# Continuous-variable nonlocality test performed over a multiphoton quantum state

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We propose to exploit a continuous-variable measurement, based on displacing the input field at different points of the phase space, over a multiphoton state produced by a high-gain optical parametric amplifier. We show that by correlating the different values of the displaced parity operators obtained from the two separated parties, it is possible to violate a Bell's inequality and thus demonstrate the nonlocality of the overall state. The robustness of the results against two independent sources of error, loss and dephasing, is also discussed.

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## I. INTRODUCTION

The discussion of nonlocality started by Einstein, Podolsky, and Rosen (EPR) in 1935 [1] has yielded a definition of entanglement as the most characteristic feature of quantum mechanics given by Erwin Schrödinger [2] up to the formulation of the Bell's inequality in order to test the nonlocal character of quantum states [3]. Following Bohm's scheme [4], the EPR correlations have been analyzed by addressing singlet pairs of two-level systems but not the two-particle continuous wave function introduced by EPR in their argument about the completeness of quantum mechanics. Theoretical and experimental studies of quantum nonlocality and entanglement have then been carried out on discrete systems [5–8], and the generalization of Bell's inequalities to quantum systems with continuous variables has represented a challenging issue for a long time.

Initially, it was believed that the possibility of observing the violation of Bell's inequality by addressing position and momentum over the EPR state was prevented by the non-negativity of its Wigner function. Indeed, according to Bell, the positivity of the Wigner function would have allowed the construction of a local-hidden-variable model simulating correlations for any observable defined as a function of phase-space points [9]. However, Banaszek and Wodkiewicz showed that in spite of the positivity of the Wigner function, the EPR state exhibits a high degree of nonlocality [10]. This study was later extended by Chen *et al.* [11], who showed that a maximal violation of Bell's inequality can be obtained by measuring pseudospin operators over the state produced by a nondegenerate optical parametric amplifier (NOPA) when the nonlinear gain of the amplifier grows and the NOPA state tends to the original EPR one. The relation between the positivity of the Wigner function and the possibility of observing a violation of Bell's inequality has then been clarified by Rezven *et al.* [12]; they focused their attention on the explicit assumptions that are made in a Bell's test and that involve the nature of the dynamical variables measured in order to violate a Bell's inequality. Reference [12] shows that only "nondispersive" dynamical variables, i.e., variables whose representatives as functions of hidden variables take as possible values the eigenvalues $a_n$ such that $|a_n| \leqslant 1$, can be considered good candidates for a local-hidden-variable theory. The violation of a Bell's inequality is then not only dependent on the system's Wigner function but also on the nature of the measured dynamical variables.

From an experimental point of view, the demonstration of Bell's inequality involving the measurement of discrete degrees of freedom requires the introduction of either the locality or the detection loophole [13]. The adoption of atomic systems allows one to close the detection loophole but not the locality one [14], and conversely, light can be sent at large distances but the inefficiency of detectors and the presence of losses along the communication channel prevent the possibility of closing the detection loophole. A path toward a Bell's test on bipartite multiphoton systems could involve the adoption of homodyne measurements, which can be performed with very high detection efficiency [15]. Recently, hybrid measurements involving both discrete and continuous-variable observables in order to demonstrate Bell's test violations have been addressed in Refs. [16] and [17]. The discussion of nonlocality in continuous-variable systems is then still an open problem in which the adoption of feasible measurements in reliable systems turns out to be the key requirement.

We propose a further step toward the understanding of the nonlocality problem in continuous-variable systems by addressing the possibility of performing continuous-variable measurements for a multiphoton system in order to observe a Bell's test violation. The exploited multiphoton-state source can be considered a paradigmatic system since it is based on an optical parametric amplifier, similar to the one analyzed by Banaszek and Wodkiewicz in Ref. [10] [in which the multiphoton state generated by a nondegenerate optical parametric amplifier was placed in relation with the continuous-variable EPR state], but with an additional degree of freedom: polarization. Recently, the quantum correlations present in the multiphoton state obtained by the high-gain, spontaneous parametric down-conversion process that cannot be read by a fuzzy measurement performed on it have been analyzed [18]. Even if in principle the nonlocal nature of the state could be observed for any value of the nonlinear gain of the amplifier,

