PHYSICAL REVIEW A **85**, 013813 (2012)

Optimal control of wave-packet localization in driven two-level systems and curved photonic lattices: A unified view

Ricardo Chacón
Departamento de Física Aplicada, Escuela de Ingenierías Industriales, Universidad de Extremadura, Apartado Postal 382, E-06006 Badajoz, Spain, EU

(Received 28 October 2011; published 11 January 2012)

It is shown that wave-packet localization in space-periodic systems subjected to generic ac forces having equidistant zeros is a highly nonlinear phenomenon that is fully controlled by the integral of the ac force over a half period (i.e., the force's impulse), and that suitably varying this impulse allows one to achieve wave-packet localization for almost any value of the force's amplitude, in contrast to the standard case of a harmonic force. In particular, the result is demonstrated in two simple universal models—driven two-level systems in the limit of small periods, and periodically curved waveguide arrays in the nearest-neighbor tight-binding approximation. Remarkably, the problem motivated the introduction of a nonlinear generalization of the zeroth-order Bessel function $J_0(z)$.

DOI: 10.1103/PhysRevA.85.013813 PACS number(s): 42.82.Et, 02.30.Gp, 63.20.Pw, 73.23.−b

I. INTRODUCTION

Over the past few decades, the control of wave-packet spreading by periodic driving fields has attracted a great deal of attention in many branches of physics, including spin systems, cold atoms in optical traps, quantum superconductor systems, and quantum computing [1–3]. In this context, dynamic localization (DL) [1,4–9] is the continual localization of an initially localized wave packet by means of the application of an ac field. DL has been demonstrated experimentally with Bose-Einstein condensates in optical lattices [6,9], and is closely related to the phenomenon known as coherent destruction of tunneling (CDT), in which the tunneling of a particle is suppressed when the amplitude and frequency of a sinusoidal field present certain values which are associated with zeros of the zeroth-order Bessel function $J_0(z)$ [10]. Remarkably, DL also occurs in arrays of harmonically curved optical waveguides with alternating curvature [11], in which the periodic bending profile of the waveguide plays the role of the ac field, and leads to the cancellation of diffraction. Once again, DL occurs when the optical parameters involved are tuned to various particular values associated with the zeros of $J_0(z)$ [11]. While it is natural to think that, with the period fixed, wave-packet localization must depend on the driving wave form, the main target of study up to now has only been their dependence on the driving amplitude because of the traditional use of sinusoidal functions. This has led to it being implicitly assumed that there is an *effective* limited discretization of the suitable values (of the driving amplitude) associated with CDT and DL owing to the aforementioned Bessel-function dependence. Recent work provides strong evidence for a different dependence of DL on sinusoidal and square-wave forces [8,9]. Since there are infinitely many different wave forms, the question naturally arises: How can one characterize quantitatively the effect of the driving wave form on DL?

In this work, I show that a relevant quantity properly characterizing the effect of the driving on the wave-packet localization in space-periodic systems subjected to generic ac forces $f(z)$ having equidistant zeros is the impulse transmitted by the force over a half period (hereafter referred to simply as the force's impulse [12] $I_f \equiv \int_0^{T/2} f(z)dz$, T being the period)—*a quantity integrating the conjoint effects of the force's amplitude, period, and wave form*, and that the aforementioned discretization is inessential in the sense that suitably varying the force's impulse allows one to achieve wave-packet localization for almost any value of the force's amplitude irrespective of the remaining parameters involved. The relevance of the force's impulse has recently been shown in the context of ratchet transport [13] (i.e., directed transport by symmetry breaking of zero-mean forces). Indeed, it has been shown that optimal enhancement of ratchet transport is achieved when maximal effective (i.e., critical) symmetry breaking occurs, which is in turn a consequence of two reshaping-induced competing effects: the increase in the degree of symmetry breaking and the decrease in the (normalized) maximal force's impulse, thus implying the existence of a universal force wave form that optimally enhances ratchet transport [13]. It has also been demonstrated in the context of adiabatically ac-driven periodic systems that the width of the separatrix chaotic layer and the adiabatic condition depend on the maximal impulse transmitted by the force over a period between two of its consecutive zeros, irrespective of its wave form [14].

