirement of positive semidefiniteness. Such an unusual result simply reflects quantum physical inconsistencies of the method of data postprocessing under consideration.

We consider an illustration of this fact for a single-photon-added thermal state (SPATS). This state is obtained from the single-mode thermal state with the mean photon number $\bar{n}_{\rm th}$ by adding a photon by using parametric down-conversion. The corresponding P function,

$$P(\alpha) = \frac{1}{\pi \bar{n}_{\rm th}^3}[(1+\bar{n}_{\rm th})|\alpha|^2 - \bar{n}_{\rm th}]e^{-\frac{|\alpha|^2}{\bar{n}_{\rm th}}}, \tag{25}$$

is regular, which allows its experimental reconstruction [21]. We also assume that fluctuating losses are caused by beam wandering only; see the appendix. This results in log-negative Weibull distribution for the PDTC [cf. Eq. (A1)].

For a consistent positive-definite density operator, the diagonal elements in the Fock-number basis, that is, photon-number distribution, should be always nonnegative. This distribution can be reconstructed from the homodyne-detection data; see Refs. [7,22]. In the strong-turbulence regime, the postprocessing noise may result in negative values of the photon-number distribution; see Fig. 2(a). However, such fake effects can be substantially reduced in the case of weak turbulence; see Fig. 2(b).

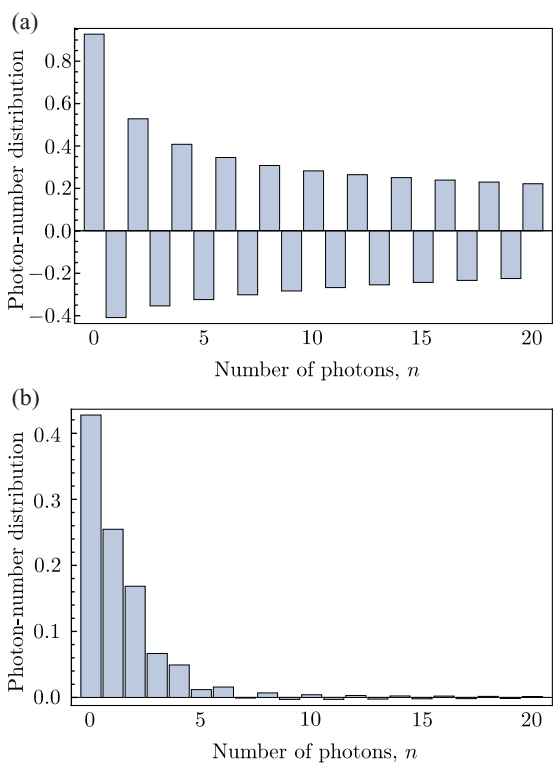

FIG. 2. (Color online) Photon-number distribution (diagonal elements of the effective density operator) for the SPATS with $n_{\rm th} = 1.11$ (such as realized in Ref. [21]) disturbed by beam wandering for the scheme with unmonitored transmission coefficient. In the corresponding turbulence model, cf. the appendix, the beam-spot radius, W, on the receiver aperture of radius a is $W = 0.9a$, and the standard deviations of beam deflection are (a) $\sigma = a$ (strong turbulence) and (b) $\sigma = 0.5a$ (weak turbulence). The corresponding reference transmission coefficient is $T_{\rm ref} = \sqrt{\langle T^2\rangle} \approx 0.586$ and $T_{\rm ref} = \sqrt{\langle T^2\rangle} \approx 0.822$, respectively. For both cases, we use the detection efficiency $\eta = 0.5$. Negative probabilities demonstrate fake effects caused by imperfect postprocessing.

IV. MONITORED TURBULENCE

In order to exclude unwanted and fake effects of the postprocessing noise, we propose to modify the scheme in Fig. 1. For this purpose, the signal and the local oscillator can be split, and a part of the local oscillator can be used for monitoring the current value of the transmission coefficient; see Fig. 3. In the most general case, the monitoring will also be affected by different kinds of noise, for example by the shot noise of the detector D_3. In terms of the previous section, this means that the value $T_{\rm ref}$ can now be replaced by the measured transmission coefficient $T_{\rm meas}$, which randomly fluctuates and correlates with fluctuating values of T. The corresponding noisy density operator is obtained similarly to Eq. (21). However, the PDTC must now be replaced with the joint PDTC $\mathcal{P}(T,T_{\rm meas})$ and integrated over both variables T and $T_{\rm meas}$. The joint PDTC can be given as