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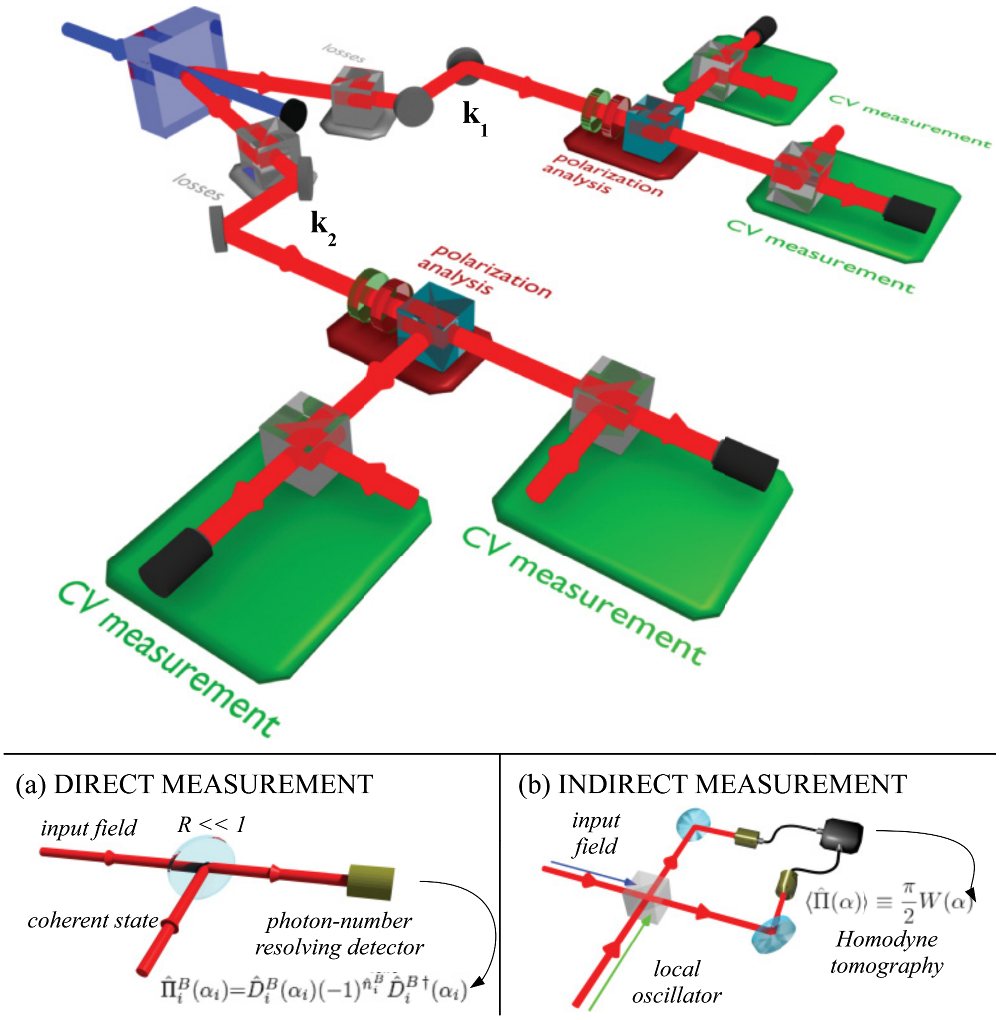


FIG. 1. (Color online) Scheme of the multiphoton source and of the detection apparatuses for the measurement of the continuous-variable Bell's test. The two spatial modes are split according to polarization, resulting in four modes which can be analyzed with independent detection systems. Losses are modeled by an additional beam-splitter. (a)–(b) Apparatuses for the direct (a) and indirect (b) measurement of the displaced parity operators.

i.e., for the increasing size of the system, the inability to determine the structure of the state with a proper measurement makes the observation of quantum correlations impossible. In that paper the inability to measure the multiphoton state in an efficient way was interpreted as one of the main causes for the quantum-to-classical-transition phenomenon [19,20].

Here, we address the problem of analyzing the multiphoton state through an efficient measurement method in order to observe the violation of a Bell's inequality. We study the violation of the Bell's test in the form proposed by Banaszek and Wodkiewicz in Ref. [10] based on the measurement of the displaced parity operators, which permit to analyze the correlations at different points of the phase space. A similar test, based on the measurement of the $Q$ function, has been analyzed in Ref. [21] in the state generated by a NOPA. By correlating the average values of the displaced parity operators at different points of the phase space, we study the possibility of violating the Bell's inequality either in the absence or in the presence of losses, and we relate the results with the value of the nonlinear gain of the amplifier, i.e., the size of the measured state. In Sec. II we introduce the physical system under investigation: the multiphoton state produced by spontaneous parametric down-conversion working as an EPR type-II source, and we address its Wigner function. In Sec. III we analyze the possibility of performing a nonlocality test upon the state generated by an optical parametric amplifier (OPA) by adopting continuous-variable measurements. We address both the lossless (in Sec. III A) and the lossy case (in Sec. III B) by studying the amount of violation as a function of the loss parameter for different values of the nonlinear gain of the amplifier. Finally, in Sec. III C we analyze the action of a dephasing channel.

## II. MULTIPHOTON STATE GENERATED THROUGH HIGH-GAIN PARAMETRIC DOWN-CONVERSION

The paradigmatic system over which we perform our analysis is the one obtained by an optical parametric amplifier, working through spontaneous parametric down-conversion (SPDC) as an EPR type-II source [22,23] (see Fig. 1). The low-gain regime of such a system has been experimentally realized and deeply studied in the past few years [23,24].