For the sake of clarity, the effect of the force's impulse on the wave-packet localization will be discussed here in the absence of any ratchet effect. Remarkably, the consideration of the force's impulse provides an additional degree of freedom to maximally optimize the control of DL and CDT. In particular, the result is demonstrated in two simple universal models—driven two-level systems [2] in which CDT is related to the presence of crossings in the spectrum of Floquet quasienergies, and periodically curved waveguide arrays in the nearest-neighbor tight-binding (NNTB) approximation [11] in which DL corresponds to periodic diffraction cancellation. The relevance of these two models is now clear. On the one hand, periodically driven two-level systems appear in many physical contexts including superconductivity, structural glasses, magnetism, and quantum information theory [2,15–18]. For example, a qubit itself is a two-level system, and the problem of its control under an external time-dependent excitation is essential in the field of quantum computing. On the other hand, DL of light in periodically curved waveguide arrays [11,19] provides the optical analog of DL for electrons in periodic potentials subjected to ac electric fields [1], thereby giving the attractive possibility of coherent manipulation of light in photonic lattices.

II. CDT IN A DRIVEN TWO-LEVEL SYSTEM

Consider first the case of a charged particle confined to a double-quantum-dot system [20], whose Hamiltonian may be transformed to the standard two-level form

$$H = \frac{\Delta\sigma_z + EF(t)\sigma_x}{2}, \tag{1}$$

where E is the amplitude of the driving field, σ_i are the standard Pauli matrices, Δ represents the splitting between the two eigenstates existing in the absence of a driving field ($E = 0$), and

$$F(t) = F(t;m,T) \equiv N(m)\,\mathrm{sn}\left[\frac{4Kt}{T}\right]\mathrm{dn}\left[\frac{4Kt}{T}\right], \tag{2}$$

in which $\mathrm{sn}(\cdot) \equiv \mathrm{sn}(\cdot;m)$ and $\mathrm{dn}(\cdot) \equiv \mathrm{dn}(\cdot;m)$ are Jacobian elliptic functions of parameter m [$K \equiv K(m)$ is the complete elliptic integral of the first kind] [21] and $N(m)$ is a normalization function (see Fig. 1, top) which is introduced for the force to have the same amplitude 1 and period T, for any wave form (i.e., $\forall\, m \in [0,1]$). When $m = 0$, then $F(t;m=0,T) = \sin(2\pi t/T)$, i.e., one recovers the previously studied case of a harmonic excitation [1,6], while for the limiting value $m = 1$ the force vanishes. Note that, as a function of m, the force's impulse

$$I_F = I_F(m,T) \equiv \frac{ETN(m)}{2K} \tag{3}$$

presents a single maximum at $m = m_{\max}^{\text{impulse}} \simeq 0.717$ (see Fig. 1, bottom). Since the driving field is periodic in time, Floquet theory allows one to write solutions of the time-dependent Schrödinger equation as

$$\psi(t) = \exp[-i\epsilon_j t]\phi_j, \tag{4}$$

where $j = L$ (R) denotes the left (right) quantum dot, while ϕ_j and ϵ_j are the Floquet state and its corresponding quasienergy, respectively. Noting that the driving field $F(t)$ exhibits the shift symmetry $F(t) = -F(t + T/2)$, the von Neumann–Wigner theorem [22] allows the two quasienergies to cross as an external parameter, such as the field strength E *or the shape parameter* m, is varied. Also, the limit of small periods is assumed to obtain accurate theoretical predictions of the quasienergies. Now, following standard perturbation schemes [23], one may consider approximate solutions of the eigenvalue equation

$$\left[H(t) - i\frac{\partial}{\partial t}\right]\phi_j(t) = \epsilon_j\phi_j(t) \tag{5}$$

to obtain (see Ref. [20] for further details) the quasienergies in the simple form

$$\epsilon_\pm = \pm\Delta|\langle\phi_+^2\rangle_T|,$$
$$\langle\phi_+^2\rangle_T \equiv T^{-1}\int_0^T \exp\left[iE\int_{T/2}^t F(\tau)\,d\tau\right]dt. \tag{6}$$