$$\mathcal{P}(T,T_{\rm meas}) = \mathcal{P}(T)\,\mathcal{P}(T_{\rm meas}|T), \tag{26}$$

where $\mathcal{P}(T_{\rm meas}|T)$ is the probability distribution of the measured transmission coefficient under the condition that its real value is T. This implies that the input-output relation for the considered experimental scheme reads as

$$\begin{aligned} P_{\rm noisy}(\alpha) = &\int_0^1 dT \int_0^1 dT_{\rm meas}\,\mathcal{P}(T)\,\mathcal{P}(T_{\rm meas}|T)\,\frac{T_{\rm meas}^2}{T^4\eta} \\ &\times \exp\left[\left(\frac{T^2 - T_{\rm meas}^2}{8T_{\rm meas}^2} + \frac{\bar{N}_{\rm nc}}{4|R_2|^2 r^2 T_{\rm meas}^2\eta}\right)\Delta_\alpha\right] \\ &\times P\left(\frac{T_{\rm meas}}{T^2\sqrt{\eta}}\,\alpha\right), \end{aligned} \tag{27}$$

where R_2 is the reflection coefficient of the beam splitter BS_2.

Let us consider the situation when the noise of monitoring is caused by the shot-noise of the detector D_3. In this case, the detected number of photocounts n_3 is related to the measured transmission coefficient $T_{\rm meas}$ as $n_3 = r^2\eta_3|T_2|^2T_{\rm meas}^2 + \bar{N}_3$. Here η_3 and $\bar{N}_3$ are the efficiency and the mean number of noise counts of detector D_3, respectively; T_2 is the transmission coefficient of the beam splitter BS_2. The measured transmission coefficient $T_{\rm meas}$ is thus obtained from the number of photocounts n_3 via

$$T_{\rm meas}^2(n_3) = \frac{1}{r^2|T_2|^2}\frac{n_3 - \bar{N}_3}{\eta_3}. \tag{28}$$

Since detector D_3 records a fraction of the local oscillator, n_3 obeys the Poissonian statistics

$$p(n_3|T) = \frac{(\eta_3 r^2|T_2|^2T^2 + \bar{N}_3)^{n_3}}{n_3!}e^{-\eta_3 r^2|T_2|^2T^2 - \bar{N}_3}. \tag{29}$$

This is the probability to get the photocount number n_3, conditioned on the value T of the transmission coefficient.

The conditional PDTC $\mathcal{P}(T_{\rm meas}|T)$ is obtained, by transforming random variables $n_3 \to T_{\rm meas}$ and using Eq. (28), in the form

$$\mathcal{P}(T_{\rm meas}|T) = \sum_{n_3=0}^{+\infty} p(n_3|T)\delta[T_{\rm meas} - T_{\rm meas}(n_3)]. \tag{30}$$