The polarization singlet state emitted in the single-pair regime was first exploited by Kwiat *et al.* [22] in order to obtain the violation of Bell's inequalities. More recent works have studied the increasing size-state properties; the presence of polarization entanglement in states up to 12 photons has been proved by studying the high-loss regime where at most one photon per branch was detected [23]. Subsequently, Caminati *et al.* [24] reconstructed theoretically and experimentally the density matrix of this two-photon state, demonstrating that it coincides with that of a Werner state (WS), i.e., a weighted mixture of a maximally entangled singlet state with a fully mixed state. However, no theoretical or experimental demonstration of entanglement and nonlocality has been given in the multiphoton regime while a study concerning the possibility of observing quantum correlations through a dichotomic measurement performed over the multiphoton state has been recently addressed in Ref. [18]. We are now interested in analyzing the possibility of violating a Bell's test with such a quantum system when its size is increased and the radiation is measured through a continuous-variable measurement.

Let us introduce the effective interaction Hamiltonian of the multiphoton system

$$\hat{\mathcal{H}}_{\text{int}} = \imath\hbar\chi(\hat{a}^{\dagger}_{\pi}\hat{b}^{\dagger}_{\pi_\perp} - \hat{a}^{\dagger}_{\pi_\perp}\hat{b}^{\dagger}_{\pi}) + \text{H.c.}, \tag{1}$$

where $\hat{a}^{\dagger}_{\pi}$ and $\hat{b}^{\dagger}_{\pi}$ are the creation operators corresponding to the generation of a $\pi$-polarized photon in spatial modes $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, respectively, and $\chi$ is the constant describing the strength of the interaction. The output state reads [23]

$$|\Psi^-\rangle = \frac{1}{C^2}\sum_{n=0}^{\infty}\Gamma^n\sqrt{n+1}|\psi_n^-\rangle, \tag{2}$$

with

$$|\psi_n^-\rangle = \sum_{m=0}^{n}\frac{(-1)^m}{\sqrt{n+1}}|(n-m)_{\pi},m_{\pi_\perp}\rangle_1|m_{\pi},(n-m)_{\pi_\perp}\rangle_2, \tag{3}$$

where $\Gamma = \tanh g$; $C = \cosh g$ (for future use, $S = \sinh g$); and $g = \chi t$ is the nonlinear gain of the process. Hence, the output state can be written as the weighted coherent superposition of singlet-spin $\frac{n}{2}$ states $|\psi_n^-\rangle$. The mean number of generated photons per polarization per mode is related to the nonlinear gain $g$ by the exponential relation $\overline{n} = S^2$, and the overall number of photons per pulse is then given by $\langle n\rangle = 4\overline{n}$. A maximum value of $g_{\text{expt}} = 3.5$, corresponding to $\langle n\rangle = 1080$ per pulse, was experimentally reached in Ref. [18].

### A. Wigner function of the multiphoton quantum state

The Wigner function of the multiphoton state can be obtained in the same way as the one addressed in Ref. [25]. We consider the presence of losses by introducing a lossy channel with transmittivity $T$, simulated by the presence of a beam splitter along the propagation of the radiation field. We assume that the channel efficiency $T$ is equal for the four modes of the source. The Wigner function in this lossy scenario of the state $\hat{\rho}^- = \mathcal{L}_T(|\Psi^-\rangle\langle\Psi^-|)$, where $\mathcal{L}_T$ is the map describing the action of a lossy channel with efficiency $T$, can then be written as [25]

$$W_T\{\alpha,\beta,g,T\} = \mathcal{N}\exp\left[-\overline{\varepsilon}\sum_{\pi=H,V}(|\alpha_\pi|^2+|\beta_\pi|^2)\right] \times \exp\{-\overline{\mu}[2\text{Re}(\alpha_V\beta_H) - 2\text{Re}(\alpha_H\beta_V)]\}. \tag{4}$$

Here, the $\{\alpha_\pi\}_{\pi=H,V}$ quadratures correspond to the spatial mode $\boldsymbol{k}_1$, the $\{\beta_\pi\}_{\pi=H,V}$ quadratures correspond to the spatial mode $\boldsymbol{k}_2$, and

$$\overline{\varepsilon} = \frac{\varepsilon(1+2S^2) - \mu 2CS}{\varepsilon^2 - \mu^2}, \tag{5a}$$

$$\overline{\mu} = \frac{\varepsilon 2CS - \mu(1+2S^2)}{\varepsilon^2 - \mu^2}, \tag{5b}$$

$$\mathcal{N} = \frac{1}{\pi^4}\left(\frac{1}{\varepsilon^2-\mu^2}\right)^2, \tag{5c}$$

where

$$\varepsilon = \tfrac{1}{2}[1 + 2(1-T)S^2], \tag{5d}$$

$$\mu = (1-T)CS. \tag{5e}$$

The lossless case can then be recovered by setting $T = 1$. We keep the same definitions of the calculation reported in Ref. [25] in which many details can be found. We observe that the four-mode Wigner function of the multiphoton state produced by the OPA is positive as is the two-mode one produced by the NOPA addressed by Banaszek and Wodkiewicz in Ref. [10]. We will show that in spite of such a positivity, it is possible to demonstrate the violation of a Bell's inequality by performing continuous-variable measurements on the state.