Finally, after some simple algebraic manipulation (see the Appendix), one obtains

$$\epsilon_\pm = \pm\frac{\Delta}{2}J^*(\alpha_{m=0},m), \tag{7}$$

$$J^*(\alpha_{m=0},m) \equiv J_{\mathrm{cn}}(\alpha;m), \tag{8}$$

where

$$\alpha = \frac{\alpha_{m=0}\pi N(m)}{2K(m)},$$
$$\alpha_{m=0} = \frac{E}{2\pi/T}, \tag{9}$$

while $J_{\mathrm{cn}}(\alpha;m)$ is the generalized Bessel function (A1)–(A3). Note that $J^*(\alpha_{m=0},m=0) = J_0(\alpha_{m=0})$, i.e., one recovers the standard case of a sinusoidal driving field [20], as expected. Figure 2 shows a contour plot of the numerically calculated Floquet quasienergy $|2\epsilon_+/\Delta|$ together with the zeros of the function $J^*(\alpha_{m=0},m)$ in the plane $\alpha_{m=0}$-m. One finds exact agreement between the theoretical prediction (8) and the numerical results obtained using the method described in [23]. Such zeros correspond to crossings in the spectrum of Floquet quasienergies (7), which in turn have been shown [10] to be closely related to CDT. Thus, the inhibition of wave-packet spreading, and hence the occurrence of CDT, occurs on one-dimensional manifolds of zeros of the function $J^*(\alpha_{m=0},m)$ in the plane $\alpha_{m=0}$-m, which begin at the zeros of

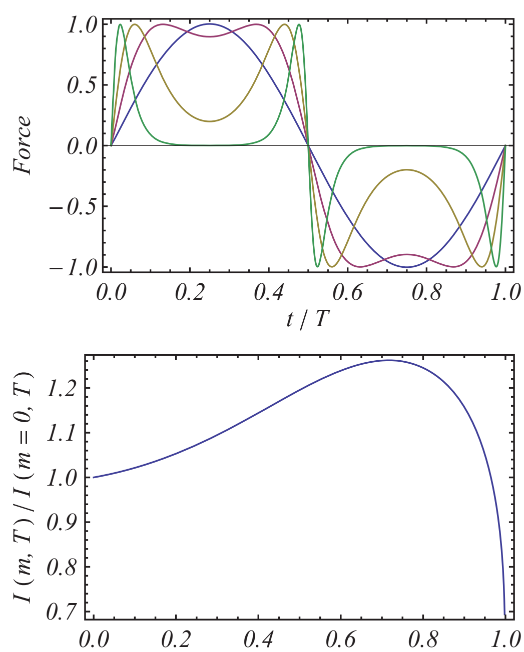

FIG. 1. (Color online) Top: Force $F(t)$ [cf. Eq. (2)] versus t/T, where T is the temporal period and $N(m) \equiv 1/\{a + b/[1 + \exp(\{m - c\}/d)]\}$, with $a \equiv 0.439\,32, b \equiv 0.697\,96, c \equiv 0.3727, d \equiv 0.268\,83$, for four values of the shape parameter: $m = 0$ (sinusoidal pulse), $m = 0.72$ (nearly square-wave pulse), $m = 0.99$ (double-humped pulse), and $m = 0.999\,999$ (sharp double-humped pulse). Bottom: The normalized force's impulse $I_F(m,T)/I_F(0,T) \equiv N(m)K(0)/[N(0)K(m)]$ versus m. The quantities plotted are dimensionless.