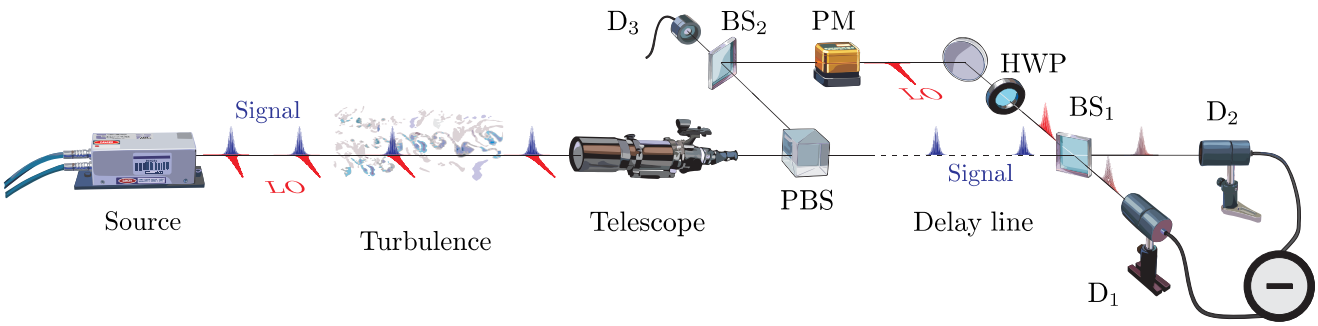

FIG. 3. (Color online) Homodyne detection of quantum light passing through the turbulent atmosphere with monitoring the transmission coefficient. The signal and the local oscillator are sent in two orthogonally polarized modes. After passing through the atmospheric channel and collection by a telescope, the signal and the local oscillator are split by the polarization beam splitter PBS. A part of the local oscillator transmitted through the beam-splitter BS_2 is used for monitoring the transmission coefficient with the detector D_3. Another part, after setting the needed phase by the phase modulator PM and conversion of the polarization direction by the half-wave plate HWP, is combined with the signal on a standard homodyne detector, which consists of the 50 : 50 beam splitter BS_2 and two photodetectors D_1 and D_2.

From this distribution, the conditional expectation value E of $T^2_{\rm meas}$ is found to be equal to T^2,

$$\mathrm{E}\big(T^2_{\rm meas}\big|T\big) = T^2. \tag{31}$$

The corresponding conditional variance reads as

$$\mathrm{Var}\big(T^2_{\rm meas}|T\big) = \frac{T^2}{\eta_3|T_2|^2r^2} + \frac{\bar{N}_3}{\eta_3^2|T_2|^4r^4}. \tag{32}$$

Equations (31) and (32) yield the relative error of T^2,

$$\epsilon = \frac{\sqrt{\mathrm{Var}\big(T^2_{\rm meas}|T\big)}}{\mathrm{E}\big(T^2_{\rm meas}|T\big)} = \sqrt{\frac{1}{\eta_3\,|T_2|^2\,T^2r^2} + \frac{\bar{N}_3}{\eta_3^2\,|T_2|^4\,T^4r^4}}. \tag{33}$$

From this expression, it follows that for the measurement of T^2 with the relative error ϵ one has to use the local-oscillator amplitude

$$r = \frac{1}{T|T_2|\sqrt{\eta_3}\epsilon}\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \epsilon^2\bar{N}_3}}. \tag{34}$$

Hence a problem of monitoring appears for tiny T values, for which the value of the local-oscillator amplitude should be really large. However, for this domain the signal is of poor quality anyway. We can choose a minimal value $T_{\rm min}$ of the transmission coefficient T and postselect the values $T \geqslant T_{\rm min}$. Based on this assumption, Eq. (30) with the relative error ϵ for T^2 reduces to

$$\mathcal{P}(T_{\rm meas}|T) = \delta(T - T_{\rm meas}), \tag{35}$$

in which connection the local-oscillator amplitude should be chosen according to Eq. (34) with $T = T_{\rm min}$. Under these conditions, the shot noise error of detector D_3 becomes small.

Utilizing the δ-function form (35) for the conditional PDTC in Eq. (27), one gets

$$P_{\rm noisy}(\alpha) = \int_0^1 dT\mathcal{P}(T)\exp\left[\frac{\bar{N}_{\rm nc}}{4|R_2|^2r^2T^2\eta}\Delta_\alpha\right]\frac{1}{T^2\eta}P\left(\frac{\alpha}{T\sqrt{\eta}}\right). \tag{36}$$

The effect of noise counts can be omitted if the local-oscillator amplitude obeys the condition

$$r^2 \gg \frac{\bar{N}_{\rm nc}}{|R_2|^2T^2_{\rm min}\eta}. \tag{37}$$

In this case, the input-output relation (36) reduces to

$$P_{\rm noisy}(\alpha) = \int_0^1 dT\mathcal{P}(T)\frac{1}{T^2\eta}P\left(\frac{\alpha}{T\sqrt{\eta}}\right), \tag{38}$$

which appears to be similar to the case of an independently controlled local oscillator (cf. Ref. [11]). However, an important difference is that this relation does not contain phase noise of the signal after passing through the turbulent channel. It is also worth noting that the δ-function form of the conditional PDTC (35) excludes the postprocessing noise of the measured data. For this reason, no fake quantum effects appear in Eqs. (36) and (38).