## III. VIOLATION OF THE BELL'S TEST

Let us recall briefly the test performed by Banaszek and Wodkiewicz on the NOPA state in Ref. [10]. Their nonlocality proof starts from the observation about the possibility of writing the two-mode Wigner function as

$$W(\alpha;\beta) = \frac{4}{\pi^2}\Pi(\alpha;\beta), \tag{6}$$

where $\Pi(\alpha;\beta) = \langle\hat{\Pi}(\alpha;\beta)\rangle$ is the expectation value of the displaced parity operator, i.e.,

$$\hat{\Pi}(\alpha;\beta) = \hat{D}_1(\alpha)(-1)^{\hat{n}_1}\hat{D}_1^{\dagger}(\alpha)\otimes\hat{D}_2(\beta)(-1)^{\hat{n}_2}\hat{D}_2^{\dagger}(\beta), \tag{7}$$

where $\hat{D}_1(\alpha)$ and $\hat{D}_2(\beta)$ are displacement operators for the two spatial modes $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, respectively, and $\hat{n}_1$ and $\hat{n}_2$ are the corresponding photon number operators. Since a parity-operator measurement gives a $\pm 1$ result, it fits perfectly for the CHSH inequality [26] and can be used to show nonlocality of the NOPA wave function. Using displacements in the phase space, the correlation between the two parties can be written as

$$E(\boldsymbol{a};\boldsymbol{b}) = \Pi(\alpha;\beta). \tag{8}$$

The nonlocality parameter can then be written as

$$B = \Pi(0;0) + \Pi(\sqrt{\mathcal{I}};0) + \Pi(0;-\sqrt{\mathcal{I}}) - \Pi(\sqrt{\mathcal{I}},-\sqrt{\mathcal{I}}), \tag{9}$$

where $\mathcal{I}$ is a positive parameter. For local theories, the inequality $-2 \leqslant B \leqslant 2$ holds. In Ref. [10] it is shown that this inequality is violated by the NOPA state even for the squeezing parameter going to $\infty$ and for the NOPA state approximating the EPR one.

In this work we investigate the nonlocality of a more general multiphoton state produced by an OPA in which the correlations are present in two degrees of freedom, the spatial and the polarization ones. We then need to generalize Eq. (9) to the four-dimensional case in which the Wigner function is expressed as a function of $\alpha = (\alpha_H,\alpha_V)$ and $\beta = (\beta_H,\beta_V)$, where the subscripts $H$ and $V$ stand for the horizontal and vertical polarizations, respectively. The $B$ parameter can then be rewritten as $B(\alpha_H,\alpha_V;\beta_H,\beta_V)$, and the violation results are functions of the nonlinear gain of the amplifier and the displacement of the state in the eight-dimensional phase space.

Let us conclude this section by discussing how the displaced parity operators can be experimentally realized. A direct method to measure the displaced parity operators of Eq. (7) exploits the displacement of the input field by a complex parameter $\alpha$, using a beam splitter to combine the incoming field with a coherent state, and the measurement of the parity of the output field. The $(+1)$ and the $(-1)$ outcomes are assigned at each shot of the experiment at Alice and Bob sites. In this scenario, no assumptions are required for both the state and the detection apparatus; hence, the proposed inequality is a genuine nonlocality test. An alternative approach exploits the high detection efficiency of homodyne measurements and relies on the connection between the displaced parity operators and the Wigner function of the state. In this case, the value of the nonlocality parameter $B$ can be retrieved by performing a homodyne measurement on the output field and by evaluating $B$ from the reconstructed Wigner function. This methodology requires a physical assumption for the detection apparatus, namely, it assumes that the Wigner function describes the optical radiation, thus reducing the set of local-hidden-variable models that can be rejected by the violation of Eq. (9).