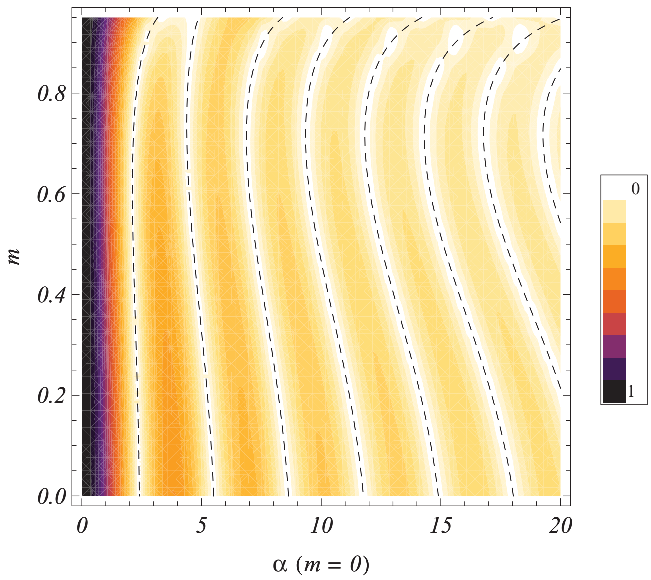

FIG. 2. (Color online) Contour plot of the numerically calculated quasienergy $|2\epsilon_+/\Delta|$ vs $\alpha\ (m=0) \equiv E/(2\pi/T)$ and m together with the one-dimensional manifolds associated with zeros of the function $J^*(\alpha_{m=0},m)$ [cf. Eq. (12), dashed curves] in the plane $\alpha(m=0)$-m. The quantities plotted are dimensionless.

$J_0(\alpha_{m=0})$ for $m=0$. This inhibition occurs on each of such manifolds at a *minimal* value of the ratio $E/(2\pi/T)$ when the shape parameter is very near $m_{\max}^{\text{impulse}} \simeq 0.717$, meaning that the force's impulse is the quantity properly controlling CDT in two-level quantum systems, at least in the limit of small periods. Remarkably, the larger the ratio $E/(2\pi/T)$, the more sensitive is the localization scenario to the force's impulse (i.e., the greater is the curvature of the one-dimensional manifolds; cf. Fig. 2). This means that the localization scenario is highly *nonlinear* in the sense that it is different for small and large amplitudes, and that the standard Bessel-function-induced discretization of the suitable values of $E/(2\pi/T)$ for CDT is only a particular case associated with the sinusoidal wave form.

III. OPTICAL DL

Consider now light propagation in a simple photonic lattice: a one-dimensional periodically curved array of coupled waveguides [24]. For single-mode waveguides, in the NNTB approximation and assuming that the lowest Bloch band of the array is excited, the following coupled-mode equations can be derived [11,24]:

$$i\frac{d\psi_n}{dz} + C(\lambda)[\psi_{n+1}+\psi_{n-1}] = \frac{n_0 d\ddot{x}_0(z)\, n\psi_n}{\lambda} \qquad (10)$$

for the amplitudes ψ_n of the field in the individual waveguides, where the overdot indicates the derivative with respect to z (the propagation distance along the waveguides), n is the waveguide number, n_0 is the refractive index of the medium, the coefficient $C(\lambda)$ is the coupling strength between