As in the previous section, we illustrate the method with the single-photon-added thermal states [cf. Eq. (25)]. Besides, we suppose that this state is displaced with the coherent amplitude γ (cf. Ref. [11]) such that its P function is

$$P(\alpha) = \frac{1}{\pi\bar{n}_{\rm th}^3}[(1+\bar{n}_{\rm th})|\alpha-\gamma|^2 - \bar{n}_{\rm th}]e^{-\frac{|\alpha-\gamma|^2}{\bar{n}_{\rm th}}}. \tag{39}$$

As has been shown in Ref. [11], increasing the displacement amplitude γ results in diminishing the nonclassicality in the scenario when the signal and local oscillator are radiated from different sources. It turns out that this rule does not apply in the considered case. The absence of phase fluctuations in input-output relation (38) lifts strong restrictions for the coherent amplitude of nonclassical states considered in Ref. [11]. The quantum state in this case may preserve its nonclassical properties. In Fig. 4, we show the P function for the scenario with the monitored transmission coefficient. It is clearly seen that the state is still nonclassical even for large values of the coherent displacement amplitude γ.

V. QUADRATURE SQUEEZING

Quadrature squeezing is a remarkable property of quantum light, which can be observed by homodyne detection. In this

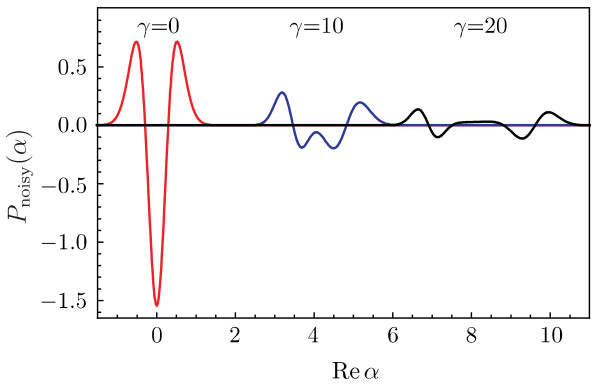

FIG. 4. (Color online) P function of the displaced SPATS for $\mathrm{Im}\alpha = 0$, with $n_{\rm th} = 1.11$ (such as realized in Ref. [21]) and different values of γ, disturbed by beam wandering and detected with monitored transmission coefficient. In the corresponding turbulence model (cf. the appendix), the beam-spot radius, W, on the receiver aperture of radius a is $W = 0.9a$, and the standard deviation of beam-deflection is $\sigma = 10a$. The detection efficiency is $\eta = 0.5$.

section, we consider how the disturbance of the signal and the local oscillator by the atmosphere affects the detection of quadrature squeezing. We consider two orthogonal quadratures [cf. Eq. (13)] for a certain value of the local-oscillator phase φ,

$$\hat{x}_1 = \hat{x}(\varphi), \tag{40}$$

$$\hat{x}_2 = \hat{x}\left(\varphi + \frac{\pi}{2}\right). \tag{41}$$

A well-known relation,

$$\langle \Delta\hat{x}_i \Delta\hat{x}_j \rangle = \tfrac{1}{2}\delta_{i,j} + \langle : \Delta\hat{x}_i \Delta\hat{x}_j : \rangle, \tag{42}$$

$i, j = 1, 2$, connects the covariance matrix with its normally ordered form. If a diagonal element (variance) of the latter becomes negative for properly chosen φ, the state is quadrature squeezed.

Let us first consider the case with monitored transmission coefficient; see Sec. IV. Based on Eq. (36), we can write the input-output relation for the covariance matrix,

$$\langle : \Delta\hat{x}_i \Delta\hat{x}_j : \rangle_{\rm noisy} = \eta\langle T^2\rangle\langle : \Delta\hat{x}_i \Delta\hat{x}_j : \rangle + \eta\langle\hat{x}_i\rangle\langle\hat{x}_j\rangle\langle\Delta T^2\rangle + \frac{\bar{N}_{\rm nc}\langle T^{-2}\rangle}{\eta r^2 |R_2|^2}\delta_{i,j}. \tag{43}$$

The first term of this relation resembles the standard attenuation. The second term is caused by the atmospheric turbulence. The third term of the equation describes the disturbance effect of noise counts on the quadrature squeezing.

The contribution from noise counts disappears for a sufficiently strong local oscillator when $r^2 \gg \frac{\bar{N}_{\rm nc}\langle T^{-2}\rangle}{\eta |R_2|^2}$. However, due to large contributions of events with small T, the value of $\langle T^{-2}\rangle$ can be really large. For example, if fluctuating losses are caused by beam wandering (cf. the appendix and Ref. [16]), this term is infinite. In practice, however, the measured transmission coefficient is bounded by its minimal value $T_{\rm min}$. This implies that the third term of Eq. (43) becomes negligible for the local-oscillator amplitude satisfying condition (37).