### A. Violation in absence of losses

We first analyze the perfect case in which the multiphoton state is not affected by decoherence or loss. In this case the Wigner function in Eq. (4) reads

$$\begin{aligned} W_0\{\alpha,\beta,g\} &= \left(\frac{2}{\pi}\right)^4 \exp\left[-2(1+2S^2)\sum_{\pi=H,V}(|\alpha_\pi|^2+|\beta_\pi|^2)\right] \\ &\quad\times \exp\{-2CS[2\mathrm{Re}(\alpha_V\beta_H)-2\mathrm{Re}(\alpha_H\beta_V)]\}. \end{aligned} \quad (10)$$

The displaced parity can be written as

$$\Pi_0(\alpha_H,\alpha_V,\beta_H,\beta_V,g) = \left(\frac{\pi}{2}\right)^4 W_0(\alpha_H,\alpha_V,\beta_H,\beta_V,g). \quad (11)$$

Now the Bell parameter can be written as a function of eight phase-space variables and the nonlinear gain:

$$\begin{aligned} B_0(z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8,g) &= \Pi_0(z_1,z_2,z_3,z_4,g) + \Pi_0(z_1,z_2,z_7,z_8,g) \\ &\quad + \Pi_0(z_5,z_6,z_3,z_4,g) + -\Pi_0(z_5,z_6,z_7,z_8,g), \end{aligned} \quad (12)$$

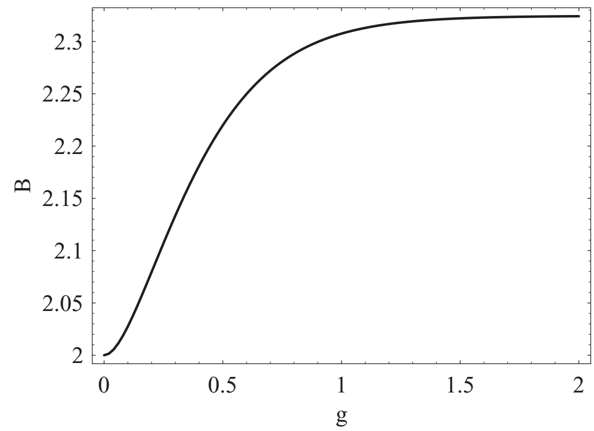


FIG. 2. Trend of the violation of Bell's test in the absence of losses as a function of the nonlinear gain $g$ of the amplifier.

where $z_i$, with $i = 1,\ldots,8$, represents the displacements in the phase space for the fixed $\{\pi_H,\pi_V\}$ polarization basis, where $z_1,z_2$ and $z_5,z_6$ are the displacements relative to the $\boldsymbol{k}_1$ mode while $z_3,z_4$ and $z_7,z_8$ are relative to the $\boldsymbol{k}_2$ spatial mode. We have then maximized the value of $B$ with respect to $z_i$ for different fixed values of the nonlinear gain $g$. We found numerically that for $g = 2$ the maximum violation is obtained for real displacements given by $z_1 = -0.0241$, $z_2 = -0.0066$, $z_3 = -0.0066$, $z_4 = 0.0241$, $z_5 = 0.0725$, $z_6 = 0.0198$, $z_7 = 0.0198$, and $z_8 = -0.0725$ and corresponds to a violation equal to

$$B_0^{\max} = 2.32. \quad (13)$$

Figure 2 reports the trend of the Bell's inequality violation as a function of the nonlinear gain. We observe that for low values of $g$ we have a small violation since Gaussian states with no squeezing cannot violate this inequality. This can be shown by noting that for $g = 0$ the Wigner distribution of the state is Gaussian and equally distributed according to

$$W_0(\alpha_i) = \mathcal{N}\exp\left(-\frac{1}{\sigma^2}\sum_i |\alpha_i|^2\right) = \mathcal{N}\prod_i \exp\left(-\frac{|\alpha_i|^2}{\sigma^2}\right), \quad (14)$$

and in such a case the $B$ expression can be grouped as

$$\begin{aligned} B_0(z_1,z_2,\ldots,z_8) &= \Pi_0(z_1,z_2)[\Pi_0(z_3,z_4)+\Pi_0(z_7,z_8)] \\ &\quad + \Pi_0(z_5,z_6)[\Pi_0(z_3,z_4)-\Pi_0(z_7,z_8)]. \end{aligned} \quad (15)$$

As each displaced-parity mean value obeys $|\Pi_0(z)| \leqslant 1$, we have $|B_0| \leqslant 2$.

For $g \geqslant 1$, the amount of violation progressively saturates and reaches its maximum value equal to $B_0^{\max}$ in Eq. (13). We observe that the points at which we can observe the maximal violation of the inequality depend on the nonlinear gain of the amplifier since it changes the squeezing of the generated state. Increasing the value of $g$, we obtain displacement values closer to the origin of the phase space, creating an experimental challenge.