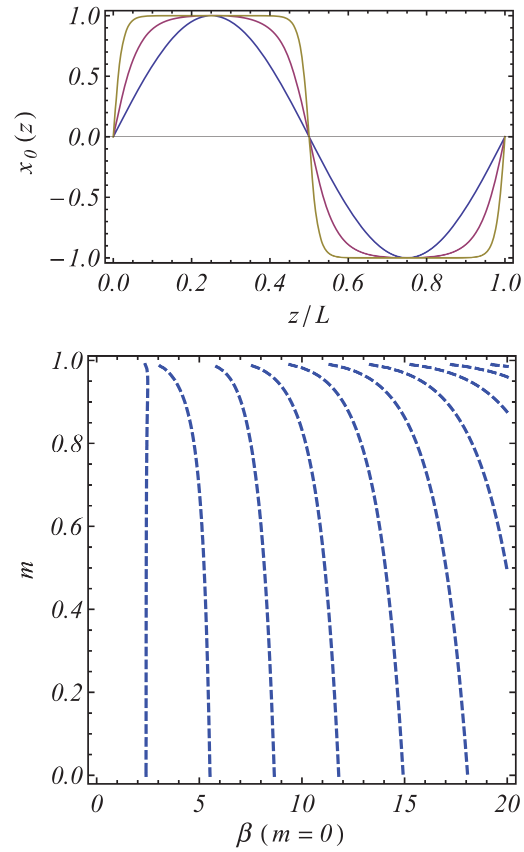

FIG. 3. (Color online) Top: Bending profile $x_0(z)$ [cf. Eq. (11)] versus z/L, with L being the spatial period, for $A=1$ and three values of the shape parameter: $m=0$ (sinusoidal bending), $m=0.9999$ (intermediate bending), and $m=1-10^{-14}$ (nearly square-wave bending). Bottom: One-dimensional manifolds associated with zeros of the function $J^*(\beta_{m=0},m)$ [cf. Eq. (14)] in the plane $\beta(m=0)$-m, with $m\in[0,0.99]$. The quantities plotted are dimensionless.

the neighboring waveguides, $\lambda \equiv \lambda/(2\pi)$ is the normalized wavelength, d is the waveguide spacing, and

$$x_0(z) = x_0(z;m,L) \equiv \frac{A\arcsin\{\sqrt{m}\,\mathrm{sn}[4Kz/L;m]\}}{\arcsin(\sqrt{m})} \qquad (11)$$

is a function describing periodic waveguide bending in our curved array, in which A is the bending amplitude, L is the bending period, and the shape parameter m controls the bending wave form (see Fig. 3, top). When $m=0$, then $x_0(z;m=0,L) = A\sin(2\pi z/L)$, i.e., one recovers the previously studied case of a sinusoidally bent array [11,24], while for the limiting value $m=1$ one has a square-wave bending profile. Note that the area enclosed by the periodic bending profile and the z axis plays the role of the force's impulse, $I_{x_0} = I_{x_0}(m,L)$, which is a monotonically increasing function of m reaching its supremum (least upper bound) at $m = m_{\max}^{\text{area}} = 1$. Optical DL corresponds to periodic self-imaging in the planes $z = 0, L, 2L, 3L, \ldots$. In the NNTB approximation, the condition for DL is

$$\int_0^L \cos\left[\frac{n_0 d\dot{x}_0(\zeta)}{\lambda}\right] d\zeta = 0. \qquad (12)$$

By substituting

$$\dot{x}_0(\zeta) = \frac{4A\sqrt{m}K}{L\,\arcsin(\sqrt{m})}\mathrm{cn}[4K\zeta/L;m] \tag{13}$$

into Eq. (12), and after some simple algebraic manipulation (see the Appendix), one obtains the final condition for DL:

$$J^*(\beta_{m=0},m) \equiv J_{\mathrm{cn}}(\beta;m) = 0, \tag{14}$$

where

$$\beta = \frac{2\beta_{m=0}\sqrt{m}K}{\pi\,\arcsin(\sqrt{m})}, \qquad \beta_{m=0} = \frac{2\pi n_0 d A}{\lambda L}, \tag{15}$$