It is readily seen from the second term of Eq. (43) that the disturbance effect of the turbulence on the quadrature squeezing increases with increasing mean value of the quadrature $\langle\hat{x}_i\rangle$. This means that states with a small coherent amplitude have better chances of preserving this nonclassical property. For the states with $\langle\hat{x}_i\rangle = 0$, the squeezing of the ith quadrature is disturbed in the same way as for standard attenuation.

Next we consider the case of unmonitored transmission coefficient; see Sec. III. The corresponding input-output relation for the covariance matrix is obtained from Eq. (21) and reads as

$$\langle : \Delta\hat{x}_i \Delta\hat{x}_j : \rangle_{\rm noisy} = \eta\langle T_{\rm eff}^2\rangle\langle : \Delta\hat{x}_i \Delta\hat{x}_j : \rangle + \eta\langle\hat{x}_i\rangle\langle\hat{x}_j\rangle\langle\Delta T_{\rm eff}^2\rangle + \left(\frac{\langle T_{\rm eff}\rangle - 1}{2} + \frac{\bar{N}_{\rm nc}}{\eta r^2 T_{\rm ref}^2}\right)\delta_{i,j}, \tag{44}$$

where $T_{\rm eff}$ is given by Eq. (23). In contrast to the case of monitored transmission coefficient, the third term in this equation does not disappear even in the case of weak noise counts.

The postprocessing noise may result in such an effective covariance matrix that the corresponding density operator is not positive semidefinite anymore. This means that the Heisenberg uncertainty relation,

$$\langle\Delta\hat{x}_1^2\rangle_{\rm noisy}\langle\Delta\hat{x}_2^2\rangle_{\rm noisy} \geqslant \tfrac{1}{4}, \tag{45}$$

is not satisfied; see Fig. 5. When the real transmission coefficient appears to be much less compared with the reference transmission coefficient $T_{\rm ref}$, the negative effective noise [cf. Eq. (24)] leads to such a nonphysical squeezing.

The covariance matrix completely characterizes quantum states in the Gaussian approximation. They can be used, for

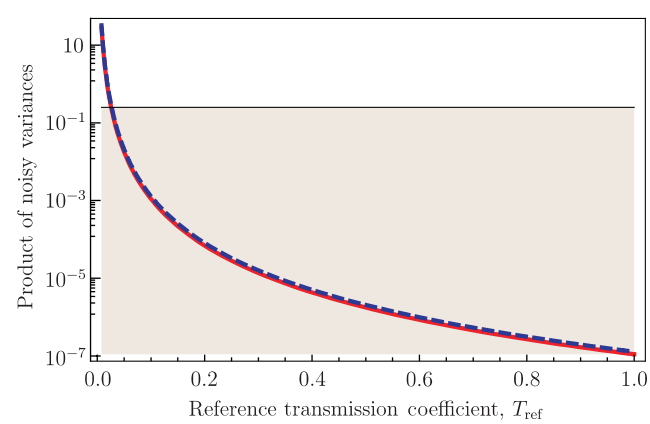

FIG. 5. (Color online) Product of noisy variances, $\langle\Delta\hat{x}_1^2\rangle_{\rm noisy}\langle\Delta\hat{x}_2^2\rangle_{\rm noisy}$, vs the reference transmission coefficient, $T_{\rm ref}$, for the scheme with unmonitored turbulence. The solid (dashed) line corresponds to the vacuum state (8-dB squeezed vacuum state) at the transmitter. We suppose that fluctuating losses are caused by beam wandering with the standard deviation of beam deflection $\sigma = 40a$ and with the beam-spot radius $W = 0.95a$, which leads to 35-dB mean losses. The detection efficiency is $\eta = 0.5$. The shaded area corresponds to the violation of the Heisenberg uncertainty relation.

example, for continuous-variable protocols of quantum-key distribution with coherent and squeezed states [23]. Evidently, in order to get a consistent effective density operator in this approximation, one should use an appropriate reference transmission coefficient $T_{\rm ref}$. This means that the effect of the postprocessing noise may preserve consistency of the noisy density operator even in the case of using the scheme with unmonitored transmission coefficient.