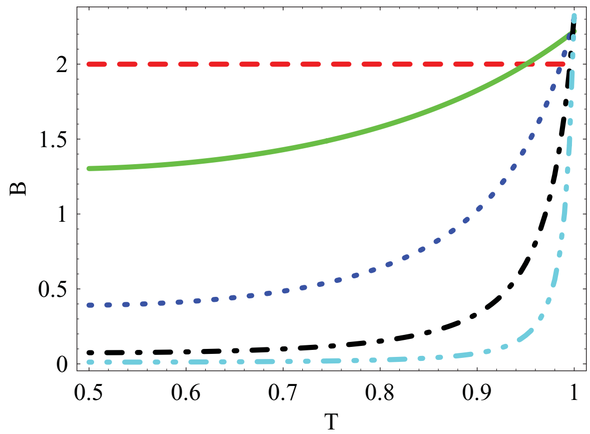


FIG. 3. (Color online) Trend of the violation of Bell's test as a function of the transmittivity $T$ for different values of the nonlinear gain $g$. Red dashed curve: $g = 0.01$. Green solid curve: $g = 0.5$. Blue dotted curve: $g = 1$. Black dash-dotted curve: $g = 1.5$. Cyan dash-dot-dotted curve: $g = 2$.

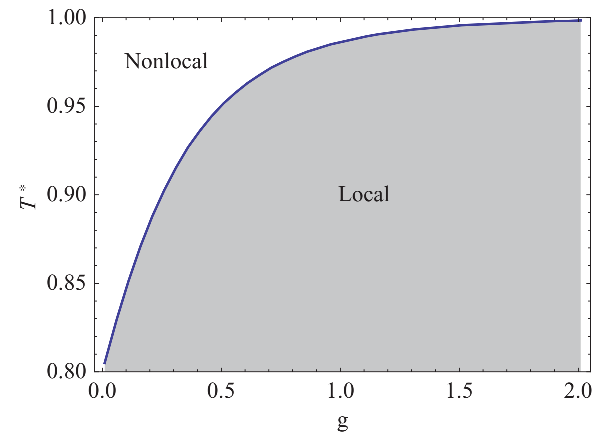


FIG. 4. (Color online) Limiting value of $T = T^*$ for which $B = 2$ as a function of the nonlinear gain $g$. The white region identifies the range of parameters where nonlocality can be detected.

### B. Violation in the presence of losses

Let us consider now the case in which the state undergoes a loss process, simulated by the presence of a beam splitter in Fig. 1. The loss contribution is taken into account by the parameter $T$, and the Wigner function in the lossy case is given by Eq. (4). The displaced parity of the phase space is given by

$$\Pi(\alpha_H,\alpha_V,\beta_H,\beta_V,g,T) = \left(\frac{\pi}{2}\right)^4 W_T(\alpha_H,\alpha_V,\beta_H,\beta_V,g,T), \tag{16}$$

and the violation turns out to be dependent on the loss parameter. Similarly to the perfect case, we define a Bell parameter by

$$\begin{aligned} &B(z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8,g,T) \\ &\quad = \Pi(z_1,z_2,z_3,z_4,g,T) + \Pi(z_1,z_2,z_7,z_8,g,T) \\ &\qquad + \Pi(z_5,z_6,z_3,z_4,g,T) - \Pi(z_5,z_6,z_7,z_8,g,T), \end{aligned} \tag{17}$$

and we maximize it with respect to $z_i$ for fixed values of $g$ and $T$. In Fig. 3 we report the trend of violation as a function of $T$ for different values of the nonlinear gain. We observe that the amount of violation decreases rapidly as a function of $T$, and the maximum value of $T = T^*$ for which we cannot observe a violation strongly depends on $g$. Figure 4 reports the trend of $T^*$, function of $g$, such that $B(z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8,g,T^*) = 2$ is a function of $g$. We observe that the value of $T^*$ increases with the increasing size of the system, and for high values of $g$, it becomes practically impossible to observe a violation even in the presence of a small loss. We stress that while the increase of nonlinear gain $g$ produces a larger squeezing of the multiphoton state, the presence of loss is responsible for the broadening of the Wigner function [27]. This interplay between the two effects determines the points at which we can see the maximal violation of the Bell's inequality.

### C. Violation in the presence of dephasing noise

In this section we consider the possibility of observing a violation of the analyzed Bell's inequality in the presence of dephasing noise. The main cause of dephasing for an optical field is given by uncontrolled fluctuations in the optical path of the beam. Hence, the action of this process can be modeled by adding a random phase shift in the optical mode. This can be evaluated by adding a unitary rotation $\hat{U}(\phi)$ in the field and by averaging over $\phi$ with a Gaussian probability distribution. The width $\sigma$ of the Gaussian distribution is related to the strength of the dephasing process. For large $\sigma$, this model corresponds to the complete loss of the phase properties of the field, described by the suppression of the nondiagonal elements of the density matrix $\hat{\rho} = \sum_{m,n} \rho_{m,n} |m\rangle\langle n|$, according to $\rho_{m,n} \rightarrow \rho_{m,n}\delta_{m,n}$.