while $J_{\mathrm{cn}}(\beta;m)$ is the generalized Bessel function (A1)–(A3). Note that $J^*(\beta_{m=0},m=0) = J_0(\beta_{m=0})$, i.e., one recovers the standard case of a sinusoidally bent array [11,24], as expected. Figure 3 (bottom) shows a contour plot of the zeros of the function $J^*(\beta_{m=0},m)$ in the plane $\beta_{m=0}$-m. One sees that optical DL occurs on one-dimensional manifolds of zeros of the function $J^*(\beta_{m=0},m)$ in the plane $\beta_{m=0}$-m, which begin at the zeros of $J_0(\beta_{m=0})$ for $m=0$. This cancellation of diffraction occurs on each of such manifolds at *ever smaller* values of the ratio $2\pi n_0 dA/(\lambda L)$ as the shape parameter approaches $m_{\max}^{\mathrm{area}} = 1$, meaning that the aforementioned enclosed area or "the force's impulse" is the quantity properly controlling DL in periodically curved waveguide arrays, at least in the NNTB approximation. In the limiting case $m=1$ associated with an *exact* square-wave waveguide, DL is impossible since for pieces of straight waveguides the curvature $\ddot{x}_0$ is zero. Indeed, one has $\lim_{m\to 1}\beta = \infty$ [cf. Eq. (15)], and hence condition (14) is improperly satisfied according to properties (A5) and (A6) (see Fig. 4). Remarkably, the larger the ratio $2\pi n_0 dA/(\lambda L)$, the more sensitive is the localization scenario to the force's impulse (i.e., the greater is the curvature of the one-dimensional manifolds; cf. Fig. 3, bottom)—again showing its highly nonlinear character in the sense that it is different for small and large values of the ratio $2\pi n_0 dA/(\lambda L)$.

IV. CONCLUSIONS

In summary, for ac-driven space-periodic systems, it has been shown that the impulse transmitted by the (effective) force is an essential quantity for the optimal control of the phenomena of coherent destruction of tunneling and dynamic localization. The result, demonstrated here for driven two-level systems in the limit of small periods and periodically curved waveguide arrays in the nearest-neighbor tight-binding approximation, reveals a common nonlinear scenario of wave-packet localization—in contrast to the standard Bessel-function-based scenario—and provides a powerful principle to systematically develop the notion of quasienergy band engineering [9], while it can be easily implemented experimentally, for example in photonic lattices [24]. This principle, which can be straightforwardly applied to other phenomena, such as field-induced barrier transparency [25], inhibition of light tunneling in waveguide arrays [26], and dynamics of ultracold atoms held in optical-lattice potentials [27], opens up new avenues for applications of the wave-packet localization effect in diverse physical contexts including optical lattices, semiconductor superlattices, waveguide arrays, and quantum dots.

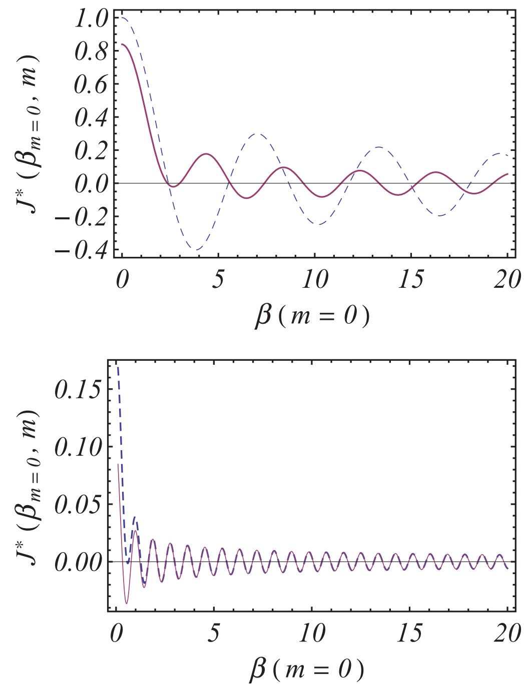

FIG. 4. (Color online) Top: Function $J^*(\beta_{m=0},m)$ [cf. Eq. (14)] versus $\beta(m=0)$ for two values of the shape parameter: $m=0$ [$J_0(z)$, dashed line] and $m=0.99$. Bottom: Function $J^*(\beta_{m=0},m)$ (dashed line) and asymptotic behavior [cf. Eq. (A6)] versus $\beta(m=0)$ for $m = 1-10^{-14}$. The quantities plotted are dimensionless.