VI. SUMMARY AND CONCLUSIONS

Homodyne detection of quantum light passing through the turbulent atmosphere with the local oscillator sent in the orthogonally polarized mode is a promising technique for long-distance quantum communication based on continuous variables. In this connection, a problem appears of how to connect a measured photocount difference with the field quadrature. Indeed, when the receiving local-oscillator amplitude is a fluctuating variable, this question is not trivial anymore. We consider two possible solutions for this problem. One possibility could be based on some reference value for the local-oscillator amplitude transmission coefficient (e.g., its mean value). Alternatively, here we proposed a method based on the monitoring of the transmitted local oscillator with the aim of having control over the fluctuating transmission coefficient of the turbulent atmosphere. In both cases, the quantum state of the received light can be characterized by a noisy density operator, which includes also information about shortcomings of the measurement and postprocessing procedures of the used methods.

When the local-oscillator amplitude (and thus the transmission coefficient) is monitored, the main limitations are caused by stray-light and dark-count noise. These effects can be, in principle, eliminated by a sufficiently strong local oscillator and the postselection of events with an appropriately chosen threshold value of the transmission coefficient. In the simpler procedure, using a fixed reference value of the local-oscillator amplitude, the shortcomings caused by the resulting postprocessing noise are much more dramatic. The resulting disadvantages cannot be eliminated anymore. In such a scenario, the noisy density operator may even violate the fundamental requirement of positivite semidefiniteness. Thus fake quantum effects may occur due to the used postprocessing procedure. However, even based on this technique one may obtain a consistent noisy density operator in the Gaussian approximation, provided that the reference transmission coefficient is properly chosen. We believe that these methods may be of some interest in the context of continuous-variable quantum key distribution.

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APPENDIX: PDTC FOR BEAM WANDERING

Here we remind readers of some results of Ref. [16], in particular the explicit form of the PDTC when the fluctuating losses are caused by beam wandering. If the beam is randomly deflected around the aperture center according to a two-dimensional Gaussian distribution with the variance σ^2, the PDTC is given by the log-negative Weibull distribution,

$$\mathcal{P}(T)=\frac{2R^2}{\sigma^2\lambda T}\left(2\ln\frac{T_0}{T}\right)^{\frac{2}{\lambda}-1}\exp\left[-\frac{1}{2\sigma^2}R^2\left(2\ln\frac{T_0}{T}\right)^{\frac{2}{\lambda}}\right] \tag{A1}$$

for $T\in[0,T_0]$ and 0 else. Here the parameters T_0, λ, and R are expressed in terms of the beam-spot radius at the aperture W and the aperture radius a,

$$T_0=\sqrt{1-\exp\left[-2\frac{a^2}{W^2}\right]}, \tag{A2}$$

$$\lambda=8\frac{a^2}{W^2}\frac{\exp\left[-4\frac{a^2}{W^2}\right]\mathrm{I}_1\left(4\frac{a^2}{W^2}\right)}{1-\exp\left[-4\frac{a^2}{W^2}\right]\mathrm{I}_0\left(4\frac{a^2}{W^2}\right)}\times\left[\ln\left(\frac{2T_0^2}{1-\exp\left[-4\frac{a^2}{W^2}\right]\mathrm{I}_0\left(4\frac{a^2}{W^2}\right)}\right)\right]^{-1} \tag{A3}$$

$$R=a\left[\ln\left(\frac{2T_0^2}{1-\exp\left[-4\frac{a^2}{W^2}\right]\mathrm{I}_0\left(4\frac{a^2}{W^2}\right)}\right)\right]^{-\frac{1}{\lambda}}. \tag{A4}$$

In the case when the turbulence is weak and the beam is focused on the aperture, the beam-deflection variance can be approximately evaluated as

$$\sigma^2\approx 1.919\,C_n^2z^3(2W_0)^{-1/3}, \tag{A5}$$

where C_n^2 is the index-of-refraction structure constant, W_0 is the beam-spot radius at the radiation source, and z is the distance between source and receiver aperture [5,24]. Integration with the PDTC $\mathcal{P}(T)$ must be performed in the Lebesgue sense with respect to the measure $d[R(2\ln\frac{T_0}{T})^{\frac{1}{\lambda}}]$.

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