Let us start by considering isotropic dephasing at one spatial mode. The ideal $\hat{\rho}_{\Psi^-} = |\Psi^-\rangle\langle\Psi^-|$ is then degraded to

$$\hat{\rho}^{\sigma}_{\Psi^-} = \int_{-\infty}^{\infty} d\phi p_\sigma(\phi)\big(\hat{U}^{(A)}_\phi \hat{\rho}_{\Psi^-} \hat{U}^{(A)\dagger}_\phi\big), \tag{18}$$

where

$$\hat{U}^{(A)}_\phi = \hat{U}^{AH}_\phi \otimes \hat{U}^{AV}_\phi \otimes \mathbb{1}^{BH} \otimes \mathbb{1}^{BV}, \tag{19}$$

and the superscripts $i$ and $j$ with $i = A,B$ and $j = H,V$ over the rotation operators stand for the spatial and polarization modes, respectively. The Wigner function for the state $\hat{\rho}^{\sigma}_{\Psi^-}$ of Eq. (18) after the action of dephasing noise can be written as

$$\begin{aligned} &W_\sigma(\alpha_H,\alpha_V,\beta_H,\beta_V,\sigma) \\ &\quad = \int_{-\infty}^{\infty} d\phi p_\sigma(\phi) W_0(\alpha_H e^{\imath\phi},\alpha_V e^{\imath\phi},\beta_H,\beta_V). \end{aligned} \tag{20}$$

This model corresponds to the case in which the random phase shift is equal for both polarization modes on each shot of the experiment. We can then define a Bell's parameter analogously to the lossless case according to

$$\begin{aligned} &B(z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8,g,\sigma) \\ &\quad = \Pi(z_1,z_2,z_3,z_4,g,\sigma) + \Pi(z_1,z_2,z_7,z_8,g,\sigma) \\ &\qquad + \Pi(z_5,z_6,z_3,z_4,g,\sigma) - \Pi(z_5,z_6,z_7,z_8,g,\sigma), \end{aligned} \tag{21}$$

where

$$\Pi(\alpha_H,\alpha_V,\beta_H,\beta_V,g,\sigma) = \left(\frac{\pi}{2}\right)^4 W_\sigma(\alpha_H,\alpha_V,\beta_H,\beta_V,g,\sigma). \tag{22}$$

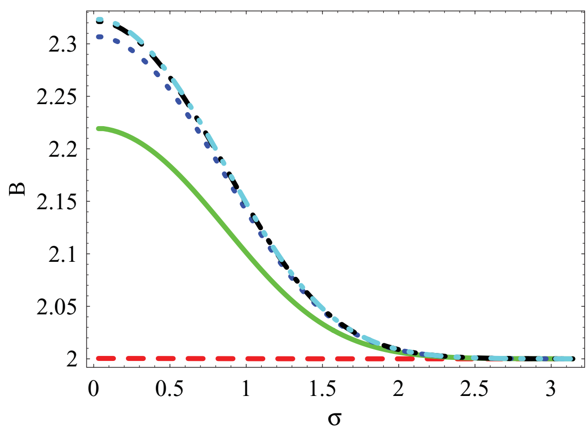


FIG. 5. (Color online) Bell parameter $B$ as a function of the dephasing strength $\sigma$ for different values of the gain $g$ of the amplifier. Red dashed curve: $g = 0.01$. Green solid curve: $g = 0.5$. Blue dotted curve: $g = 1$. Black dash-dotted curve: $g = 1.5$. Cyan dash-dot-dotted curve: $g = 2$.

We evaluated numerically the Bell's parameter by maximizing over the parameters $\{z_i\}$ as a function of the dephasing strength $\sigma$. The results are shown in Fig. 5. We observe that the amount of violation decreases with an increasing value of the noise strength $\sigma$. One can note that asymptotically the classical value 2 is approached from above: $B \to 2_+$. This means that in the presence of dephasing, the amount of violation detectable with the present measurement strategy progressively decreases, and the local-hidden-variable limit is reached for phase uncertainty at about or above $\pi$.

The present model can also be extended to the nonisotropic case, corresponding to phase fluctuations uncorrelated between the two polarization modes. We consider statistically independent phase fluctuations of the same strength,

$$\hat{U}^{(A)}_{\phi_H,\phi_V} = \hat{U}^{AH}_{\phi_H} \otimes \hat{U}^{AV}_{\phi_V} \otimes \mathbb{1}^{BH} \otimes \mathbb{1}^{BV}, \tag{23}$$

implying that the Wigner function after the dephasing process reads

$$\begin{aligned} W'_\sigma(\alpha_H,\alpha_V,\beta_H,\beta_V,\sigma) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\phi_H d\phi_V\, p_\sigma(\phi_H) p_\sigma(\phi_V) \\ &\quad \times W_0(\alpha_H e^{\imath\phi_H},\alpha_V e^{\imath\phi_V},\beta_H,\beta_V). \end{aligned} \tag{24}$$

We then analyzed numerically the violation of the Bell's inequality $|B| \leqslant 2$, where $B$ is the parameter defined in Eq. (21). We found that the results for the nonisotropic case present the same trend as that of the isotropic case reported in Fig. 5. Hence, the violation of the Bell's inequality after the action of a dephasing process is lost only for asymptotically large values of the noise strength $\sigma$. These results show that the present approach is robust to phase noise. This robustness with respect to this source of decoherence is obtained since the correlations in the output state of the noncollinear OPA are imprinted both in the diagonal and in the nondiagonal part of the state. The action of dephasing reduces the correlations in the nondiagonal part without affecting the diagonal part of the density matrix. For asymptotically large noise, only the diagonal terms are left, leading to a value of $|B| = 2$ of the Bell's parameter lying at the boundary between the local and the nonlocal regions.