ACKNOWLEDGMENTS

The author is grateful to Charles Creffield for useful discussions and for developing the numerical code used to calculate the Floquet quasienergies, and to Yuri Kivshar for useful discussions and kind hospitality at the Canberra Nonlinear Physics Centre where part of this work was carried out.

APPENDIX : ELLIPTIC GENERALIZATION OF THE BESSEL FUNCTION $J_0(Z)$

The cn-based elliptic generalization of the Bessel function $J_0(z)$ is defined by the integral

$$J_{\mathrm{cn}}(z;m) \equiv \frac{1}{2\pi}\int_0^{2\pi} \exp\{-iz\,\mathrm{cn}[2K\theta/\pi;m]\}d\theta, \tag{A1}$$

which, after changing to variables $\theta = \pi p/(2K)$ and $\zeta = \mathrm{am}(p;m)$ (elliptic amplitude [21]), and using the relationship $1 = m\,\mathrm{sn}^2(p;m) + \mathrm{dn}^2(p;m)$, can be recast into the form

$$J_{\mathrm{cn}}(z;m) = \frac{1}{4K}\int_0^{2\pi} \frac{\exp(-iz\sin\zeta)}{\sqrt{1-m\sin^2\zeta}}d\zeta. \tag{A2}$$

By expanding the square root in a power series and using standard integral tables, one finally obtains

$$J_{\rm cn}(z;m) = \frac{\pi}{2K}\sum_{n=0}^{\infty} c_n \left(\frac{m}{z}\right)^n J_n(z), \tag{A3}$$

where $c_n = \binom{n-1/2}{n}(2n-1)!!$ and $J_n(z)$ is the nth-order Bessel function. By using standard properties of the Bessel functions [28], one straightforwardly obtains

$$J_{\rm cn}(z;m=0) = J_0(z), \tag{A4}$$

$$\lim_{m\to 1} J_{\rm cn}(z;m) = 0, \tag{A5}$$

$$J_{\rm cn}(z\to\infty;m) \sim \sqrt{\frac{\pi}{2zK^2}}\cos\left(z-\frac{\pi}{4}\right), \tag{A6}$$

$$J_{\rm cn}(z\to 0;m) \sim \frac{\pi}{2K}\sum_{n=0}^{\infty}\frac{c_n}{n!}\left(\frac{m}{2}\right)^n. \tag{A7}$$

[1] D. H. Dunlap and V. M. Kenkre, Phys. Rev. B **34**, 3625 (1986).

[2] M. Grifoni and P. Hänggi, Phys. Rep. **304**, 229 (1998); S. Kohler *et al.*, *ibid.* **406**, 379 (2005).

[3] C. H. Bennett and D. P. DiVicenzo, Nature (London) **404**, 247 (2000).

[4] M. Holthaus, Phys. Rev. Lett. **69**, 351 (1992); X.-G. Zhao, J. Phys.: Condens. Matter **6**, 2751 (1994); K. Drese and M. Holthaus, Chem. Phys. **217**, 201 (1997).

[5] M. Holthaus and D. Hone, Phys. Rev. B **47**, 6499 (1993).

[6] K. W. Madison, M. C. Fischer, R. B. Diener, Q. Niu, and M. G. Raizen, Phys. Rev. Lett. **81**, 5093 (1998).

[7] M. M. Dignam and C.M. deSterke, Phys. Rev. Lett. **88**, 046806 (2002).

[8] C. E. Crefield, Europhys. Lett. **66**, 631 (2004).

[9] A. Eckardt, M. Holthaus, H. Lignier, A. Zenesini, D. Ciampini, O. Morsch, and E. Arimondo, Phys. Rev. A **79**, 013611 (2009).

[10] F. Grossmann, T. Dittrich, P. Jung, and P. Hänggi, Phys. Rev. Lett. **67**, 516 (1991); F. Grossmann and P. Hänggi, Europhys. Lett. **18**, 571 (1992); A. Eckardt, C. Weiss, and M. Holthaus, Phys. Rev. Lett. **95**, 260404 (2005); J. Gong, L. Morales-Molina, and P. Hänggi, *ibid.* **103**, 133002 (2009); S. Longhi, Phys. Rev. A **83**, 034102 (2011).

[11] S. Longhi, M. Marangoni, M. Lobino, R. Ramponi, P. Laporta, E. Cianci, and V. Foglietti, Phys. Rev. Lett. **96**, 243901 (2006); F. Dreisow *et al.*, Opt. Express **16**, 3474 (2008); A. Szameit *et al.*, Nat. Phys. **5**, 271 (2009).

[12] See, e.g., M. Alonso and E. J. Finn, *Physics* (Addison-Wesley, New York, 1992), p. 118.

[13] R. Chacón, J. Phys. A **40**, F413 (2007); **43**, 322001 (2010); P. J. Martínez and R. Chacón, Phys. Rev. Lett. **100**, 144101 (2008); M. Rietmann, R. Carretero-Gonzalez, and R. Chacon, Phys. Rev. A **83**, 053617 (2011).

[14] R. Chacón, M. Yu. Uleysky, and D. V. Makarov, Europhys. Lett. **90**, 40003 (2010).

[15] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).

[16] P. W. Anderson, B. Halperin, and C. Varma, Philos. Mag. **25**, 1 (1972).

[17] U. Weiss, *Quantum Dissipative Systems*, 3rd ed. (World Scientific, Singapore, 2008).

[18] M. A. Nielsen and I. L. Chuang, *Quantum Computations and Quantum Information* (Cambridge University Press, Cambridge, 2002).

[19] I. L. Garanovich, A. A. Sukhorukov, and Y. S. Kivshar, Phys. Rev. E **74**, 066609 (2006).

[20] C. E. Creffield, Phys. Rev. B **67**, 165301 (2003).

[21] See, e.g., P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer-Verlag, Berlin, 1971).

[22] J. von Neumann and E. Wigner, Phys. Z **30**, 467 (1929).

[23] C. E. Creffield and G. Platero, Phys. Rev. B **65**, 113304 (2002), and references therein.

[24] I. L. Garanovich, S. Longhi, A. A. Sukhorukov, and Y. S. Kivshar, Phys. Rep. (to be published).

[25] I. Vorobeichik, R. Lefebvre, and N. Moiseyev, Europhys. Lett. **41**, 111 (1998); S. Longhi, Phys. Rev. B **82**, 205123 (2010).

[26] G. DellaValle, M. Ornigotti, E. Cianci, V. Foglietti, P. Laporta, and S. Longhi, Phys. Rev. Lett. **98**, 263601 (2007); I. L. Garanovich *et al.*, Opt. Express **15**, 9737 (2007); A. Szameit, Y. V. Kartashov, F. Dreisow, M. Heinrich, T. Pertsch, S. Nolte, A. Tunnermann, V. A. Vysloukh, F. Lederer, and L. Torner, Phys. Rev. Lett. **102**, 153901 (2009).

[27] O. Morsch and M. Oberthaler, Rev. Mod. Phys. **78**, 179 (2006); H. Lignier, C. Sias, D. Ciampini, Y. Singh, A. Zenesini, O. Morsch, and E. Arimondo, Phys. Rev. Lett. **99**, 220403 (2007); C. Sias, H. Lignier, Y. P. Singh, A. Zenesini, D. Ciampini, O. Morsch, and E. Arimondo, Rev. Mod. Phys. **100**, 040404 (2008); C. E. Creffield and F. Sols, Phys. Rev. A **84**, 023630 (2011).

[28] See, e.g., *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972), Chap. 9.