## IV. CONCLUSIONS

In conclusion, we have theoretically addressed the problem of observing nonlocality by performing continuous-variable measurements on a multiphoton paradigmatic state, the one produced by an OPA showing correlations both in the spatial and in the polarization degrees of freedom. We have generalized the Bell's test proposed in Ref. [10] for a NOPA state for an enlarged four-mode multiphoton state. In Sec. II we reviewed the basic notions about the quantum system under investigation, and we have derived its Wigner function (Sec. II A). We have then introduced the nonlocality test in Sec. III by addressing both the lossless (Sec. III A) and the lossy cases (Sec. III B). We have shown that in the lossless case a maximum violation of $B_{\max} = 2.32$ can be reached for the increasing size of the investigated system while in the presence of loss, the amount of violation quickly decreases by increasing the nonlinear gain or the parameter of loss. This renders it extremely difficult to observe experimentally the quantum features for a system of increasing size even if an efficient measurement is performed on it. Finally, in Sec. III C we considered the possibility of observing a violation of the Bell's inequality in the presence of dephasing noise. We found that in the presence of dephasing, the investigated Bell's inequality is violated for any value of the number of photons, approaching the classical bound for large noise. These results demonstrate that the present approach is extremely robust to phase noise and suggest that the adoption of a measurement with high quantum efficiency seems to be a crucial requirement to observe nonlocality in the present system. In conclusion, we believe that our study should facilitate a deeper understanding of the problem of the observability of nonlocality by adopting continuous-variable measurements over quantum states of increasing size.

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[1] A. Einstein, B. Podolski, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] E. Schrodinger, Naturwissenschaften **23**, 807 (1935).
[3] J. S. Bell, Physics **1**, 195 (1964).
[4] D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1965).

[5] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).
[6] A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. **49**, 91 (1982).
[7] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **81**, 5039 (1998).
[8] W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, Phys. Rev. Lett. **81**, 3563 (1998).
[9] J. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).
[10] K. Banaszek and K. Wodkiewicz, Phys. Rev. A **58**, 4345 (1998).
[11] Z.-B. Chen, J.-W. Pan, G. Hou, and Y.-D. Zhang, Phys. Rev. Lett. **88**, 040406 (2002).
[12] M. Revzen, P. A. Mello, A. Mann, and L. M. Johansen, Phys. Rev. A **71**, 022103 (2005).
[13] P. M. Pearle, Phys. Rev. D **2**, 1418 (1970).
[14] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, Nature (London) **409**, 791 (2001).
[15] R. García-Patrón, J. Fiurášek, N. J. Cerf, J. Wenger, R. Tualle-Brouri, and P. Grangier, Phys. Rev. Lett. **93**, 130409 (2004).
[16] D. Cavalcanti, N. Brunner, P. Skrzypczyk, A. Salles, and V. Scarani, Phys. Rev. A **84**, 022105 (2011).
[17] N. Spagnolo, C. Vitelli, M. Paternostro, F. De Martini, and F. Sciarrino, Phys. Rev. A **84**, 032102 (2011).
[18] C. Vitelli, N. Spagnolo, L. Toffoli, F. Sciarrino, and F. De Martini, Phys. Rev. A **81**, 032123 (2010).
[19] J. Kofler and C. Brukner, Phys. Rev. Lett. **99**, 180403 (2007).
[20] J. Kofler and C. Brukner, Phys. Rev. Lett. **101**, 090403 (2008).
[21] C. Invernizzi, S. Olivares, M. G. A. Paris, and K. Banaszek, Phys. Rev. A **72**, 042105 (2005).
[22] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih, Phys. Rev. Lett. **75**, 4337 (1995).
[23] H. S. Eisenberg, G. H. Khoury, G. A. Durkin, C. Simon, and D. Bouwmeester, Phys. Rev. Lett. **93**, 193901 (2004).
[24] M. Caminati, F. De Martini, R. Perris, F. Sciarrino, and V. Secondi, Phys. Rev. A **73**, 032312 (2006).
[25] N. Spagnolo, C. Vitelli, T. De Angelis, F. Sciarrino, and F. De Martini, Phys. Rev. A **80**, 032318 (2009).
[26] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[27] R. C. Drumond, L. A. M. Souza, and M. Terra Cunha, Phys. Rev. A **82**, 042302 (2010).